Math 816
Exam #2
Due Friday, December 14th by 2pm

The rules:

(a) You may not discuss the problems with any living person other than the instructor. *No collaboration of any kind.*

(b) You may use any reference books you like, but everything you need to know can be found in the notes and the problem sets. Any ideas taken from other sources (including Dummit and Foote) must be properly credited to their original source; that is to say, you can get ideas from books but you can't pass them off as your own ideas.

I have read the above rules and understand that failure to follow them constitutes cheating.

Signature Date

Name

Problem 1 (10pts). Let R be a commutative ring with one. An *idempotent* in R is an element $e \in R$ such that $e^2 = e$. Prove that if e is an idempotent then so is $1 - e$, and that there is an isomorphism of rings

$$R \cong S \times T$$

where $S = eR$ and $T = (1 - e)R$.

Problem 2 (10pts). Do exercise 11 from Dummit and Foote §9.2.

Problem 3 (10pts). Consider the Gaussian integers $\alpha = -9 + 46i$ and $\beta = 32 + 43i$. Use Euclid's algorithm to compute the greatest common divisor, δ, of α and β, and find $x, y \in \mathbb{Z}[i]$ such that

$$\alpha x + \beta y = \delta.$$

Problem 4 (20pts). Define ideals $I, J \subset \mathbb{Q}[x]$ by

$$I = (x^3 - 2) \qquad J = (x^2 + x + 1).$$

(a) Show that I and J are comaximal and deduce that there is an isomorphism

$$\mathbb{Q}[x]/(IJ) \cong \mathbb{Q}[x]/I \times \mathbb{Q}[x]/J.$$

(b) Find $f(x) \in \mathbb{Q}[x]$ with the property that

$$f(x) + IJ \mapsto ((x^2 + 3) + I, (x - 4) + J)$$

under the isomorphism of (a).

Problem 5 (20pts). Consider the ring

$$R = \mathbb{Q}[x]/(f)$$

where

$$f(x) = (x^2 + 1)^2 (x^2 + x + 1)(x^2 - 3)^2.$$

(a) List all ideals in R. Which ones are prime? Which ones are maximal?
(b) Find all nilpotent elements of R.

Problem 6 (10pts). Let F be a field and define the *ring of Laurent series* $F((x))$ with coefficients in F to be the set of all expressions of the form

$$a_n x^n + a_{n+1} x^{n+1} + \cdots$$

with $n \in \mathbb{Z}$ and $a_n, a_{n+1}, \ldots \in F$. Note that we allow n to be negative; thus a Laurent series is like a power series, except we allow finitely many negative powers of x to appear. Prove that $F((x))$ is a field, and that in fact $F((x))$ is isomorphic to the fraction field of the power series ring $F[[x]]$.

Problem 7 (20pts). Let R be an integral domain and $I \subset R$ a prime ideal. Define a subset of the fraction field of R by

$$S = \{a/b \mid a, b \in R, b \notin I\}.$$

(a) Prove that S is a subring the fraction field of R.
(b) Prove that S has exactly one maximal ideal.