Average bending of convex pleated planes in hyperbolic three-space

Martin Bridgeman

May 14, 1997

Abstract

If $\mathcal{P}$ is a pleated plane in 3-dimensional hyperbolic space H^3 and α a geodesic in its intrinsic metric we define $B(\mathcal{P}, \alpha)$, the average bending of $\mathcal{P}$ in the direction α. We show that if $\mathcal{P}$ is a convex pleated plane embedded in H^3 then $B(\mathcal{P}, \alpha) \leq K$ for some universal K. Furthermore if $\mathcal{P}_\Gamma$ is a boundary component of the convex hull of a quasi-Fuchsian group Γ then $B(\mathcal{P}_\Gamma, \alpha) = B(\Gamma)$ almost everywhere, where $B(\Gamma)$ is a constant times the length of the bending lamination β_Γ of the pleated surface $X_\Gamma = \mathcal{P}_\Gamma/\Gamma$. We use these to prove a number of results about quasi-Fuchsian groups including a universal bound on the Lipschitz constant for the map to infinity and a bound on the length of β_Γ by a constant times the Euler characteristic of X_Γ.

1 Introduction

In this paper we study the bending of pleated planes in 3-dimensional hyperbolic space H^3. If $\mathcal{P}$ is a pleated plane with bending measured lamination β and α is a geodesic in the intrinsic hyperbolic metric on $\mathcal{P}$ then we define the *average bending* of α, $B(\mathcal{P}, \alpha)$, as the transverse measure of α per unit length. This gives us a measure of the bending of $\mathcal{P}$ in the direction of α. The main result is that if $\mathcal{P}$ is a convex embedded pleated plane then $B(\mathcal{P}, \alpha) \leq K$ where K is universal.

If Γ is a quasi-Fuchsian group then let $\mathcal{P}_\Gamma$ be a component of the boundary of the convex hull of Γ. Then $\mathcal{P}_\Gamma$ is a convex pleated plane embedded in H^3 and therefore $B(\mathcal{P}_\Gamma, \alpha) \leq K$. Furthermore using the group action on $\mathcal{P}_\Gamma$ we show that $B(\mathcal{P}_\Gamma, \alpha) = B(\Gamma)$ almost everywhere where $B(\Gamma)$ is a constant depending only on Γ and almost everywhere means in almost every direction. The surface $X_\Gamma = \mathcal{P}_\Gamma/\Gamma$ has a hyperbolic metric coming from the metric on $\mathcal{P}_\Gamma$ and has bending measured lamination β_Γ. We show that $B(\Gamma) = L_{X_\Gamma}(\beta_\Gamma)/(\pi^2 \cdot |\chi(X_\Gamma)|)$ where $L_{X_\Gamma}(\beta_\Gamma)$ is the length of β_Γ in X_Γ. Thus let X be a hyperbolic surface and β a measured lamination on X. Then a necessary condition for (X, β) to correspond to a component of the convex core of a quasi-Fuchsian group is that $L_X(\beta) \leq K \cdot \pi^2 \cdot |\chi(X)|$. We further use the universal bound on average bending to show that $1 + K$ is a universal bound on the Lipschitz constant between the hyperbolic metric on the convex core boundary and the hyperbolic metric on the quotient of the domain of of discontinuity.

2 Acknowledgements

I would like to thank Rich Schwartz for all his help and advice; Francis Bonahon, Dick Canary, Joe Christie, Curt McMullen and Bill Thurston who provided many useful ideas and suggestions. Also I would like to thank I.H.E.S. and Loyola University for their support while researching this paper. Finally I would like to thank the referee for their comments and criticism.

3 Pleated planes, Average bending

Informally a convex pleated plane is a hyperbolic plane H^2 which is mapped into H^3 bent along some disjoint geodesics. A simple example is if we have a geodesic α on H^2 and we map H^2 into H^3 so that it is bent along α by some angle t. The pair (t, α) corresponds to the measured lamination $t\alpha$ which assigns to any arc transverse to α, t times the intersection number with α. Note that the map of H^2 into H^3 is isometric in the complement of α and the amount of bending is given by t. Also the pair $(H^2, t\alpha)$ uniquely determines the pleated plane up to post-composition by a Moebius transformation.

