Random Geodesics

Martin Bridgeman [*]
Boston College

1 Background

Let $\mathbf{H}^n$ be n-dimensional hyperbolic space and let S^{n-1}_∞ denote the sphere at infinity. A *Kleinian group* is a discrete subgroup Γ of the group $\mathrm{Iso}^+(\mathbf{H}^n)$ of orientation preserving isometries of $\mathbf{H}^n$. Thus Γ acts discontinuously on $\mathbf{H}^n$ so that, providing Γ has no elements of finite order, the quotient $\mathbf{H}^n/\Gamma$ is a Riemannian manifold of constant sectional curvature -1.

A Kleinian group also acts as a discrete group of conformal automorphisms on the sphere S^{n-1}_∞. The action of the group partitions S^{n-1}_∞ into two disjoint sets. The *regular set* Ω_Γ is the maximal open set upon which Γ acts discontinously, the *limit set* L_Γ is its complement. If x is any point in $\mathbf{H}^n$ then L_Γ is also equal to the set of accumulation points of the orbit of x under Γ on the sphere S^{n-1}_∞. The points of L_Γ are called *limit points*. Furthermore a point ξ of S^{n-1}_∞ is called a *conical limit point* if it is the limit of a sequence of orbit points which stay a bounded hyperbolic distance from a geodesic with endpoint ξ.

The *convex hull* $CH(L_\Gamma)$ of L_Γ is the smallest convex set in $\mathbf{H}^n$, so that every hyperbolic geodesic with endpoints in L_Γ is in $CH(L_\Gamma)$. If one takes the quotient of $CH(L_\Gamma)$ by the action of the group, then the resulting submanifold $C(\Gamma) = CH(L_\Gamma)/\Gamma$ is the smallest convex submanifold in $N = \mathbf{H}^n/\Gamma$ so that the inclusion map is a homotopy equivalence. This submanifold is called the *convex core* of Γ. A finitely generated Kleinian group Γ is *geometrically finite* if $\mathrm{vol}\, C(\Gamma) < \infty$. This is but one of many definitions of the concept of geometric finiteness; which agree in dimension two and mostly agree in dimension three (see Bowditch [2] for a full discussion of the various different formulations).

In this paper, we will describe various applications of geodesic currents and random geodesics; much of this work was done in collaboration with either Dick Canary or Edward Taylor. We hope to give the reader an understanding of how random geodesics arise and how they can be applied to obtain interesting geometric measurements. Some proofs are given but, for many of the proofs, the reader is directed to the original sources.

Geodesic Currents

In [1], Bonahon developed the theory of geodesic currents as an extension of the notion of closed geodesics. We will briefly summarize some of Bonahon's results. We let $G(\mathbf{H}^n)$ be the space of

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unoriented geodesics in $\mathbf{H}^n$. When we characterize a geodesic by its pair of endpoints on the sphere at infinity, we have that

$$G(\mathbf{H}^n) \cong (S^{n-1}_\infty \times S^{n-1}_\infty \setminus \Delta)/\mathbf{Z}_2$$

where Δ is the diagonal and $\mathbf{Z}_2$ acts by switching the entries in the ordered pairs.

Let S be a closed oriented hyperbolic surface. Then $S = \mathbf{H}^2/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Iso}^+(\mathbf{H}^2)$. We let $p : \mathbf{H}^2 \to S$ be the covering map obtained by taking the quotient of $\mathbf{H}^2$ by Γ. Let α be a closed geodesic on S then its preimage $\tilde{\alpha} = p^{-1}(\alpha)$ is a Γ invariant collection of geodesics in $\mathbf{H}^2$. We obtain a measure on $G(\mathbf{H}^2)$ by taking the Dirac measure on the discrete set $\tilde{\alpha}$. In this way we can naturally identify closed geodesics on S with Γ invariant measures on $G(\mathbf{H}^2)$.

Definition A *geodesic current* on $S = \mathbf{H}^2/\Gamma$ is a Γ invariant measure on $G(\mathbf{H}^2)$. The space of geodesic currents is denoted $\mathcal{C}(S)$ and given the weak* topology.

Theorem 1 *(Bonahon [1]) There is a continuous bilinear function $i : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbf{R}$ extending the geometric intersection number on closed geodesics.*

This description of geodesic currents mirrors the description of measured laminations introduced by Thurston (see [15]). In particular, a measured lamination is precisely a geodesic current supported on a closed set of disjoint geodesics.

