

4. DUE WEDNESDAY, OCT. 3

Exercise 4.1. Write down the left cosets of $H = \langle (1\ 2\ 3\ 4) \rangle$ in S_4 , and find a complete set of coset representatives. Now do the same for the right cosets. Are the left and right cosets the same?

Exercise 4.2. Consider the group D_{16} of symmetries of the regular octagon. Show that there is a homomorphism $D_{16} \rightarrow \{\pm 1\}$ taking all rotations to 1 and all reflections to -1 . Hint: The eight rotations form a subgroup of index 2 in D_{16} , so let D_{16} act on...

Exercise 4.3. Define $\rho, \tau \in D_{16}$ by letting ρ be clockwise rotation by $\pi/4$ and letting τ be reflection across your favorite diagonal a . Let D_{16} act on the set X of diagonals of the regular octagon (so $|X| = 4$). Determine the stabilizer of a , write down the cosets of Stab_a , and write down the bijection $D_{16}/\text{Stab}_a \rightarrow X$ of the Fundamental Theorem of Group Actions.

Exercise 4.4. Let G be a group and H a subgroup, and let G act on G/H in the usual way: $g \square xH = (gx)H$. Show that the stabilizer of the coset xH is the subgroup xHx^{-1} .

Exercise 4.5.

- (a) Show that if G is a finite group then $g^{|G|} = e$ for any $g \in G$.
- (b) Deduce the Little Fermat Theorem: if p is prime and $a \in \mathbb{Z}$ is not divisible by p then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Exercise 4.6. Prove that if G is a group and $A, B \subset G$ are subgroups with $|A|$ and $|B|$ relatively prime then $A \cap B = \{e\}$.

Exercise 4.7. Let $f : G \rightarrow H$ be a homomorphism of groups. Prove that for any $a, b \in G$

$$a \cdot \ker(f) = b \cdot \ker(f) \iff f(a) = f(b).$$

In words: the cosets of the kernel are the fibers (in the sense of Exercise 1.1) of the homomorphism.

Exercise 4.8. Prove that if p is a prime and G is a group of order p^n with $n > 0$ then G has an element of order p . Hint: first, why is there an element whose order is a positive power of p ?

Exercise 4.9. For any $\sigma \in S_n$ let $c(\sigma)$ denote the number of disjoint cycles of σ (including the singleton cycles). For example if $n = 8$ and $\sigma = (2)(5)(1\ 8\ 3)(4\ 7\ 6)$ then $c(\sigma) = 4$. Show that for any $\sigma \in S_n$ and any transposition $t \in S_n$

$$c(\sigma t) = c(\sigma) \pm 1.$$