

Saddle tangencies and the distance of Heegaard splittings

TAO LI

We give another proof of a theorem of Scharlemann and Tomova and of a theorem of Hartshorn. The two theorems together say the following. Let M be a compact orientable irreducible 3-manifold and P a Heegaard surface of M . Suppose Q is either an incompressible surface or a strongly irreducible Heegaard surface in M . Then either the Hempel distance $d(P) \leq 2 \text{genus}(Q)$ or P is isotopic to Q . This theorem can be naturally extended to bicompressible but weakly incompressible surfaces.

[57N10; 57M50](#)

1 Introduction

Let P be a closed orientable surface of genus at least 2. The curve complex of P , introduced by Harvey [6], is the complex whose vertices are the isotopy classes of essential simple closed curves in P , and $k + 1$ vertices determine a k -simplex if they are represented by pairwise disjoint curves. We denote the curve complex of P by $\mathcal{C}(P)$. For any two vertices in $\mathcal{C}(P)$, the distance $d(x, y)$ is the minimal number of 1-simplices in a simplicial path joining x to y . To simplify notation, unless necessary, we do not distinguish a vertex in $\mathcal{C}(P)$ from a simple closed curve in P representing this vertex.

Let M be a compact orientable irreducible 3-manifold and P an embedded connected separating surface in M with $\text{genus}(P) \geq 2$. Let U and V be the closure of the two components of $M - P$. We may view $\partial U = \partial V = P$. As in Scharlemann–Tomova [14], we say P is *bicompressible* if P is compressible in both U and V . Let \mathcal{U} and \mathcal{V} be the set of vertices in $\mathcal{C}(P)$ represented by curves bounding compressing disks in U and V respectively. The distance $d(P)$ is defined to be the distance between \mathcal{U} and \mathcal{V} in the curve complex $\mathcal{C}(P)$. If P is a Heegaard surface, then $d(P)$ is the distance defined by Hempel [7]. We say P is *strongly irreducible* or following the definition in [14], say P is *weakly incompressible* if $d(P) \geq 2$, ie every compressing disk in U intersects every compressing disk in V .

Let Q be another closed orientable surface embedded in M . Let $g(Q)$ be the genus of Q . A theorem of Hartshorn [5] says that if Q is incompressible and P is a strongly irreducible Heegaard surface, then $d(P) \leq 2g(Q)$. In [14], Scharlemann and Tomova showed that if both P and Q are connected, separating, bicompressible and strongly irreducible, then either $d(P) \leq 2g(Q)$ or P and Q are well-separated or P and Q are isotopic. In particular, if both P and Q are strongly irreducible Heegaard surfaces, either P and Q are isotopic or $d(P) \leq 2g(Q)$.

Combining Hartshorn's theorem and the theorem of Scharlemann and Tomova, we have the following Theorem.

Theorem 1.1 *Suppose M is a compact orientable irreducible 3-manifold and P is a separating bicompressible and strongly irreducible (or weakly incompressible) surface in M . Let Q be an embedded closed orientable surface in M and suppose Q is either incompressible or separating, bicompressible but strongly irreducible. Then either*

- (1) $d(P) \leq 2g(Q)$, or
- (2) after isotopy, $P_t \cap Q = \emptyset$ for all t , where P_t ($t \in [0, 1]$) is a level surface in a sweep-out for P , see [Section 2](#) for definition, or
- (3) P and Q are isotopic.

Remark The statement of [Theorem 1.1](#) is basically the same as the main theorem of [14]. If Q is separating, bicompressible but strongly irreducible and $P_t \cap Q = \emptyset$ for all $t \in [0, 1]$, then it is easy to see that P and Q are well-separated. Note that part (3) of the theorem never happens if Q is incompressible.

In this paper, we give another proof of [Theorem 1.1](#). Some arguments were originally used in a different proof of the main theorem by the author [9]. The motivation for this paper is a conjecture in [9] which generalizes both the main theorem of [9] and the theorem of Scharlemann and Tomova. We hope this proof and the techniques in [9, 10] can lead to a solution of this conjecture. Some arguments in the proof are similar to those in [1, 14].

I would like to thank Marty Scharlemann for pointing out a mistake in an earlier version of the paper. The research was partially supported by NSF grant DMS-0406038.

2 Saddle tangencies

Notation Throughout this paper, we denote the interior of X by $\text{int}(X)$ for any space X .

Let P be a bicompressible surface and let U and V be the closure of the two components of $M - P$ as above. Let P^U and P^V be the possibly disconnected surfaces obtained by maximal compressing P in U and V respectively. Since M is irreducible, after capping off 2-sphere components by 3-balls, we may assume P^U and P^V do not contain 2-sphere components. Moreover, we may also assume $P^U \subset \text{int}(U)$ and $P^V \subset \text{int}(V)$. Since P is strongly irreducible, as in Casson–Gordon [3], P^U and P^V are incompressible in M . Furthermore, $P^U \cup P^V$ bounds a submanifold M_P of M and P is a strongly irreducible Heegaard surface of M_P . Note that if U is a handlebody, then $P^U = \emptyset$. If P is a Heegaard surface of M , then we may view $M_P = M$.

The surface P cuts M_P into a pair of compression bodies $U \cap M_P$ and $V \cap M_P$. There are a pair of properly embedded graphs $G^U \subset U \cap M_P$ and $G^V \subset V \cap M_P$ which are the spines of the two compression bodies. The endpoints of the graphs G^U and G^V lie in P^U and P^V respectively. Let $\Sigma_U = P^U \cup G^U$ and $\Sigma_V = P^V \cup G^V$, then $M_P - (\Sigma_U \cup \Sigma_V)$ is homeomorphic to $P \times (0, 1)$.

We consider a sweepout $H: P \times (I, \partial I) \rightarrow (M_P, \Sigma_U \cup \Sigma_V)$, see [11], where $I = [0, 1]$ and $H|_{P \times (0,1)}$ is an embedding. We denote $H(P \times \{x\})$ by P_x for any $x \in I$. We may assume $P_0 = \Sigma_U$, $P_1 = \Sigma_V$ and each P_x ($i \neq 0, 1$) is isotopic to P . To simplify notation, we will not distinguish $H(P \times (0, 1))$ from $P \times (0, 1)$.

Let $\pi: P \times I \rightarrow P$ be the projection. To simplify notation, we do not distinguish between an essential simple closed curve γ in P_x and the vertex represented by $\pi(\gamma)$ in the curve complex $\mathcal{C}(P)$.