A special geodesic current, associated with the hyperbolic metric on S, is called the Liouville geodesic current L. Using the product structure on $G(\mathbf{H}^2)$, L is defined by

$$L([a,b] \times [c,d]) = \left| \log \left| \frac{(a-c)(b-d)}{(a-d)(b-c)} \right| \right|$$

As L is defined using the cross-ratio, we have that L is invariant under the full isometry group $\mathrm{Iso}(\mathbf{H}^2)$. This property uniquely defines L up to a multiplicative constant (see Bonahon [1]).

The Liouville geodesic current has the following property;

Theorem 2 *(Bonahon [1]) If α is a closed geodesic then $i(L, \alpha) = l(\alpha)$ where $l(\alpha)$ is the length of α. Thus the function $l : \mathcal{C}(S) \to \mathbf{R}$ defined by $l(\mu) = i(L, \mu)$ is a continuous extension of the length function on closed geodesics. Furthermore, the Liouville current L satisfies $l(L) = \pi^2|\chi(S)|$.*

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L is a Random Geodesic

L is random geodesic in following sense; Let C be the hyperbolic diameter of S and v lie in the unit tangent space $T_1(S)$. We obtain a closed curve $c_t(v)$ by travelling at unit speed along the geodesic tangent to v for time t and then joining the endpoints by a geodesic of length $\leq C$. We let $\alpha_t(v)$ be the closed geodesic in the homotopy class of $c_t(v)$ (see Figure 1).

Theorem 3 *(Bonahon [1]) For almost every v (with respect to the volume measure on $T_1(S)$),*

$$\lim_{t \to \infty} \frac{\alpha_t(v)}{l(\alpha_t(v))} = \overline{L}$$

where $\overline{L}$ is the unique multiple of the Liouville geodesic current having unit length.

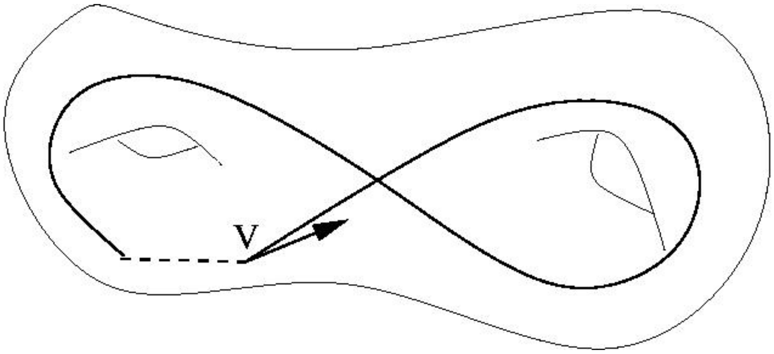

Figure 1: Random geodesic

2 Applications of random geodesics

We now consider quasifuchsian groups.

Definition A Kleinian group is *quasifuchsian* if L_Γ is topologically a circle and is *strictly quasifuchsian* if L_Γ is not a geometric circle.

The *convex hull* of a Kleinian group is the smallest convex set in $\mathbf{H}^n$ containing all geodesics with both endpoints in the limit set L_Γ. We have the following theorem which describes the geometry of the convex hull of a strictly quasifuchsian group.

Theorem 4 *(Epstein-Marden [9]) The convex hull of a strictly quasifuchsian Kleinian group* Γ *has the following description.*

1. *The convex hull* $CH(L_\Gamma)$ *is a* 3*-dimensional submanifold of* $\mathbf{H}^3$*, so that the boundary of* $CH(L_\Gamma)$ *consists of two simply connected components* C^+ *and* C^-*, each homeomorphic to an open disk.*

2. *Both* C^+ *and* C^- *consists of pieces of hyperbolic planes meeting along infinite geodesic lines. The hyperbolic metric in* $\mathbf{H}^3$ *induces an intrinsic metric on each boundary component, so that both components are complete* 2*-dimensional hyperbolic spaces with respect to these intrinsic metrics.*

Let Γ be quasifuchsian and ρ be the hyperbolic metric on $\mathbf{H}^3$. We consider a single boundary component of the convex hull $CH(L_\Gamma)$. Choosing C^+, we let ρ^+ be the intrinsic metric on C^+ and b_Γ^+ be the set of bending lines on C^+. Then we obtain a measured lamination β_Γ^+ with support b_Γ^+ by assigning to each arc α on C^+, transverse to b_Γ^+, the amount of bending along α (see [15] for details). We denote the amount of bending along α by $i(\alpha, \beta_\Gamma^+)$.