Formally a convex pleated plane $\mathcal{P}$ is given by a triple $(X_\mathcal{P}, \beta_\mathcal{P}, f_\mathcal{P} : X_\mathcal{P} \rightarrow H^3)$ where $X_\mathcal{P}$ is isometric to the hyperbolic plane H^2, $\beta_\mathcal{P}$ is a measured geodesic lamination on $X_\mathcal{P}$ and $f_\mathcal{P}$ is a continuous map with the following properties. The map $f_\mathcal{P}$ is an isometry in the complement of $\beta_\mathcal{P}$, the image $f_\mathcal{P}(X_\mathcal{P})$ is convex in the sense that all bending is in the same direction and the bending lamination is precisely $\beta_\mathcal{P}$.

Another description is that there is a well defined normal map $n : X_\mathcal{P} - \beta_\mathcal{P} \rightarrow T_1(H^3)$ which assigns to each point x the unique normal vector $n(x)$ such that the triple of vectors $(f_{\mathcal{P}*}(e_1), f_{\mathcal{P}*}(e_2), n(x))$ is an oriented orthonormal basis whenever (e_1, e_2) is an oriented orthonormal basis. The map n induces a measure on curves transverse to $\beta_\mathcal{P}$ on $X_\mathcal{P}$ by measuring the total angle of turning of n along the transverse curve. The convexity of $\mathcal{P}$ corresponds to this being a non-negative transverse measure and the bending measure being exactly $\beta_\mathcal{P}$ means they are equal as transverse measures.

The map $f_\mathcal{P}$ is unique up to post-composition with a Moebius transformation ([T1]), so we will identify $\mathcal{P}$ with the pair $(X_\mathcal{P}, \beta_\mathcal{P})$. Also in discussion of a metric on $\mathcal{P}$ we are referring to the metric $\mathcal{P}$ inherits from $X_\mathcal{P}$.

For a more detailed description of convex pleated planes see the article by Epstein and Marden ([EM]).

To measure how bent a convex pleated plane is, we introduce the notion of *average bending*. Let α be a finite length geodesic arc on $\mathcal{P}$.

Definition: 1 *The* average bending *of α, denoted $B(\mathcal{P}, \alpha)$, is given by*

$$B(\mathcal{P}, \alpha) = \frac{i(\alpha, \beta_\mathcal{P})}{L_\mathcal{P}(\alpha)}$$

where $L_\mathcal{P}(\alpha)$ is the length of α in $\mathcal{P}$.

An alternative description is that $B(\mathcal{P}, \alpha)$ is the bending per unit length of α.

If α is an infinite length geodesic then we define the *average bending of α centered at x* by

$$B(\mathcal{P}, \alpha_x) = \overline{\lim_{t \to \infty}} B(\mathcal{P}, \alpha_x^t)$$

where α_x^t is the closed subarc of α of length $2t$ centered at $x \in \alpha$.

If $\mathcal{P}$ is a convex pleated plane then we define the notion of *support plane* in much the same way as is done for convex polyhedra.

Definition: 2 *A* support plane *P to a pleated plane $\mathcal{P}$ is a hyperbolic plane in H^3 which is the boundary of a half-space H_P such that $H_P \cap f_{\mathcal{P}}(X_{\mathcal{P}}) \subseteq P$.*

Thus a support plane P to a pleated plane $\mathcal{P}$ does not pass through $\mathcal{P}$ but has a glancing intersection. In general the intersection of P and $\mathcal{P}$ can either be a single geodesic, called an *edge*, or a flat piece of the pleated plane bounded by a set of disjoint geodesics, called a *flat*.

The following Lemma, due to Thurston, shows how support planes can be used to bound $i(\alpha, \beta_{\mathcal{P}})$.

Lemma: 1 *([T1]) Let $\alpha : [0,1] \to \mathcal{P}$ be a geodesic arc on a pleated plane $\mathcal{P}$ and $\{P_{t_i}\}_{i=0}^n$ be support planes to $\mathcal{P}$ with $\alpha(t_i) \in P_{t_i}$ for $0 = t_0 \leq t_1 \leq \ldots \leq t_n = 1$. If $P_{t_{i-1}}$ and P_{t_i} intersect with exterior dihedral angle b_i then*

$$i(\alpha, \beta_{\mathcal{P}}) \leq \sum_{i=1}^{n} b_i \leq n\pi$$

We say a geodesic arc α has an *$n\pi$-roof* if it is as in the above Lemma.