Definition 2.1 Let Q be a properly embedded compact surface in M . We say Q is in *regular position with respect to $P \times I$* if

- (1) $Q \cap G^U$ and $Q \cap G^V$ consist of finitely many points and Q is transverse to $P^U \cup P^V$ and
- (2) Q is transverse to each P_x , $x \in (0, 1)$, except for finitely many critical levels $t_1, \dots, t_n \in (0, 1)$ and
- (3) at each critical level t_i , Q is transverse to P_{t_i} except for a single saddle or center tangency.

If $x \in (0, 1)$ is not one of the t_i 's, then we say x or P_x is a regular level. Clearly every embedded surface Q can be isotoped into a regular position.

Definition 2.2 We say Q is *irreducible with respect to $P \times I$* if

- (1) Q is in regular position with respect to $P \times I$ and

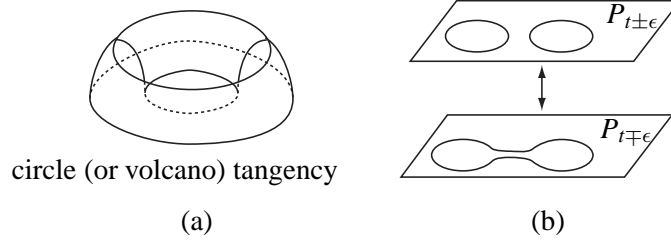


Figure 2.1

- (2) at each regular level P_x , if a component γ of $Q \cap P_x$ is trivial in P_x , then γ is also trivial in Q .

In this section, we assume Q is irreducible with respect to the sweepout $P \times I$. We first perform some isotopy on Q to eliminate center tangencies and trivial intersection curves. [Lemma 2.1](#) can be viewed as a special case of a theorem of Thurston [15] and [4, Theorem 7.1].

Lemma 2.1 *Let Q be an embedded surface in M and suppose Q is irreducible with respect to the sweepout $P \times I$. Then, one can perform an isotopy on Q so that*

- (1) $Q \cap (G^U \cup G^V)$ consists of finitely many points, Q is transverse to $P^U \cup P^V$, and $Q \cap (P^U \cup P^V)$ consists of curves essential in Q ;
- (2) Q is transverse to each P_x , $x \in (0, 1)$, except for finitely many critical levels $t_1, \dots, t_n \in (0, 1)$;
- (3) at each critical level t_i , Q is transverse to P_{t_i} except for a saddle or circle tangency, as shown in [Figure 2.1\(a\)](#);
- (4) at each regular level x , every component of $Q \cap P_x$ is an essential curve in P_x .

Proof Since $P^U \cup P^V$ is incompressible in M and M is irreducible, after some standard isotopy we may assume condition (1) in the lemma holds.

Note that the intersection of Q with $P \times I$ yields a (singular) foliation of $Q \cap M_P$ with each leaf a component of $Q \cap P_x$ for some $x \in I$. A singular point in the foliation is either a point in $Q \cap (G^U \cup G^V)$ or a saddle or center tangency.

Let x be a regular level and suppose a component γ of $Q \cap P_x$ is trivial in P_x . Suppose γ is innermost in P_x , ie the disk bounded by γ in P_x does not contain any other intersection curve with Q . Since Q is irreducible with respect to $P \times I$, γ bounds a

disk D_γ in Q . If the induced foliation on D_γ contains more than one singular point, since γ is trivial in P_x , we can construct a disk $D' \subset P \times (x - \epsilon, x + \epsilon)$ for some small ϵ such that

- (1) $\partial D' = \gamma$,
- (2) the induced foliation of $D' \cap (P \times I)$ consists of parallel circles except for a singular point corresponding to a center tangency,
- (3) $(Q - D_\gamma) \cup D'$ is embedded in M and irreducible with respect to $P \times I$.

Since M is irreducible, $(Q - D_\gamma) \cup D'$ is isotopic to Q . Moreover, the induced foliation on $(Q - D_\gamma) \cup D'$ has fewer singular points. So after finitely many such operations, we may assume that for any regular level x and for any component γ of $Q \cap P_x$ that is trivial in P_x , the disk bounded by γ in Q lies in M_P and is transverse to $P \times (0, 1)$ except for a single center tangency.

Let t be a critical level and suppose $Q \cap P_t$ contains a saddle tangency. Let ϵ be a sufficiently small number. So the component of $Q \cap (P \times [t - \epsilon, t + \epsilon])$ that contains the saddle tangent point is a pair of pants F . [Figure 2.1\(b\)](#) is a picture of the curves changing from $F \cap P_{t-\epsilon}$ to $F \cap P_{t+\epsilon}$.

We first claim that at most one component of ∂F is trivial in the corresponding level surface $P_{t \pm \epsilon}$. Let γ_1, γ_2 and γ_3 be the 3 components of ∂F and suppose γ_1 and γ_2 are both trivial in the corresponding level surfaces. Then by the change of $F \cap P_x$ near the saddle tangency as shown in [Figure 2.1\(b\)](#), γ_3 must also be trivial in the corresponding level surface $P_{t \pm \epsilon}$. Since Q is irreducible with respect to $P \times I$, γ_1 and γ_2 bound disks D_1 and D_2 in Q respectively. By the assumption above, the disk D_i does not contain any saddle tangency and hence $F \cap D_i = \gamma_i$, $i = 1, 2$. Thus $F \cup D_1 \cup D_2$ is a disk in Q bounded by γ_3 and $F \cup D_1 \cup D_2$ contains a saddle tangent point. This contradicts the assumption above. Thus at most one component of ∂F is trivial in $P_{t \pm \epsilon}$.

Let F and γ_i be as above. Suppose γ_1 and γ_2 lie in $P_{t-\epsilon}$ and γ_3 lies in $P_{t+\epsilon}$. If γ_1 is trivial in $P_{t-\epsilon}$ and let D_1 be the disk in Q bounded by γ_1 , then $F \cap D_1 = \gamma_1$ as above and $F \cup D_1$ is an annulus in Q bounded by $\gamma_2 \cup \gamma_3$. Since D_1 is isotopic to a disk in $P_{t-\epsilon}$, we can first push D_1 into $P \times [t - \epsilon, t + \epsilon]$, then as shown in [Figure 2.2\(a\)](#), we may perform another isotopy on Q canceling the center tangency in D_1 and the saddle tangency in F . If γ_3 is trivial in $P_{t+\epsilon}$, by the assumption above, both γ_1 and γ_2 are essential in $P_{t-\epsilon}$. Hence γ_1 and γ_2 must be parallel in $P_{t-\epsilon}$. Let D_3 be the disk in Q bounded by γ_3 . As above, $F \cap D_3 = \gamma_3$ and $F \cup D_3$ is an annulus in Q bounded by $\gamma_1 \cup \gamma_2$. Since D_3 is isotopic to the disk in $P_{t+\epsilon}$ bounded by γ_3 , we can first push the annulus $F \cup D_3$ into a ∂ -parallel annulus in $P \times [t - \epsilon, t + \epsilon]$. Then an isotopy as