We choose $x, y \in C^+$ and let α be the geodesic in C^+ joining x to y (see Figure 2).

We have three geometric quantities associated with x, y, namely

$$\rho(x,y) \qquad \rho^+(x,y) \qquad i(\alpha, \beta_\Gamma^+)$$

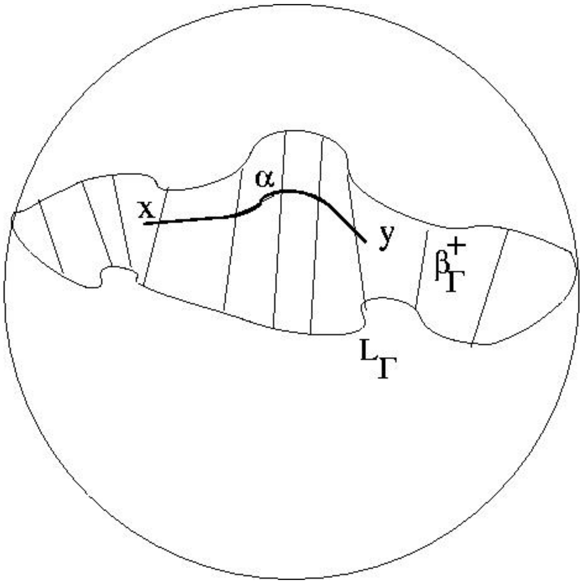

Figure 2: Convex hull boundary component C^+

We now consider how these quantities are related as x and y tend to infinity. We do this by taking the ratio of two of the quantities and studying it's ergodic properties. We now describe two ratios, *average bending* and *length distortion*.

$$\text{Average Bending: } \frac{i(\alpha, \beta_\Gamma^+)}{\rho^+(x,y)}$$

$$\text{Length Distortion: } \frac{\rho(x,y)}{\rho^+(x,y)}$$

3 Average Bending

For a complete description of average bending see [4], [5] and [6].

Let $l(\alpha) = \rho^+(x,y)$. Then the average bending of α, denoted by $B(\alpha)$ is

$$B(\alpha) = \frac{i(\alpha, \beta_\Gamma^+)}{l(\alpha)}$$

We let $S = C^+/\Gamma$. Then S is a closed hyperbolic surface. As β_Γ^+ is invariant under Γ we obtain a measured lamination $\beta_N^+ = \beta_\Gamma^+$. If α is a geodesic arc in S then we define $B(\alpha) = B(\overline{\alpha})$ where $\overline{\alpha}$ is a lift of α.

Theorem 5 *(Bending Bound [5]) If* $l(\alpha) \leq 2\sinh^{-1} 1$ *then* $i(\alpha, \beta_\Gamma^+) \leq 2\pi$.

Corollary 6 *If α is a closed geodesic in S then*

$$B(\alpha) \leq \frac{\pi}{\sinh^{-1} 1} = K$$

equivalently

$$i(\alpha, \beta_N^+) \leq K \cdot l(\alpha)$$

Thus, considered as an inequality on the space of geodesic currents $\mathcal{C}(S)$, $i(\alpha, \beta_N^+) \leq K \cdot l(\alpha)$ holds for α any closed geodesic. By linearity of the inequality in α, the inequality holds on the set of multiples of closed geodesics which form a dense set in $\mathcal{C}(S)$ (see [1]). Therefore, by continuity of the intersection and length functions, the inequality extends to $\mathcal{C}(S)$. The Corollary is then applicable to the Liouville current and we obtain

$$i(L, \beta_N^+) \leq K \cdot l(L)$$

This gives the following result

Theorem 7 *([5])*

$$l(\beta_N^+) \leq K\pi^2 |\chi(S)|$$

If we consider a pair (S, β) as the input data to build a quasifuchsian group by bending the hyperbolic surface S along β then the above theorem quantifies the observation that if you bend S too much then you don't get a discrete group.

