5. Due Wednesday, Oct. 10

Exercise 5.1. Given $x \in G$ define the *centralizer* of x

$$\text{cent}_x = \{g \in G \mid gxg^{-1} = x\}.$$

Prove that cent_x is a subgroup of G, and that if G is finite then

$$|\text{cent}_x| \cdot |C_x| = |G|.$$

Exercise 5.2. Do exercises 5 (hint: use the previous problem) and 12 in Dummit and Foote §4.3.

Exercise 5.3. Prove Proposition 2.5.9 in the notes.

Exercise 5.4.

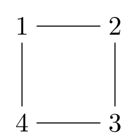
(a) Determine the conjugacy classes in D_8.
(b) The subgroups of D_8 are listed on p. 69 of Dummit and Foote: which of these subgroups are conjugate to which? For each of the subgroups, compute the normalizer.

Exercise 5.5. Define elements of S_4

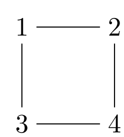
$$a = (12)(34) \qquad b = (13)(24) \qquad c = (14)(23)$$

and let $X = \{a, b, c\}$. From the notes we know that X is a single conjugacy class in S_4, and so S_4 acts transitively on X by conjugation. Verify that the resulting homomorphism $S_4 \to S_X$ is surjective. Hint: why is it enough to show that the image contains an element of order 2 and an element of order 3?

Exercise 5.6. Consider the usual action of D_8 on the vertices of the square. If we label the vertices as



then this action allows us to view D_8 as a subgroup $A \subset S_4$. If we instead label the vertices as



then D_8 can be viewed as a different subgroup $B \subset S_4$.

(a) Find a $\tau \in S_4$ such that $B = \tau A \tau^{-1}$.
(b) How many subgroups of S_4 are conjugate to A?

Exercise 5.7. This is a continuation of exercises from the last two problem sets. Given $\sigma \in S_n$ define the *sign* (or *signature*) of σ, denoted $\text{sgn}(\sigma)$, by

$$\text{sgn}(\sigma) = (-1)^{n - c(\sigma)}$$

where $c(\sigma)$ is the number of disjoint cycles (including the singleton cycles) of σ. Prove that

$$\text{sgn}(t_1 \cdots t_r) = (-1)^r$$

for any transpositions $t_1, \ldots, t_r$, and deduce that $\text{sgn} : S_n \to \{\pm 1\}$ is a homomorphism. Finally show that every cycle of length k in S_n has signature $(-1)^{k+1}$.