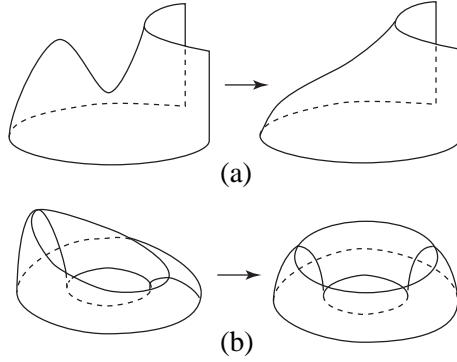


Figure 2.2

shown in [Figure 2.2\(b\)](#) can cancel the center tangency in D_3 and the saddle tangency in F , changing $F \cup D_3$ into an annulus with a circle (or volcano) tangency. Note that the circle tangency is an essential curve in the corresponding level surface P_x .

Note that condition (1) of the lemma implies that for a small ϵ , $Q \cap P_\epsilon$ and $Q \cap P_{1-\epsilon}$ consist of essential curves in P_ϵ and $P_{1-\epsilon}$ respectively. Since Q is not a 2-sphere, a curve of $Q \cap P_x$ that is trivial in P_x will eventually meet and cancel with a saddle tangency. Thus after a finite number of isotopies as above, we can eliminate all the curves of $Q \cap P_x$ that are trivial in P_x , and get a surface Q satisfying all the conditions in the lemma. \square

Note that a circle tangency does not create any singularity in the foliation of $Q \cap M_P$ induced from $P \times I$. Thus, if Q satisfies the conditions in [Lemma 2.1](#), a singular point in the foliation of $Q \cap M_P$ corresponds to either a saddle tangency or a point in $Q \cap (G^U \cup G^V)$. It is possible that Q does not intersect $M_P = P \times I$, ie $P_t \times Q = \emptyset$ for all t , after isotopy.

Lemma 2.2 *Let P and Q be as above and assume Q satisfies the conditions in [Lemma 2.1](#). Suppose $Q \cap \Sigma_U \neq \emptyset$ and $Q \cap \Sigma_V \neq \emptyset$. Then the distance $d(P) = d(U, V) \leq 2g(Q)$.*

Proof Since Q is connected and P is separating, $Q \cap \Sigma_U \neq \emptyset$ and $Q \cap \Sigma_V \neq \emptyset$ imply $Q \cap P_t \neq \emptyset$ for every t .

Claim 1 Let t be a critical level and ϵ a sufficiently small number. Let σ and w be any components of $Q \cap P_{t-\epsilon}$ and $Q \cap P_{t+\epsilon}$ respectively. Then $d(\sigma, w) \leq 1$.

Proof of Claim 1 The claim is obvious if P_t contains a circle tangency. So we suppose P_t contains a saddle tangency. Let F be the component of $Q \cap (P \times [t - \epsilon, t + \epsilon])$ that contains the saddle tangency. Then F is a pair of pants and all other components of $Q \cap (P \times [t - \epsilon, t + \epsilon])$ are essential vertical annuli in $P \times [t - \epsilon, t + \epsilon]$. If σ is a boundary curve of a vertical annulus, then σ is isotopic to a component of $Q \cap P_{t+\epsilon}$ and hence $d(\sigma, w) \leq 1$ for any curve w in $Q \cap P_{t+\epsilon}$. If neither σ nor w is a boundary curve of a vertical annulus, then σ and w are components of ∂F and $d(\sigma, w) = 1$ as shown in Figure 2.1(b). \square

Let $s_0 < \dots < s_n$ be a collection of regular levels such that $s_0 = \delta$, $s_n = 1 - \delta$ for a small δ and there is exactly one saddle or circle tangency in each $P \times (s_i, s_{i+1})$. Let $\Gamma_i = Q \cap P_{s_i}$ for each i .

Recall that $P_0 = \Sigma_U = P^U \cup G^U$ and $P_1 = \Sigma_V = P^V \cup G^V$. Since $s_0 = \delta$ for a small δ , we may assume $d(\mathcal{U}, \Gamma_0)$ is either 0 or 1, and if $d(\mathcal{U}, \Gamma_0) = 1$ then $d(\mathcal{U}, \sigma) = 1$ for any component σ of Γ_0 . Similarly, $d(\mathcal{V}, \Gamma_n)$ is either 0 or 1, and if $d(\mathcal{V}, \Gamma_n) = 1$ then $d(\mathcal{V}, w) = 1$ for any component w of Γ_n .

Suppose $d(\mathcal{U}, \mathcal{V}) > 2g(Q)$ and hence $d(\mathcal{U}, \mathcal{V}) > 2$. Let k be the smallest integer such that $d(\mathcal{U}, \Gamma_k) \neq 0$ and l the largest integer such that $d(\Gamma_l, \mathcal{V}) \neq 0$. Since $d(\mathcal{U}, \Gamma_0) \leq 1$ and $d(\mathcal{V}, \Gamma_n) \leq 1$, by Claim 1 above, $d(\mathcal{U}, \Gamma_k) = d(\Gamma_l, \mathcal{V}) = 1$ and $k \leq l$. Without loss of generality, we assume $k < l$. Next we show that every curve in Γ_k is essential in Q . Suppose a curve γ in Γ_k is trivial in Q and let D be the disk bounded by γ in Q . Since P^U and P^V are incompressible, we may assume $D \subset M_P$. Since P is a strongly irreducible Heegaard surface of M_P , by the no-nesting lemma of Scharlemann [12, Lemma 2.2], γ must bound a disk in one of the two compression bodies, ie either $\gamma \in \mathcal{U}$ or $\gamma \in \mathcal{V}$. However, $\gamma \in \mathcal{U}$ contradicts $d(\mathcal{U}, \Gamma_k) \neq 0$, and $\gamma \in \mathcal{V}$ contradicts $d(\mathcal{U}, \mathcal{V}) > 2$. Thus every curve in Γ_k must be essential in Q . Similarly every curve in Γ_l is also essential in Q .

Let $Q' = Q \cap (P \times [s_k, s_l])$, and let U' and V' be the two components of $M - P \times (s_k, s_l)$ containing G^U and G^V respectively, $F_U = Q \cap U'$ and $F_V = Q \cap V'$. Since Γ_k and Γ_l are essential in Q , F_U , Q' and F_V are essential subsurfaces of $Q = F_U \cup Q' \cup F_V$.

