THE WALDHAUSEN CONJECTURE

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A *Heegaard* splitting of a closed and orientable 3–manifold M is a decomposition $M = H_1 \cup_S H_2$, where $S = \partial H_1 = \partial H_2 = H_1 \cap H_2$ is a closed embedded separating surface and each H_i ($i = 1, 2$) is a handlebody. The surface S is called a *Heegaard surface*, and the genus of S is the genus of this Heegaard splitting. Every closed and orientable 3–manifold has a Heegaard splitting.

A Heegaard splitting is *reducible* if there is an essential curve in the Heegaard surface that bounds compressing disks in both handlebodies. A Heegaard splitting $M = H_1 \cup_S H_2$ is *weakly reducible* [1] if there exist a pair of compressing disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$. If a Heegaard splitting is not reducible (resp. weakly reducible), then it is *irreducible* (resp. *strongly irreducible*). A lemma of Haken [4] says that if M is reducible, then every Heegaard splitting is reducible. Casson and Gordon [1] showed that if a Heegaard splitting of a non-Haken 3–manifold is irreducible, then it is strongly irreducible.

A conjecture of Waldhausen asserts that a closed orientable 3–manifold has only a finite number of Heegaard splittings of any given genus, up to homeomorphism. Johannson [6, 7] proved this conjecture for Haken manifolds. If M contains an incompressible torus, one may construct an infinite family of homeomorphic but non-isotopic Heegaard splittings using Dehn twists along the torus. The so-called generalized Waldhausen conjecture says that a closed, orientable and atoroidal 3–manifold has only finitely many Heegaard splittings of any genus, up to isotopy. This is also proved to be true for Haken manifolds by Johannson [6, 7]. In this report, we outline a proof of the (generalized) Waldhausen conjecture, see [8] for details. A much stronger theorem for non-Haken 3–manifolds is proved in [9].

Theorem 1. *A closed, orientable, irreducible and atoroidal 3–manifold has only finitely many Heegaard splittings in each genus, up to isotopy.*

The first ingredient of the proof is normal surface theory. A normal disk in a tetrahedron is a triangle or a quadrilateral that meets each edge in at most one point. A surface is called a normal surface, if the intersection of the surface with each tetrahedron consists of normal disks. Any incompressible surface is isotopic to a normal surface [3].

An almost normal piece in a tetrahedron is either an octagon, or an annulus obtained by connecting two normal disks using an unknotted tube, see [14] for a picture. An embedded surface S is *almost normal* if S is normal except in one tetrahedron T, where $T \cap S$ consists of normal disks and at most one almost normal piece. A theorem of Rubinstein and Stocking [12, 14] says that every strongly irreducible Heegaard surface is isotopic to an almost normal surface.

The second ingredient of the proof is the theory of branched surfaces. A branched surface is a 2-dimensional generalization of a train track, see [2, 11] for a picture

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and basic properties. Given an almost normal surface, by identifying the parallel normal disks, we get a branched surface, and we say that this branched surface *fully carries* the almost normal surface. Since there are only a finitely number of normal disk types, there are only finitely many different such branched surfaces. This is basically a construction in [2], where the authors showed that every incompressible surface is fully carried by one of a finite collection of branched surfaces. It follows trivially that, after isotopy, every strongly irreducible Heegaard surface is fully carried by one of a finite collection of branched surfaces.

For any branched surface B, there is a one-to-one correspondence between compact surfaces carried by B and nonnegative integer solutions to the system of branch equations, see [11] for basic definitions. Given any two compact surfaces F_1 and F_2 carried by the B, the sum F_1+F_2 is obtained by a canonical cutting-and-pasting along the double curves. The sum F_1+F_2 is also a surface carried by B and its corresponding integer solution is the vector sum of the two integer solutions corresponding to F_1 and F_2.

A crucial part of the proof is to analyze normal surfaces with nonnegative Euler characteristic. Since the 3–manifold is orientable, if a branched surfaces does not carry any normal 2–sphere or torus, then it dose not carry any closed normal surface with nonnegative Euler characteristic.

Theorem 2. *Let M be a closed orientable, irreducible and atoroidal 3–manifold, and suppose M is not a small Seifert fiber space. Then, M has a finite collection of branched surfaces, such that*

(1) *each branched surface in this collection is obtained by gluing together normal disks and at most one almost normal piece, similar to* [2]*,*
(2) *up to isotopy, each strongly irreducible Heegaard surface is fully carried by a branched surface in this collection,*
(3) *no branched surface in this collection carries any normal 2–sphere or normal torus.*

Proof of Theorem 1 using Theorem 2. Suppose a branched surface B in Theorem 2 fully carries an infinite family of almost normal surfaces of genus g. Then one can find two surfaces S_1 and S_2 in this family such that $S_2=S_1+T$, where T is a closed surface carried by B. Since S_1 and S_2 have the same genus, $\chi(S_1)=\chi(S_2)$ and hence $\chi(T)=0$. This implies that a component of T has nonnegative Euler characteristic. Moreover, T must be a normal surface, since S_1 and S_2 have the same almost normal piece. This contradicts Theorem 2. □

The first two conditions in Theorem 2 follow trivially from the construction of Floyd and Oertel mentioned above. To prove Theorem 2, we show that, given a branched surface B, one can perform some splittings and obtain a finite collection of branched surfaces such that any Heegaard surface fully carried by B is carried by a branched surface in this collection and no branched surface in this collection carries any normal 2–sphere of torus.