To prove these results we need to analyse the measured lamination β_Γ^+. The measure on β_Γ^+ is defined in terms of support planes (see [9] for details). A *support plane* to the convex hull $CH(L_\Gamma)$ is a plane P such that $P \cap CH(L_\Gamma) = P \cap \partial CH(L_\Gamma) \neq \emptyset$. Thus P has "glancing intersection"with the convex hull. The set $P \cap \partial CH(L_\Gamma)$ is either a single geodesic or a region bounded by geodesics called a *flat*. The half space H_P is the unique half space with boundary P such that the interior of H_P is disjoint from the convex hull $CH(L_\Gamma)$.

Let α be a geodesic arc on C^+. Take a 1-parameter family of support planes along α, $P_t, 0 \leq t \leq 1$. An elementary consequence of the definition of the bending measure is the following;

Lemma 8 *([5]) If $P_t \cap P_0 \neq \emptyset$ for all t then $i(\alpha, \beta_\Gamma^+) \leq \theta$, where θ is the exterior dihedral angle between P_0 and P_1.*

We now prove that the bending bound $l(\alpha) \leq 2\sinh^{-1} 1$ implies $i(\alpha, \beta_\Gamma) \leq 2\pi$.

Proof: We first make an observation. Let H_1, H_2, H_3 be three mutually disjoint half spaces in $\mathbf{H}^3$. We consider a curve c in $\mathbf{H}^3$ containing a point of each half space. Then it is an easy exercise to show that the length of c must be at least $2\sinh^{-1} 1$ with the minimum attained by the piecewise geodesic in an ideal hyperbolic triangle as shown in Figure 3.

We now assume $l(\alpha) < 2\sinh^{-1} 1$. If α has three mutually disjoint support planes P_1, P_2, P_3, then the associated half spaces $H_{P_1}, H_{P_2}, H_{P_2}$ are mutually disjoint. But as $l(\alpha) < 2\sinh^{-1} 1$, it follows that α cannot have three mutually disjoint support planes.

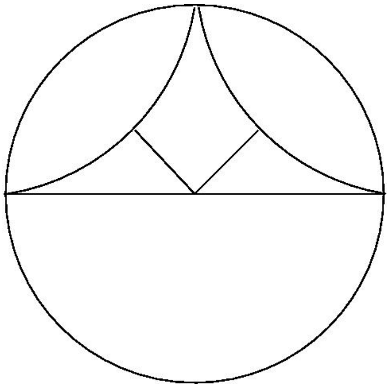

Figure 3: Shortest geodesic intersecting three disjoint half spaces

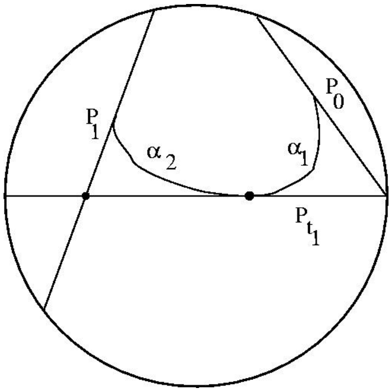

Figure 4: Planes P_0, P_{t_1}, P_1

We take a 1-parameter family of support planes along α, $P_t, 0 \leq t \leq 1$ as above. Let P_{t_1} be first support plane that doesn't intersect P_0. The geodesic α splits into subarcs α_1 and α_2 at $\alpha(t_1)$. As P_0 and P_{t_1} have a common point on $\mathbf{S}^2_\infty$, it follows from the previous lemma that $i(\alpha_1, \beta_\Gamma) \leq \pi$.

Assume that there is a support plane $P_t, t > t_1$ such that P_{t_1} and P_t do not intersect. Then we let P_{t_2} be the first plane that doesn't intersect P_{t_1} for $t_2 > t_1$. As α cannot have three mutually disjoint support planes, then P_{t_2} and P_0 must intersect. Considering the configuration of the half-spaces $H_{P_0}, H_{P_{t_1}}, H_{P_{t_2}}$ we obtain a non-trivial annulus in the compliment of L_Γ. Thus L_Γ must be disconnected. But this contradicts our definition of a quasifuchsian group. Therefore $P_t \cap P_{t_1} \neq \emptyset$ for $t > t_1$. Therefore it follows from the previous lemma that $i(\alpha_2, \beta_\Gamma) \leq \theta$ where θ is exterior dihedral angle between P_{t_1} and P_1 (see Figure 4). Thus

$$i(\alpha, \beta_\Gamma) = i(\alpha_1, \beta_\Gamma) + i(\alpha_2, \beta_\Gamma) \leq 2\pi$$

The case when $l(\alpha) = 2\sinh^{-1} 1$ follows as a limiting case of this inequality. ∎

4 Length distortion

An extensive description of length distortion appears in [7] and [8].