Claim 2 Let σ_k be any component of Γ_k , then $d(\sigma_k, \mathcal{U}) \leq 1$.

Proof of Claim 2 By the definition of k and the argument above, Claim 2 holds if $k = 0$. If $k > 0$, then $d(\mathcal{U}, \Gamma_{k-1}) = 0$ and $d(\mathcal{U}, \Gamma_k) = 1$. Let w be a component of Γ_{k-1} that represents a vertex in \mathcal{U} . By Claim 1, for any component σ_k of Γ_k , $d(\sigma_k, \mathcal{U}) \leq d(\sigma_k, w) \leq 1$. \square

Claim 3 There is a component σ_k of Γ_k and a component σ_l of Γ_l such that $d(\sigma_k, \sigma_l) \leq -\chi(Q')$.

Proof of Claim 3 Let $t_1 < \dots < t_N$ be the levels in (s_k, s_l) that contain the saddle tangencies. For a sufficiently small ϵ , $P \times [t_i + \epsilon, t_{i+1} - \epsilon]$ contains no saddle tangency for each i (to simplify notation we set $t_0 + \epsilon = s_k$ and $t_{N+1} - \epsilon = s_l$). So by the conditions in [Lemma 2.1](#), $Q \cap (P \times [t_i + \epsilon, t_{i+1} - \epsilon])$ consists of annuli for each $i = 0, \dots, N$. If $Q \cap (P \times [t_i + \epsilon, t_{i+1} - \epsilon])$ consists of ∂ -parallel annuli, then $Q \cap P_t = \emptyset$ for some t after isotopy, a contradiction to our assumption at the beginning. Thus an annulus component A_i of $Q \cap (P \times [t_i + \epsilon, t_{i+1} - \epsilon])$ is vertical. We choose γ_i to be a meridian circle in A_i for each i and assume $\sigma_k = \gamma_0 = A_0 \cap P_{s_k} \subset \Gamma_k$ and $\sigma_l = \gamma_N = A_N \cap P_{s_l} \subset \Gamma_l$. Since each A_i is vertical, γ_i is parallel to a component of $Q \cap P_{t_{i+1} - \epsilon}$. Similarly γ_{i+1} is parallel to a component of $Q \cap P_{t_{i+1} + \epsilon}$. By [Claim 1](#), $d(\gamma_i, \gamma_{i+1}) \leq 1$ and hence $d(\sigma_k, \sigma_l) = d(\gamma_0, \gamma_N) \leq N$. Moreover, since the only singular points in the induced foliation of Q' are the saddle tangencies, by a standard index argument, $-\chi(Q') = N$ and hence $d(\sigma_k, \sigma_l) \leq -\chi(Q')$. \square

Since Q' , F_U and F_V are essential subsurfaces of Q , $\chi(Q') \geq \chi(Q)$. By [Claim 2](#), $d(\sigma_k, \mathcal{U}) \leq 1$ and similarly $d(\sigma_l, \mathcal{V}) \leq 1$. Therefore, $d(\mathcal{U}, \mathcal{V}) \leq d(\mathcal{U}, \sigma_k) + d(\sigma_k, \sigma_l) + d(\sigma_l, \mathcal{V}) \leq 1 - \chi(Q') + 1 \leq 2 - \chi(Q) = 2g(Q)$. \square

Corollary 2.3 Let P and Q be as in [Theorem 1.1](#). Then [Theorem 1.1](#) holds if Q is incompressible.

Proof If Q is incompressible, then Q can be isotoped to be irreducible with respect to $P \times I$. Moreover, if $Q \cap \Sigma_U = \emptyset$, then since Q is incompressible, Q can be isotoped out of the compression body $M_P - N(\Sigma_U)$. Hence $Q \cap M_P = \emptyset$ after isotopy and part (2) of [Theorem 1.1](#) holds. Now [Corollary 2.3](#) follows from [Lemma 2.1](#) and [Lemma 2.2](#). \square

3 The graphics of sweepouts

In this section, we suppose Q is separating, bicompressible and strongly irreducible.

Let X and Y be the closure of the 2 components of $M - Q$. Let Q^X and Q^Y be the possibly disconnected surfaces obtained by maximal compressing Q in X and Y respectively and capping off 2-sphere components by 3-balls. Similar to the argument on P^U and P^V above, we may assume $Q^X \subset \text{int}(X)$ and $Q^Y \subset \text{int}(Y)$ are incompressible

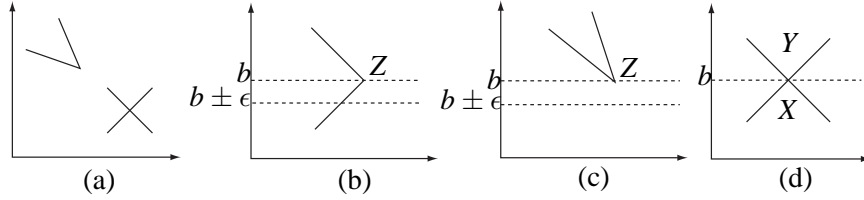


Figure 3.1

in M . Furthermore, $Q^X \cup Q^Y$ bounds a submanifold M_Q of M and Q is a strongly irreducible Heegaard surface of M_Q . If X is a handlebody, then $Q^X = \emptyset$. If Q is a Heegaard surface of M , we may view $M_Q = M$.

As in Section 2, the surface Q cuts M_Q into a pair of compression bodies $X \cap M_Q$ and $Y \cap M_Q$. Let graphs $G^X \subset X \cap M_Q$ and $G^Y \subset Y \cap M_Q$ be the spines of the two compression bodies and let $\Sigma_X = Q^X \cup G^X$ and $\Sigma_Y = Q^Y \cup G^Y$. Then $M_Q - (\Sigma_X \cup \Sigma_Y)$ is homeomorphic to $Q \times (0, 1)$.

Now we consider the two sweepouts $H: P \times (I, \partial I) \rightarrow (M_P, \Sigma_U \cup \Sigma_V)$ and $H': Q \times (I, \partial I) \rightarrow (M_Q, \Sigma_X \cup \Sigma_Y)$. Let $P_t = H(P \times \{t\})$ and $Q_t = H'(Q \times \{t\})$, $t \in I$. We may assume $Q_0 = \Sigma_X$, $Q_1 = \Sigma_Y$ and Q_x is isotopic to Q for each $x \in (0, 1)$.