Theorem: 1 *If $\mathcal{P}$ is a convex pleated plane embedded in H^3 then*

$$B(\mathcal{P}, \alpha) \leq \frac{2\pi}{\log 3}\left(1 + \frac{\log 3}{L_{\mathcal{P}}(\alpha)}\right)$$

where α is a geodesic of length $L_{\mathcal{P}}(\alpha)$.

Proof: The proof of this fact follows from an observation similar to the "insulator condition" of Gabai ([G]). The observation is that a ball $B(x,r)$ in H^3 that meets three disjoint half spaces H_1, H_2, H_3 must have radius $r \geq (\log 3)/2$. To see this, arrange by an isometry that the center x of the ball is the origin O of the Poincaré model for H^3, bounded by the unit sphere S^2 in R^3. Then each H_i touches S^2 along a spherical disk D_i centered at $x_i \in S^2$ and of spherical radius r_i. Considering the circle passing through the centers x_1, x_2, x_3, we see that at least one of the radii must satisfy $r_i \leq \pi/3$, since the D_i have disjoint interiors. By elementary trigonometry, it follows that the Euclidean distance from $x = O$ to the corresponding H_i is at most $d = 2 - \sqrt{3}$, and therefore that the hyperbolic distance from x to H_i is bounded by $\log((1+d)/(1-d)) = (\log 3)/2$. Because $B(x,r)$ meets H_i it follows that $r \geq (\log 3)/2$.

Let $\alpha : [0,1] \to H^3$ be a closed geodesic arc of length l with $l \leq \log 3$ on $\mathcal{P}$. Therefore α lies inside a ball B of radius $l/2$ centered about the midpoint of α. Any support plane to $\mathcal{P}$

which contains a point of α must intersect B. Let P_0, P_1 be support planes to $\mathcal{P}$ containing $\alpha(0), \alpha(1)$ respectively. If $P_0 \cap P_1 \neq \emptyset$ then α has a π-roof and by Lemma 1 $i(\alpha, \beta_{\mathcal{P}}) \leq \pi$.

If P_0, P_1 do not intersect, then they are interpolated by a continuous one-parameter family of support planes $\{P_s | s \in [0,1]\}$ obtained by going along α. At any point $\alpha(t)$ there will be either a single support plane or a one-parameter family of support planes and P_s is just the sum of these. Each P_s must intersect B. To see that there must be a support plane $P_{s'}$ which intersects both P_0 and P_1, consider B centered at the origin O in the Poincare model for H^3. P_0, P_1 are represented by two disjoint disks D_0, D_1 on S^2 and P_s by a continuous one-parameter family of disks D_s interpolating between D_0 and D_1. The radius of B is less than or equal $(\log 3)/2$ and therefore by the above each disk D_s must intersect either D_0 or D_1. Therefore as D_s is a continuous one-parameter family of disks interpolating between D_0 and D_1, there is a disk $D_{s'}$ intersecting both D_0 and D_1. Thus $P_0, P_{s'}, P_1$ forms a 2π-roof for α and therefore by Lemma 1 $i(\alpha, \beta_{\mathcal{P}}) \leq 2\pi$.

Thus if α is a geodesic arc with $L_{\mathcal{P}}(\alpha) \leq \log 3$ then

$$i(\alpha, \beta_{\mathcal{P}}) \leq 2\pi$$

If α is a geodesic arc then we break it up into subarcs of length less than or equal $\log 3$. If $[x]^+$ denotes the smallest integer greater than or equal to x then α splits into $[L_{\mathcal{P}}(\alpha)/\log 3]^+$ such sunbarcs. By the above, the intersection of each of these subarcs is bounded by 2π and therefore

$$i(\alpha, \beta_{\mathcal{P}}) \leq 2\pi \left[\frac{L_{\mathcal{P}}(\alpha)}{\log 3}\right]^+ \leq 2\pi \left(\frac{L_{\mathcal{P}}(\alpha)}{\log 3} + 1\right)$$

Dividing by $L_{\mathcal{P}}(\alpha)$ we get

$$B(\mathcal{P}, \alpha) \leq \frac{2\pi}{\log 3}\left(1 + \frac{\log 3}{L_{\mathcal{P}}(\alpha)}\right)$$

■

We define functions $i, b : R_+ \to R_+$ by

$$i(t) = 2\pi \left(\frac{t}{\log 3} + 1\right) \qquad b(t) = \frac{2\pi}{\log 3}\left(1 + \frac{\log 3}{t}\right)$$

Then by Theorem 1 any geodesic arc of length t on a convex pleated plane has intersection with the bending lamination bounded by $i(t)$ and average bending bounded by $b(t)$.