Given a Kleinian group Γ, we have the Poincaré series

$$g_s(x,y) = \sum_{\gamma \in \Gamma} e^{-s\rho(x,\gamma y)}$$

The critical exponent $\delta(\Gamma)$ is given by

$$\delta(\Gamma) = \inf\{s : g_s < \infty\}$$

We let $D(L_\Gamma)$ be the Hausdorf dimension of the limit set L_Γ.

Theorem 9 *(Patterson [12], Sullivan [14], Tukia [16]) If Γ is geometrically finite then $\delta(\Gamma)$ is the Hausdorf dimension of the limit set L_Γ.*

As before, we let Γ be a quasifuchsian group and C^+ be a boundary component of the convex hull $CH(L_\Gamma)$. As the intrinsic metric on C^+ is hyperbolic, we let Γ^+ be the fuchsian group given by the action of Γ on C^+ and let $S = C^+/\Gamma$. Then by the above theorem we have $\delta(\Gamma) = D(L_\Gamma)$. Also, as S is a closed hyperbolic surface, $\delta(\Gamma^+) = D(L_{\Gamma^+}) = 1$.

It follows that Poincaré series for the hyperbolic metrics ρ and ρ^+ satisfy

$$\sum_{\gamma \in \Gamma} e^{-s\rho(x,\gamma y)} < \infty \qquad s > \delta(\Gamma)$$

$$\sum_{\gamma \in \Gamma} e^{-s\rho^+(x,\gamma y)} < \infty \qquad s > 1$$

Bowen considered the question of the extent to which L_Γ is a fractal curve. He proved

Theorem 10 *(Bowen [3]) If Γ is strictly quasifuchsian, then $\delta(\Gamma) > 1$.*

We define

$$R(x,y) = \frac{\rho(x,y)}{\rho^+(x,y)}$$

Let M be a Riemannian manifold. Let $Exp_t : T_1(M) \to T_1(M)$ be time t geodesic flow on M and $exp_t : T_1(M) \to M$ be defined by $exp_t = p \circ Exp_t$ where $p : T_1(M) \to M$ is the tangent bundle map.

We identify $\mathbf{H}^2$ with C^+ and, as above, let Γ^+ be the fuchsian action on $\mathbf{H}^2$ corresponding to the action of Γ on C^+. If $v \in T_1(\mathbf{H}^2)$ we define $R_t(v) = R(p(v), exp_t(v))$. Then R_t descends to a function on $T_1(S)$ which we will also call R_t for convenience.

We let L be the Liouville geodesic current on the surface S. As Γ is quasifuchsian, there is a natural quasiconformal map $f : \mathbf{S}^2_\infty \to \mathbf{S}^2_\infty$ conjugating the action of Γ^+ to the action of Γ. Using f we can push geodesic currents on S into the convex core of Γ (see section 5 for details).

The length distortion $\mathcal{R}^+$ of C^+ is then defined to be

$$\mathcal{R}^+ = \frac{l(L)}{l^+(L)}$$

where $l^+(L)$ is the length of L in C^+ and $l(L)$ is the length by pushing L into the convex core.

Alternately it follows easily from Theorem 3 that

$$\mathcal{R}^+ = \lim_{t\to\infty} R_t(v) \tag{1}$$

for almost every $v \in T_1(S)$ (with respect to the standard volume measure on $T_1(S)$).

We proceed to relate the length distortion to the exponents of convergence of the Poincaré series discussed above.

Theorem 11 *(Length Distortion Bound [7])*

$$\mathcal{R}^+ = \lim_{t\to\infty} R_t(v) \geq \frac{1}{\delta(\Gamma)}$$

In order to prove this, we need to consider certain subsets of Γ. If $K \subset \Gamma$, we choose an $x \in \mathbf{H}^3$ and define $L_K = \overline{Kx} \cap \mathbf{S}^2_\infty$. We let CL_K be the subset of L_K which are conical limit points. The sets we consider are $\Gamma_\epsilon = \{\gamma \in \Gamma \mid R(x,\gamma y) \leq \frac{1}{\delta(\Gamma)+\epsilon}\}$.