The graphic Λ of the sweepouts, defined in [11], is the set of points $(s, t) \in (0, 1) \times (0, 1)$ such that P_s is not transverse to Q_t . We briefly describe the graphic below and refer to [11] for more details. As in [11], Cerf theory implies that after some isotopy, we may assume that Λ is a graph in $(0, 1) \times (0, 1)$ whose edges are the set of points (s, t) for which P_s is transverse to Q_t except for a single saddle or center tangency. There are two types of vertices in Λ , birth-and-death vertices and crossing vertices, as shown in Figure 3.1(a). Moreover, each arc $(0, 1) \times \{x\}$ contains at most one vertex, $x \in (0, 1)$. The complement of Λ , $(0, 1) \times (0, 1) - \Lambda$, is a finite collection of regions. Note that for every (s, t) in $(0, 1) \times (0, 1) - \Lambda$, P_s is transverse to Q_t , and for any two points (s, t) and (s', t') in the same region, $P_s \cap Q_t$ and $P_{s'} \cap Q_{t'}$ have the same intersection pattern.

Let $(s, t) \in (0, 1) \times (0, 1) - \Lambda$. Suppose there are disks or annuli $D_P \subset P_s$ and $D_Q \subset Q_t$ with $D_P \cap D_Q = \partial D_P = \partial D_Q \subset P_s \cap Q_t$. Suppose D_P is parallel to D_Q (fixing $\partial D_P = \partial D_Q$) in M and suppose $D_P \cup D_Q$ bounds a 3-ball or solid torus E . Moreover, suppose $Q_t \cap E = D_Q$. Then we can perform an isotopy on Q_t by pushing D_Q across E and remove the intersection $\partial D_P = \partial D_Q$. This isotopy is the same as the operation that changes Q_t to $(Q_t - D_Q) \cup D_P$ and then perturbs the resulting surface. We call such an isotopy a *trivial isotopy* on Q_t at P_s . We may view a trivial isotopy on

Q_t is associated with the disk or annulus $D_Q \subset Q_t$. Suppose we are to perform another trivial isotopy on Q_t at $P_{s'}$ and let $D'_Q \subset Q_t$ be the disk or annulus in the isotopy as above. Then D_Q and D'_Q are either disjoint or nested in Q_t . Thus either the two trivial isotopies are disjoint or we can view one isotopy as a middle step of the other.

Labelling For any Q_t , we use X_t (resp. Y_t) to denote the component of $M - Q_t$ that contains Σ_X (resp. Σ_Y). We label a region, ie a component of $(0, 1) \times (0, 1) - \Lambda$, X (resp. Y) if for a point (s, t) in the region, either (1) there is a component of $P_s \cap Q_t$ that is trivial in P_s but bounds an essential disk in X_t (resp. Y_t), or (2) Σ_U or Σ_V lies in Y_t (resp. X_t) after some *trivial isotopies* on Q_t at finitely many regular levels P_x . We label $t \in (0, 1)$ X (resp. Y) if the horizontal line segment $(0, 1) \times \{t\}$ intersects a region labelled X (resp. Y). Note that since a trivial isotopy does not increase $|\Sigma_U \cap Q_t|$ or $|\Sigma_V \cap Q_t|$, if t is not labelled, $Q_t \cap \Sigma_U \neq \emptyset$ and $Q_t \cap \Sigma_V \neq \emptyset$ after any trivial isotopies.

Lemma 3.1 *Either Theorem 1.1 holds or for a sufficiently small $\delta > 0$, δ is labelled X and $1 - \delta$ is labelled Y .*

Proof For a sufficiently small $\delta > 0$, $H'(Q \times [0, \delta])$ is a small neighborhood of $\Sigma_X = Q^X \cup G^X$. If $P_s \cap G^X \neq \emptyset$ for some s , then by definition, δ is labelled X for a sufficiently small δ . Suppose δ is not labelled X , then the graph G^X must be disjoint from $M_P = H(P \times I)$. Moreover, if $Q^X \cap P_t = \emptyset$ for some t after isotopy, since Q^X is incompressible, we can isotope Q^X out of the two compression bodies $M_P - P_t$. Hence, $Q_\delta \cap M_P = \emptyset$ after isotopy and part (2) of Theorem 1.1 holds. If $Q^X \cap P_t \neq \emptyset$ for all t , since Q^X is incompressible, by Corollary 2.3, $d(P) \leq 2g(Q^X) \leq 2g(Q)$ and Theorem 1.1 follows. The proof for $1 - \delta$ is similar. \square

Lemma 3.2 *Either Theorem 1.1 holds or no $t \in (0, 1)$ is labelled both X and Y .*

Proof We first remark that if $\Sigma_U \subset Y_t$ then one cannot move Σ_U to X_t by a trivial isotopy, since if this happens, then Σ_U must lie in E , where E is the 3-ball or solid torus in the trivial isotopy described above. However, since $g(P) \geq 2$ and P is strongly irreducible, Σ_U cannot lie in a 3-ball or solid torus by [3]. So by our labelling, if t is labelled both X and Y , then one can always find $s_1 \neq s_2$ such that (s_1, t) and (s_2, t) are points in $(0, 1) \times (0, 1) - \Lambda$ and one of the following 3 cases occurs.

(1). A component of $P_{s_1} \cap Q_t$ contains a curve bounding an essential disk D_X in X_t and a component of $P_{s_2} \cap Q_t$ contains a curve bounding an essential disk D_Y in Y_t . In

this case, since $s_1 \neq s_2$, $\partial D_X \cap \partial D_Y = \emptyset$ in Q_t , which contradicts the assumption that Q is strongly irreducible.

(2). After trivial isotopies, $\Sigma_U \subset Y_t$ and $\Sigma_V \subset X_t$. This means that $Q_t \subset P \times (0, 1) \subset M_P$ and Q_t separates Σ_U and Σ_V in M_P . If Q_t is incompressible in $P \times (0, 1)$, then Q_t is isotopic to P and [Theorem 1.1](#) holds. Suppose Q_t is compressible in $P \times (0, 1)$. Similar to the construction of M_Q earlier, by maximal compressing Q_t in $P \times (0, 1)$ on both sides and capping off 2-sphere components, we obtain a submanifold M'_Q of $P \times (0, 1)$ such that Q_t is a strongly irreducible Heegaard surface of M'_Q . Moreover, by [\[3\]](#), $\partial M'_Q$ is incompressible in $P \times (0, 1)$. So each component of $\partial M'_Q$ is parallel to P and M'_Q must be a product of P and an interval. Thus we can view Q_t as a strongly irreducible Heegaard surface of a product $P \times [0, 1]$. By Scharlemann–Thompson [\[13\]](#), either Q_t is isotopic to P or Q_t cuts $P \times [0, 1]$ into a handlebody and a compression body. In the later case, both Σ_U and Σ_V lie in Y_t (or both in X_t), a contradiction.