Lemma: 2 *Let α an infinite length geodesic on a convex plane $\mathcal{P}$ and α_1, α_2 be geodesic subarcs of lengths L_1, L_2 respectively. Let C be a bound for the maximum length of a connected component of $(\alpha_1 \cap \alpha_2^c) \cup (\alpha_2 \cap \alpha_1^c)$ such that $L_1 > 2C + 2$. Then there exist a constant K_C depending only on C such that*

$$|B(\mathcal{P}, \alpha_1) - B(\mathcal{P}, \alpha_2)| \leq \frac{K_C}{L_1 - 2C}$$

Proof: Let $A-B$ denote the set $A\cap B^c$. Each of the sets $(\alpha_1-\alpha_2),(\alpha_2-\alpha_1)$ is composed of at most two connected components of length $\leq C$. Also we have $|L_2-L_1|\leq 2C$. Therefore

$$\begin{aligned}
|B(\mathcal{P},\alpha_1)-B(\mathcal{P},\alpha_2)| &= |\frac{i(\alpha_1,\beta_S)}{L_1}-\frac{i(\alpha_2,\beta_S)}{L_2}| \\
&= |\frac{i(\alpha_1-\alpha_2,\beta_S)}{L_1}+(\frac{1}{L_1}-\frac{1}{L_2})i(\alpha_1\cap\alpha_2,\beta_S)-\frac{i(\alpha_2-\alpha_1,\beta_S)}{L_2}| \\
&\leq \frac{i(\alpha_1-\alpha_2,\beta_S)}{L_1}+\frac{|L_2-L_1|}{L_2}\frac{i(\alpha_1\cap\alpha_2,\beta_S)}{L_1}+\frac{i(\alpha_2-\alpha_1,\beta_S)}{L_2} \\
&\leq \frac{2}{L_1}i(C)+\frac{2C}{L_2}b(L_1)+\frac{2}{L_2}i(C) \\
&\leq \frac{2Cb(L_1)+4i(C)}{L_1-2C}
\end{aligned}$$

If we let $K_C=2Cb(2)+4i(C)$ then since $L_1>2$ and b is monotonic we have

$$|B(\mathcal{P},\alpha_1)-B(\mathcal{P},\alpha_2)|\leq\frac{K_C}{L_1-2C}$$

■

Lemma: 3 *Let α be an infinite length geodesic on a pleated plane $\mathcal{P}$ and $x,y\in\alpha$. Then $B(\mathcal{P},\alpha_x)=B(\mathcal{P},\alpha_y)$. Furthermore $B(\mathcal{P},\alpha_x)=\overline{\lim}_{n\to\infty}B(\mathcal{P},\alpha_x^n)$.*

Proof: Applying the above Lemma to subarcs α_x^t and α_y^t where $x,y\in\alpha$, we can take $C=$ distance from x to y on α. Thus for $t>C+1$, $|B(\mathcal{P},\alpha_x^t)-B(\mathcal{P},\alpha_y^t)|\leq K_C/(2t-2C)$. Therefore we have $B(\mathcal{P},\alpha_x)=B(\mathcal{P},\alpha_y)$.

Let $t>0$ and $n=[t]^+$. Applying the above Lemma to the subarcs α_x^t and α_x^n we can take $C=1$. Thus for $t>4$, $|B(\mathcal{P},\alpha_x^t)-B(\mathcal{P},\alpha_x^n)|\leq K_1/(2t-2)$. Therefore we have $B(\mathcal{P},\alpha_x)=\overline{\lim}_{n\to\infty}B(\mathcal{P},\alpha_x^n)$.
■

We now define the the average bending of an infinite length geodesic .