To eliminate 2–spheres, we use a 0–efficient triangulation. A 0–efficient triangulation is a triangulation with only one vertex and the only normal S^2 is the vertex-linking one. It is shown in [5] that, except for S^3 and certain Lens spaces, every closed and orientable 3–manifold admits a 0–efficient triangulation. Since the branched surfaces in our construction fully carry some Heegaard surfaces, we may assume our branched surfaces do not carry the vertex-linking normal S^2.

To eliminate normal tori, we need to consider measured laminations. In general, a measured lamination (carried by a branched surface) corresponds to an irrational point in the solution space of the branch equations. Given any measured lamination μ carried by B, the Euler characteristic of μ can be defined using a linear equation of the weights of μ at the branch sectors of B [10].

Let $\mathcal{PL}(B)$ be the projective solution space of the branch equations of B ($\mathcal{PL}(B)$ is also called the projective measured lamination space). Let $\mathcal{T}(B) \subset \mathcal{PL}(B)$ be the projective space of measured laminations with Euler characteristic 0. For any Heegaard surface S fully carried by B and $\mu \in \mathcal{T}(B)$, we may assume that S and μ lie in $N(B)$, where $N(B)$ is a regular neighborhood of B and $N(B)$ can be regarded as an I–bundle over B. We call an isotopy in $N(B)$ a B–isotopy if the isotopy is invariant on each I–fiber of $N(B)$. We call an arc $\alpha \subset S$ a *splitting arc* relative to μ if $\alpha \cap \mu \neq \emptyset$ under any B–isotopy of μ in $N(B)$. Note that if we delete a small neighborhood of α from $N(B)$, we get a regular neighborhood of a new branched surface B' that still carries S but does not carry μ. We say that B' is obtained by splitting B along α. The following lemma is an important step in the proof of Theorem 2. In the lemma, we use a combinatorial length, and one can consider the length of an arc to be the number of intersection points of the arc with the 2–skeleton.

Lemma 3. *Let $\mu \in \mathcal{T}(B)$. Then, for any strongly irreducible Heegaard surface S fully carried by B, there is a surface S' carried by B and isotopic to S in M, such that either (1) $S' \cap \mu = \emptyset$ or (2) there is a splitting arc $\alpha \subset S'$ relative to μ with length$(\alpha) < K(B, \mu)$, where $K(B, \mu)$ is a number depending on B and μ.*

A theorem in [10] says that any measured lamination is a disjoint union of a finite number of sublaminations of the following types: (1) a lamination consisting of compact leaves and (2) a minimal exceptional set (every leaf is dense). The proof of Lemma 3 is a discussion of two cases: (1) μ is a normal torus and (2) μ is an exceptional minimal set. In the first case, we use the fact that a normal torus in a 0–efficient triangulation always bounds a solid torus. In the second case, we first show that μ is contained in a nice solid torus. Then we apply a theorem in [13], which says that the intersection of a strongly irreducible Heegaard surface with a certain solid torus is very simple.

Let α be the splitting arc in Lemma 3. If we split B along α, we get a branched surface B' that carries a surface isotopic to S but does not carry μ. In fact, for each splitting arc α, there is an open neighborhood N of μ in $\mathcal{PL}(B)$ such that none of the measured laminations in N is carried by B'. Since we use a combinatorial length, up to isotopy fixing the 2–skeleton, there are only finitely many different splitting arcs with length less than $K(B, \mu)$. Thus, for each $\mu \in \mathcal{T}(B)$, we can define N_μ to be the intersection of all the neighborhoods of μ that correspond to these splitting arcs. So, N_μ is an open neighborhood of μ in $\mathcal{PL}(B)$. By compactness, there are a finite number of measured laminations $\mu_1, \dots, \mu_n$ in $\mathcal{T}(B)$ such that $\mathcal{T}(B) \subset \bigcup_{i=1}^{n} N_{\mu_i}$.

For each strongly irreducible Heegaard surface S, we have a finite set of splitting arcs $\alpha_1, \dots, \alpha_n$ relative to $\mu_1, \dots, \mu_n$ respectively. After splitting along these arcs we obtain a branched surface B_s that carries a surface isotopic to S but does not carry any measured lamination in $\bigcup_{i=1}^{n} N_{\mu_i}$. Since $\mathcal{T}(B) \subset \bigcup_{i=1}^{n} N_{\mu_i}$, B_s does not carry any normal torus. As the length of each α_i is bounded, there are only finitely many possibilities for the set $\{\alpha_i\}$. Hence, if we apply such splittings to every strongly irreducible Heegaard surface, we end up with only finitely many different branched surfaces, none of which carries any normal torus. This proves Theorem 2.

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