(3). After trivial isotopies, $\Sigma_U \subset Y_t$ and a component of $P_{s_1} \cap Q_t$ contains a curve γ that is trivial in P_{s_1} and bounds an essential disk D in Y_t . Note that if a component of $P_{s_1} \cap Q_t$ also bounds an essential disk in X_t , then this contradicts that Q is strongly irreducible as in case (1). Thus, after some isotopy on Q_t , we may assume that γ is innermost in P_{s_1} and the disk D bounded by γ in P_{s_1} is an essential disk in Y_t . Since $\Sigma_U \subset Y_t$ and $D \subset Y_t - \Sigma_U$, by maximal compressing Q_t in $Y_t - \Sigma_U$ and capping off 2-sphere components, we obtained a (possibly disconnected) surface Q_t^Y . Note that $Q_t^Y \neq \emptyset$ because Σ_U is not contained in a 3-ball. Since Q_t is strongly irreducible, by [\[3\]](#), Q_t^Y is incompressible in $M - \Sigma_U$. Note that if P is a Heegaard surface of a closed manifold M , this is already a contradiction since Q_t^Y lies in the handlebody $M - N(\Sigma_U)$ and cannot be incompressible. Q_t^Y cuts Y_t into H_1 and H_2 , where H_2 is the compression body bounded by Q_t and Q_t^Y . Since the compressions on Q_t are disjoint from Σ_U and since Σ_U does not lie in a 3-ball, $\Sigma_U \cap H_2 = \emptyset$. Hence $\Sigma_U \subset H_1$. Since Q_t^Y is incompressible in $M - \Sigma_U$, we can push Q_t^Y out of the compression body $M_P - N(\Sigma_U)$ or equivalently push $M_P - N(\Sigma_U)$ into H_1 . So we can isotope M_P into H_1 . In particular, $Q_t \cap M_P = \emptyset$ after isotopy and part(2) of [Theorem 1.1](#) holds. \square

Lemma 3.3 *If $t \in (0, 1)$ has no label and $(0, 1) \times \{t\}$ contains no vertex of Λ , then Q_t is irreducible with respect to $P \times I$ and [Theorem 1.1](#) holds.*

Proof Since $(0, 1) \times \{t\}$ contains no vertex of Λ , Q_t is in regular position with respect to $P \times I$. For any $(s, t) \notin \Lambda$, suppose a curve γ in $P_s \cap Q_t$ is trivial in P_s . If γ is an essential curve in Q_t , by assuming γ to be an innermost such curve, the disk bounded by γ in P_s can be isotoped to be an essential disk in either X_t or Y_t . Since $t \in (0, 1)$

has no label, γ must be trivial in Q_t . Thus by definition, Q_t is irreducible with respect to $P \times I$. So after isotopy we may assume Q satisfies the conditions in [Lemma 2.1](#). Moreover, since t has no label, $Q_t \cap \Sigma_U \neq \emptyset$ and $Q_t \cap \Sigma_V \neq \emptyset$ after the isotopy in the proof of [Lemma 2.1](#). So [Theorem 1.1](#) follows from [Lemma 2.2](#). \square

Suppose [Theorem 1.1](#) is not true. Then by [Lemma 3.1](#), for a small δ , δ is labelled X and $1 - \delta$ is labelled Y . As t changes from δ to $1 - \delta$, the label changes from X to Y . So by [Lemma 3.2](#) and [Lemma 3.3](#), there must be a number $b \in (0, 1)$ such that

- (1) $(0, 1) \times \{b\}$ contains a vertex of Λ and
- (2) b has no label and
- (3) $b - \epsilon$ is labelled X and $b + \epsilon$ is labelled Y for sufficiently small $\epsilon > 0$.

Let $Z = (a, b)$ be the vertex of Λ in $(0, 1) \times \{b\}$. If Z is a birth-and-death vertex, then since no region that intersects $(0, 1) \times \{b\}$ is labelled, as shown in [Figure 3.1\(b\)](#) and (c), after perturb $(0, 1) \times \{b\}$ a little, we can find a line segment $(0, 1) \times \{b \pm \epsilon\}$ that does not intersect any labelled region, a contradiction to our assumption above. Therefore, $Z = (a, b)$ must be a crossing vertex. [Figure 3.1\(d\)](#) is a picture of Z .

Since $Z = (a, b)$ is a crossing vertex, as explained in [[11](#)] (see Kobayashi–Saeki [[8](#), Figure 2.6]), P_a is transverse to Q_b except for two saddle tangencies. Since b is not labelled, for any $s \neq a$ in $(0, 1)$, either (1) $P_s \cap Q_b$ contains a single center or saddle tangency or (2) P_s is transverse to Q_b and if a component of $P_s \cap Q_b$ is trivial in P_s then it is also trivial in Q_b . Moreover, after trivial isotopies, $Q_b \cap \Sigma_U \neq \emptyset$ and $Q_b \cap \Sigma_V \neq \emptyset$. Since P is separating and Q is connected, this implies that $Q_b \cap P_s \neq \emptyset$ for all $s \in I$.

Now we consider $Q_b \cap (P \times [a - \epsilon, a + \epsilon])$ for a small ϵ . Let F be the union of the components of $Q_b \cap (P \times [a - \epsilon, a + \epsilon])$ that contain the two saddle tangencies. Thus F is either the union of two disjoint pairs of pants or a connected surface with $\chi(F) = -2$. All other components of $Q_b \cap (P \times [a - \epsilon, a + \epsilon])$, denoted by A_1, \dots, A_m , are vertical annuli in $P \times [a - \epsilon, a + \epsilon]$.

Next we consider the case that a component of $Q_b \cap P_{a \pm \epsilon}$ is trivial in $P_{a \pm \epsilon}$. If a component γ of ∂A_i , $i = 1, \dots, m$, is trivial and innermost in $P_{a \pm \epsilon}$, then by our assumption, γ bounds a disk D_γ in Q_b . We can perform a trivial isotopy on Q_b by pushing the disk $D_\gamma \cup A_i$ away from $P \times [a - \epsilon, a + \epsilon]$. Thus, after a finite number of such operations, we may assume the boundary of every annular component A_i is essential in $P_{a \pm \epsilon}$.

Suppose a component γ of ∂F is an innermost trivial curve in $P_{a\pm\epsilon}$. So γ bounds a disk D_γ in Q_b . If D_γ contains a component of F , then as in the proof of [Lemma 2.1](#), after replacing D_γ by a disk which is transverse to every P_x except for a single center tangency, we get a surface isotopic to Q_b and has at most one saddle tangency in $P \times [a - \epsilon, a + \epsilon]$. This means that after the isotopy, Q_b is irreducible with respect to $P \times I$ and [Theorem 1.1](#) follows from [Lemma 2.2](#) and [Lemma 3.3](#). So we may assume that $D_\gamma \cap F = \gamma$ for any component γ of ∂F that is trivial in $P_{a\pm\epsilon}$.