Lemma 12 *([7])*

$$D(CL_{\Gamma_\epsilon}) \leq \frac{\delta(\Gamma)}{\delta(\Gamma)+\epsilon}$$

Proof: We consider the exponents for which Poincaré series of Γ converges. Thus, we have

$$\sum_{\gamma\in\Gamma} e^{-s\rho(x,\gamma y)} < \infty \qquad s > \delta(\Gamma)$$

By definition of R we have $\rho(x,\gamma y) = R(x,\gamma y)\rho^+(x,\gamma y)$. Therefore

$$\sum_{\gamma\in\Gamma} e^{-sR(x,\gamma y)\rho^+(x,\gamma y)} < \infty \qquad s > \delta(\Gamma)$$

If we restrict to the subset $\Gamma_\epsilon \subset \Gamma$ we have

$$\sum_{\gamma\in\Gamma_\epsilon} e^{-sR(x,\gamma y)\rho^+(x,\gamma y)} < \infty \qquad s > \delta(\Gamma)$$

For $\gamma \in \Gamma_\epsilon$, $sR(x,\gamma y) \leq \frac{s}{\delta(\Gamma)+\epsilon}$ Therefore

$$\sum_{\gamma\in\Gamma_\epsilon} e^{-s\rho^+(x,\gamma y)} < \infty \qquad s > \frac{\delta(\Gamma)}{\delta(\Gamma)+\epsilon}$$

Therefore the critical exponent of Γ_ϵ satisfies $\delta(\Gamma_\epsilon) \leq \frac{\delta(\Gamma)}{\delta(\Gamma)+\epsilon}$. Then by a standard argument of Sullivan (see [13]) relating the critical exponent to the Hausdorf dimension of the limit set, we obtain

$$D(CL_{\Gamma_\epsilon}) \leq \frac{\delta(\Gamma)}{\delta(\Gamma)+\epsilon} < 1 \qquad \blacksquare$$

Proof of the Theorem 11: We consider the Poincaré disk model where $\mathbf{H}^2$ is the unit Euclidean disk in $\mathbf{R}^2$ and $\mathbf{S}^1_\infty$ is the standard unit circle. If $a, b \in \mathbf{S}^1_\infty$ then $|a-b|$ is the Euclidean distance between the two points and is called the *chordal distance.* Let w be Lebesgue measure on the unit circle $\mathbf{S}^1_\infty$. We identify the space of oriented geodesics $G'(\mathbf{H}^2)$ with $(\mathbf{S}^1_\infty \times \mathbf{S}^1_\infty \setminus \Delta)$ with the pair (ξ_-, ξ_+) corresponding to the oriented geodesic with positive endpoint ξ_+ and negative endpoint ξ_-. A fiber bundle $\pi : T_1(\mathbf{H}^2) \to G'(\mathbf{H}^2)$ is defined by $\pi(v) = g$, where g is the oriented geodesic with v as a tangent vector. The fiber $\pi^{-1}(g)$ corresponds to the tangent vectors tangent to g and has a natural measure given by hyperbolic length measure along the geodesic. The fiber bundle is easily shown to be trivial and we obtain a homeomorphism $h : T_1(\mathbf{H}^2) \to G'(\mathbf{H}^2) \times \mathbf{R}$ which maps each fiber $\pi^{-1}(g)$ isometrically to $\mathbf{R}$ with length measure ds. Thus $T_1(\mathbf{H}^2)$ is homeomorphic to $(\mathbf{S}^1_\infty \times \mathbf{S}^1_\infty \setminus \Delta) \times \mathbf{R}$. Furthermore, in terms of this parameterization, the standard volume measure on $T_1(\mathbf{H}^2)$ has the form

$$dM = \frac{2dw(\xi_-)dw(\xi_+)ds}{|\xi_+ - \xi_-|^2}$$

where dw is lebesgue measure on the unit circle $\mathbf{S}^1_\infty$, and ds is length measure on $\mathbf{R}$ (see [11] for details).

As the Hausdorf dimension of the conical limit set CL_{Γ_ϵ} is less than 1, then it's lebesgue measure $w(CL_{\Gamma_\epsilon})$ is zero. Let $v \in T_1(\mathbf{H}^2)$, with $\pi(v) = (\xi_-, \xi_+)$. If

$$\limsup_{t\to\infty} R_t(v) < \frac{1}{\delta(\Gamma)}$$

then there exists $\epsilon > 0$ such that $\xi_+ \in CL_{\Gamma_\epsilon}$. Therefore g has positive endpoint in a set of w measure zero. Then from the description of dM above, it follows that the volume measure of the set of v with $\limsup_{t\to\infty} R_t(v) < \frac{1}{\delta(\Gamma)}$ is zero. As $\lim_{t\to\infty} R_t(v)$ exists for almost every $v \in T_1(\mathbf{H}^2)$, we therefore have

$$\lim_{t\to\infty} R_t(v) \geq \frac{1}{\delta(\Gamma)} \qquad \text{a.e.} \quad v.$$

This completes the proof of the length distortion bound.

Geometric Proof of Bowen

Let Γ be strictly quasifuchsian. By the length distortion bound, we have $\mathcal{R}^+ \geq 1/\delta(\Gamma)$. Thus we need only show that $\mathcal{R}^+ < 1$ to give a proof of Bowen's theorem.

Consider the function $R_i : T_1(\mathbf{H}^2) \to \mathbf{R}$ where i is a non-negative integer. We recall that $Exp_t : T_1(\mathbf{H}^2) \to T_1(\mathbf{H}^2)$ is the time t geodesic flow on $\mathbf{H}^2$. Let $v \in T_1(\mathbf{H}^2)$ and $Exp_i(v) = v_i$. We

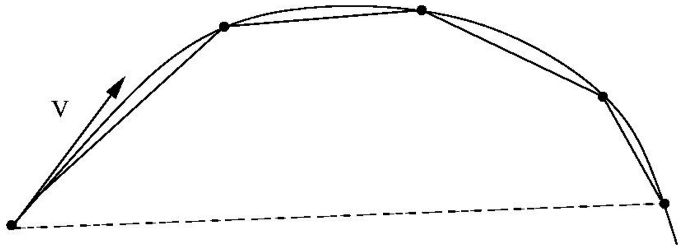

Figure 5: Time average of R_1

let x_i be the basepoint of the tangent vector v_i. By the triangle inequality we have

$$R_n(v) = \frac{\rho(x_0, x_n)}{\rho^+(x_0, x_n)} = \frac{\rho(x_0, x_n)}{n} \leq \frac{1}{n}\sum_{i=0}^{n-1} \rho(x_i, x_{i+1}) = \frac{1}{n}\sum_{i=0}^{n-1} R_1(Exp_i(v))$$

As R_1 is continuous, it is measurable with respect to the volume measure on $T_1(\mathbf{H}^2)$. In [10], Hopf proved that geodesic flow on $T_1(S)$ is ergodic. Therefore the time average of a measurable function equals the space average almost everywhere. Letting Ω be the standard unit volume measure on $T_1(S)$, we therefore have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} R_1(Exp_i(v)) = \int_{T_1(S)} R_1(w)\ d\Omega(w) \qquad \text{for almost every } v.$$

By equation 1, we have

$$\mathcal{R}^+ = \lim_{n\to\infty} R_n(v) \qquad \text{for almost every } v.$$

Therefore as

$$R_n(v) \leq \frac{1}{n}\sum_{i=0}^{n-1} R_1(Exp_i(v))$$

we combine these inequalities to obtain

$$\mathcal{R}^+ = \lim_{n\to\infty} R_n(v) \leq \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} R_1(Exp_i(v)) = \int_{T_1(S)} R_1(w)\ d\Omega(w) \qquad \text{for almost every } v.$$

In particular, we have

$$\mathcal{R}^+ \leq \int_{T_1(S)} R_1(w)\ d\Omega(w).$$

As Γ is strictly quasifuchsian, there exists some geodesic arc on C^+ bent by a definite amount. Taking v to be the tangent vector to the initial point of this geodesic arc, we have $R_1(v) < 1$. Thus R_1 is continuous, with $R_1(w) \leq 1$ for all w, and $R(v) < 1$ for some v. Then by the compactness of $T_1(S)$, we have

$$\int_{T_1(S)} R_1(w)\ d\Omega(w) < 1$$

It follows that $\mathcal{R}^+ < 1$.