Let \widehat{F} be the union of F and all the disks D_γ in Q_b bounded by ∂F as above. We may push all such disks D_γ into $P \times (a - \epsilon, a + \epsilon)$ and isotope \widehat{F} into a surface properly embedded in $P \times [a - \epsilon, a + \epsilon]$. By the construction, $\partial\widehat{F}$ is essential in $P_{a\pm\epsilon}$. So \widehat{F} has no disk component. If \widehat{F} consists of annuli, then since $\partial\widehat{F}$ is essential in $P_{a\pm\epsilon}$, each annulus is either vertical or ∂ -parallel in $P \times [a - \epsilon, a + \epsilon]$. Thus, after some isotopy, Q_b becomes irreducible with respect to $P \times I$ and [Theorem 1.1](#) follows from [Lemma 2.2](#) and [Lemma 3.3](#). So we may assume $\chi(\widehat{F})$ is either -2 or -1 , ie at most one component of ∂F is trivial in $P_{a\pm\epsilon}$.

Suppose $\chi(\widehat{F}) = -1$. If \widehat{F} is a once-punctured torus, then \widehat{F} must be incompressible in $P \times [a - \epsilon, a + \epsilon]$. Otherwise a compression on \widehat{F} yields a disk, contradicting that $\partial\widehat{F}$ is essential in $P_{a\pm\epsilon}$. As \widehat{F} is properly embedded in the product $P \times [a - \epsilon, a + \epsilon]$, \widehat{F} must be ∂ -compressible. A ∂ -compression on \widehat{F} yields an incompressible annulus with both boundary circles in $P_{a-\epsilon}$ (or $P_{a+\epsilon}$). So the resulting annulus is ∂ -parallel. Since \widehat{F} is incompressible, this implies that \widehat{F} itself is ∂ -parallel. Hence we can perform an isotopy on \widehat{F} so that Q_b becomes irreducible with respect to $P \times I$. Similarly, if \widehat{F} is a pair of pants, then \widehat{F} must be incompressible but ∂ -compressible. So a ∂ -compression on \widehat{F} yields one or two incompressible annuli. This implies that either \widehat{F} is ∂ -parallel or we can perform an isotopy on \widehat{F} so that \widehat{F} is transverse to each P_x except for a single saddle tangency. In either case, we can isotope \widehat{F} so that Q_b becomes irreducible with respect to $P \times I$ and [Theorem 1.1](#) follows from [Lemma 2.2](#) and [Lemma 3.3](#).

Therefore, we may assume $\chi(\widehat{F}) = -2$. Hence $F = \widehat{F}$ and every component of ∂F is essential in $P_{a\pm\epsilon}$.

Since b is not labelled and since every component of ∂F above is essential in $P_{a\pm\epsilon}$, at each regular level $x \in (0, 1)$, if a component of $P_x \cap Q_b$ is trivial in P_x , then it must also be trivial in Q_b . Thus, we can apply [Lemma 2.1](#) on $Q_b \cap (P \times ([0, a - \epsilon] \cup [a + \epsilon, 1]))$. So after some isotopies, Q_b satisfies all the conditions in [Lemma 2.1](#) except for the level P_a where $P_a \cap Q_b$ contains 2 saddle tangencies. Moreover, since b is not labelled, $Q_b \cap \Sigma_U \neq \emptyset$ and $Q_b \cap \Sigma_V \neq \emptyset$. Hence $Q_b \cap P_s \neq \emptyset$ for every s .

Claim A Let σ and w be any components of $Q_b \cap P_{a-\epsilon}$ and $Q_b \cap P_{a+\epsilon}$ respectively. Then $d(\sigma, w) \leq 2 = -\chi(F) = -\chi(Q_b \cap (P \times [a - \epsilon, a + \epsilon]))$.

Proof of Claim A If σ is a boundary curve of a vertical annulus component of $Q_b \cap (P \times [a - \epsilon, a + \epsilon])$, then σ is isotopic to a component of $Q \cap P_{a+\epsilon}$ and hence $d(\sigma, w) \leq 1$ for any curve w in $Q \cap P_{a+\epsilon}$. So we may assume neither σ nor w is a boundary curve of a vertical annulus. Thus σ and w are both components of ∂F .

Let Ω be the union of the components of $P_a \cap Q_b$ that contain the 2 saddle tangent points. So Ω is a possibly disconnected graph with 2 vertices of valence 4. Let $N(\Omega)$ be a regular neighborhood of Ω in P_a and let $\pi: P \times I \rightarrow P_a$ be the projection, then $\pi(\partial F) \subset N(\Omega)$ after isotopy. Since P has genus at least 2, there must be an essential curve α in P_a disjoint from $N(\Omega)$. So $d(\sigma, w) \leq d(\sigma, \alpha) + d(\alpha, w) \leq 2 = -\chi(F)$. \square

Now [Theorem 1.1](#) follows from the argument in the proof of [Lemma 2.2](#). As in the proof of [Lemma 2.2](#), let $s_0 < \dots < s_n$ be a collection of regular levels such that $s_0 = \delta$, $s_n = 1 - \delta$ for a small δ and there is exactly one critical level in each $P \times (s_i, s_{i+1})$. Let $\Gamma_i = Q \cap P_{s_i}$ for each i .

Since we assume Q is bicompressible in this section and since M is irreducible, if Q is a torus, then M must be a lens space and P and Q must be isotopic Heegaard surfaces of the lens space (see Bonahon–Otal [2]). So we may assume $g(Q) \geq 2$.

Suppose $d(\mathcal{U}, \mathcal{V}) > g(Q)$. Since $g(Q) \geq 2$, we have $d(\mathcal{U}, \mathcal{V}) > 4$. Let k be the smallest integer such that $d(\mathcal{U}, \Gamma_k) \neq 0$ and l the largest integer such that $d(\Gamma_l, \mathcal{V}) \neq 0$. By [Claim A](#) above and [Claim 1](#) in the proof of [Lemma 2.2](#), $d(\mathcal{U}, \Gamma_k)$ and $d(\Gamma_l, \mathcal{V})$ are either 1 or 2 and $k \leq l$. Without loss of generality, we assume $k < l$.