5 Random Geodesic for Kleinian groups

In [1], Bonahon defined geodesic currents for closed hyperbolic surfaces. We have the following generalization for Kleinian groups.

Definition A *geodesic current* for a Kleinian group Γ is a measure on the space of geodesics $G(\mathbf{H}^n)$ invariant under the action of Γ and with support contained in the set of geodesics with endpoints in L_Γ. We denote the space of geodesic currents for Γ by $\mathcal{C}(\Gamma)$.

In [12],[14], Patterson and Sullivan constructed a family of measures on $\mathbf{S}^{n-1}_\infty$ which are supported on the limit set of the Kleinian group. These Patterson-Sullivan measures $m_{x,y}$ are given by

$$m_{x,y} = \lim_{s \to \delta(\Gamma)^+} \frac{1}{g_s(y,y)} \sum e^{-s\rho(x,\gamma y)} D(\gamma y)$$

where $D(p)$ denotes the Dirac measure at the point p and g_s is the Poincaré series. For a detailed description of the Patterson-Sullian measure see Nicholls book [11]. Sullivan showed that $m_{x,y}$ is independent of y and therefore we denote it by m_x (see [13]).

As before we have $G(\mathbf{H}^n) \cong (S^{n-1}_\infty \times S^{n-1}_\infty \setminus \Delta)/Z_2$. The Patterson-Sullivan geodesic current μ_x is then defined by

$$d\mu_x(a,b) = \frac{dm_x(a)dm_x(b)}{(|b-a|_x)^{2\delta(\Gamma)}}$$

where $|b-a|_x$ denotes the chordal distance between the two points $a, b \in \mathbf{S}^{n-1}_\infty$ with respect to the visual metric from $x \in \mathbf{H}^n$. For different basepoints x, μ_x only differs by a constant multiple and therefore we denote a choice of Patterson-Sullivan geodesic current for Γ by μ_Γ (see Sullivan [13]).

Let $f : S^{n-1}_\infty \to S^{n-1}_\infty$ be a quasiconformal homeomorphism conjugating Γ_1 to Γ_2. Then we have a homeomorphism $f_* : \mathcal{C}(\Gamma_1) \to \mathcal{C}(\Gamma_2)$ by pushing forward measures.

To define the length of a geodesic current, we consider the natural fiber $p : T_1(\mathbf{H}^n) \to G'(\mathbf{H}^n)$, where $G'(\mathbf{H}^n)$ is the space of oriented geodesics, and $\pi(v) = g$ is the unique oriented geodesic with v as a tangent vector. The fiber $\pi^{-1}(g)$ corresponds to the set of tangent vectors tangent to g and can be given length hyperbolic length measure. The space $G'(\mathbf{H}^2)$ is a two-fold cover of $G(\mathbf{H}^n)$. Therefore, as a geodesic current μ is a measure on $G(\mathbf{H}^n)$, we can pull the measure back to obtain a measure μ' on $G'(\mathbf{H}^2)$. Then we define a measure μ^* on $T_1(\mathbf{H}^n)$ by letting $d\mu^* = d\mu' ds$ where ds is length measure along the fibers. Then the length of μ is simply defined to be $l(\mu) = \frac{1}{2}\mu^*(D)$ where D is a fundamental domain for the action of Γ on $T_1(\mathbf{H}^n)$ (the 1/2 factor is used so that the length agrees with the standard length for closed geodesics). Then the natural generalization of length distortion is

$$\mathcal{R} = \frac{l(f_*\mu_{\Gamma_1})}{l(\mu_{\Gamma_1})}$$

As μ_Γ is well defined up to constant multiple, $\mathcal{R}$ is independent of choice of μ_Γ.

Then, by comparing the Poincaré series of Γ_1, Γ_2 as in the previous section, we obtain the following generalization of the previous length distortion bound.

Theorem 13 *([8])*

$$\mathcal{R} \geq \frac{\delta(\Gamma_1)}{\delta(\Gamma_2)}.$$

Corollary 14 *If $\mathcal{R} < 1$ then $\delta(\Gamma_2) > \delta(\Gamma_1)$.*

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