Similar to the proof of [Lemma 2.2](#), Γ_k and Γ_l must be essential in Q_b . Let $Q' = Q_b \cap (P \times [s_k, s_l])$, and let U' and V' be the two components of $M - P \times (s_k, s_l)$ containing G^U and G^V respectively, $F_U = Q_b \cap U'$ and $F_V = Q_b \cap V'$. Since Γ_k and Γ_l are essential in Q_b , F_U , Q' and F_V are essential subsurfaces of $Q_b = F_U \cup Q' \cup F_V$.

Claim B Let σ_k be any component of Γ_k , then $d(\sigma_k, \mathcal{U}) \leq 1 - \chi(F_U)$.

Proof of Claim B If a component A of F_U is a ∂ -parallel annulus in U' , then we may first isotope A into $P \times (s_k - \epsilon, s_k]$. Then we isotope A so that A is transverse to each P_x except for a circle tangency. Since ∂F_U is essential in P_{s_k} , after the isotopy, Q_b still satisfies the conditions in [Lemma 2.1](#) except at the level P_a as above. Now we push A out of U' . After the isotopy, we still have $d(\mathcal{U}, \Gamma_k) \neq 0$. If k is no longer

the smallest number so that $d(\mathcal{U}, \Gamma_k) \neq 0$ after the isotopy, then we can find a new k and proceed as above. Eventually F_U does not contain any ∂ -parallel annulus after some isotopies. We can view these isotopies as trivial isotopies, so by our assumptions above, $Q_b \cap \Sigma_U \neq \emptyset$ after the isotopies.

We first show that $d(\sigma_k, \mathcal{U}) \leq 2$. As in the proof of [Lemma 2.2](#), $d(\sigma_k, \mathcal{U}) \leq 1$ if $k = 0$. So we may assume $k > 0$. By the definition of k , $d(\mathcal{U}, \Gamma_{k-1}) = 0$. Thus there is a component w of Γ_{k-1} representing a vertex in \mathcal{U} . By Claim A above and the [Claim 1](#) in the proof of [Lemma 2.2](#), $d(\sigma_k, w) \leq 2$ and hence $d(\sigma_k, \mathcal{U}) \leq 2$.

Since F_U is an essential subsurface of Q_b , $\chi(F_U) \leq 0$. Since $d(\sigma_k, \mathcal{U}) \leq 2$ and $\chi(F_U) \leq 0$, to prove the claim, we only need to consider the case that $\chi(F_U) = 0$. Suppose $\chi(F_U) = 0$. Since $d(\mathcal{U}, \Gamma_k) \neq 0$, F_U consists of incompressible annuli in U' . Let A be the component of F_U that contains σ_k . If A is also ∂ -incompressible, then A can be isotoped away from any compressing disk of U' and hence $d(\sigma_k, \mathcal{U}) \leq 1 = 1 - \chi(F_U)$. If A is ∂ -compressible, then since F_U contains no ∂ -parallel annulus, a ∂ -compression on A yields a compressing disk of U' disjoint from A . Thus, $d(\sigma_k, \mathcal{U}) \leq 1 = 1 - \chi(F_U)$ in any case. \square

Similar to Claim B, for any component σ_l of Γ_l , $d(\mathcal{V}, \sigma_l) \leq 1 - \chi(F_V)$. Although $P_a \cap Q_b$ contains 2 saddle tangencies, by Claim A and our assumptions on Q_b , [Claim 3](#) in the proof of [Lemma 2.2](#) also holds in this case, ie there is a component σ_k of Γ_k and a component σ_l of Γ_l such that $d(\sigma_k, \sigma_l) \leq -\chi(Q')$.

Since Q' , F_U and F_V are essential subsurfaces of Q_b , $d(\mathcal{U}, \mathcal{V}) \leq d(\mathcal{U}, \sigma_k) + d(\sigma_k, \sigma_l) + d(\sigma_l, \mathcal{V}) \leq 1 - \chi(F_U) - \chi(Q') + 1 - \chi(F_V) = 2 - \chi(Q) = 2g(Q)$. Thus [Theorem 1.1](#) is proved. \square

References

- [1] **D Bachman, S Schleimer**, *Surface bundles versus Heegaard splittings*, Comm. Anal. Geom. 13 (2005) 903–928 [MR2216145](#)
- [2] **F Bonahon, J-P Otal**, *Scindements de Heegaard des espaces lenticulaires*, Ann. Sci. École Norm. Sup. (4) 16 (1983) 451–466 (1984) [MR740078](#)
- [3] **A J Casson, C M Gordon**, *Reducing Heegaard splittings*, Topology Appl. 27 (1987) 275–283 [MR918537](#)
- [4] **D Gabai**, *Foliations and the topology of 3-manifolds*, J. Differential Geom. 18 (1983) 445–503 [MR723813](#)

- [5] **K Hartshorn**, *Heegaard splittings of Haken manifolds have bounded distance*, Pacific J. Math. 204 (2002) 61–75 [MR1905192](#)
- [6] **W J Harvey**, *Boundary structure of the modular group*, from: “Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)”, Ann. of Math. Stud. 97, Princeton Univ. Press, Princeton, N.J. (1981) 245–251 [MR624817](#)
- [7] **J Hempel**, *3-manifolds as viewed from the curve complex*, Topology 40 (2001) 631–657 [MR1838999](#)
- [8] **T Kobayashi, O Saeki**, *The Rubinstein–Scharlemann graphic of a 3-manifold as the discriminant set of a stable map*, Pacific J. Math. 195 (2000) 101–156 [MR1781617](#)
- [9] **T Li**, *On the Heegaard splittings of amalgamated 3-manifolds* Preprint available at <http://www2.bc.edu/~taoli/amalg.pdf>
- [10] **T Li**, *Images of the disk complex* Preprint
- [11] **H Rubinstein, M Scharlemann**, *Comparing Heegaard splittings of non-Haken 3-manifolds*, Topology 35 (1996) 1005–1026 [MR1404921](#)
- [12] **M Scharlemann**, *Local detection of strongly irreducible Heegaard splittings*, Topology Appl. 90 (1998) 135–147 [MR1648310](#)
- [13] **M Scharlemann, A Thompson**, *Heegaard splittings of $(\text{surface}) \times I$ are standard*, Math. Ann. 295 (1993) 549–564
- [14] **M Scharlemann, M Tomova**, *Alternate Heegaard genus bounds distance*, Geom. Topol. 10 (2006) 593–617 [MR2224466](#)
- [15] **W P Thurston**, *Foliations of manifolds which are circle bundles*, PhD thesis, University of California, Berkeley (1972)

Department of Mathematics, Boston College, Chestnut Hill, MA 02467, USA

taoli@bc.edu

<http://www2.bc.edu/~taoli/>