Definition: 3 *The* average bending *of an infinite length geodesic α on a pleated plane $\mathcal{P}$, denoted $B(\mathcal{P},\alpha)$, is given by $B(\mathcal{P},\alpha)=B(\mathcal{P},\alpha_x)$ where $x\in\alpha$.*

Theorem: 2 *There is a constant $K>0$ such that $B(\mathcal{P},\alpha)\leq K$ for any infinite length geodesic α on a convex pleated plane $\mathcal{P}$ embedded in H^3.*
Furthermore $K\leq 2\pi/\log 3=5.719202$.

Proof: Follows immediately by letting $L_{\mathcal{P}}(\alpha)$ tend to infinity in Theorem 1.
■

In what follows K will always refer to the universal bound on the average bending of an infinite length geodesic, which by Theorem 2 is at most 5.719202. Further work of the author has shown $K\leq 3.7$. Also there is evidence (the planar case [B]) that the optimal bound should be 1.

4 Convex pleated surfaces, quasi-Fuchsian groups

A convex pleated surface S is formally a triple $(X_S, \beta_S, f_S : \tilde{X}_S \to H^3)$ where X_S is a hyperbolic surface with universal cover $\tilde{X}_S$, β_S is a measured geodesic lamination on X_S with preimage $\tilde{\beta}_S$ in $\tilde{X}_S$ and such that $(\tilde{X}_S, \tilde{\beta}_S, f_S : \tilde{X}_S \to H^3)$ is a convex pleated plane. The *universal cover* of a pleated surface S is the pleated plane $\tilde{S}$ given by the triple $(\tilde{X}_S, \tilde{\beta}_S, f_S : \tilde{X}_S \to H^3)$. We say S is an *embedded convex pleated surface* if f_S is an embedding.

As before the map f_S is unique up to post-composition by a Moebius transformation and we therefore identify S with the pair (X_S, β_S) which we call the *pleating coordinates*.

The study of embedded convex pleated surfaces is closely related to the study of quasi-Fuchsian groups. If Γ is a quasi-Fuchsian group then a component of the boundary of the convex core S_Γ is an embedded convex pleated surface given by a pair (X_Γ, β_Γ) where X_Γ is the intrinsic metric on S_Γ and β_Γ is the bending lamination. Furthermore this pair uniquely determines the group Γ.

We define the average bending of a finite length geodesic arc α on the pleated surface S by

$$B(S, \alpha) = \frac{i(\alpha, \beta_S)}{L_S(\alpha)}$$

where $L_S(\alpha)$ is the length of α. We extend it to infinite length arcs by taking lim sup as before. Note that if $\tilde{\alpha}$ is any lift of α on $\tilde{S}$ then $B(S, \alpha) = B(\tilde{S}, \tilde{\alpha})$.

Proposition: 1 *There exists a $K > 0$ such that if S is a convex embedded pleated surface and α a geodesic on S then $B(S, \alpha) \leq K$. Therefore if α is a closed geodesic in S then*

$$i(\alpha, \beta_S) \leq K \cdot L_S(\alpha) \tag{1}$$

Proof: Let α be a geodesic on S and $\tilde{\alpha}$ a lift of α on $\tilde{S}$. Applying Theorem 2 we get $B(\tilde{S}, \tilde{\alpha}) \leq K$ and therefore $B(S, \alpha) \leq K$.

If α is a closed geodesic then

$$B(S, \alpha) = \frac{i(\alpha, \beta_S)}{L_S(\alpha)}$$

Therefore $i(\alpha, \beta_S) \leq K \cdot L_S(\alpha)$.

■

Proposition: 2 *If S is a convex embedded pleated surface then*

$$L_S(\beta_S) \leq K \cdot \pi^2 \cdot |\chi(S)|$$

Proof: By continuity of the length function and of the intersection number, the inequality of Proposition 1 extends to any geodesic current α of S in the sense of [Bon1][Bon2]. In particular, we can apply the inequality to the case where α is the Liouville geodesic current associated to the hyperbolic metric of S (see [Bon2]). Two fundamental properties of this Liouville geodesic current L are that for every geodesic current β

$$i(L,\beta) = L_S(\beta)$$

and its self-intersection number satisfies

$$i(L,L) = \pi^2|\chi(S)|$$

Substituting $\alpha = L$ and $\beta = \beta_S$ into the inequality in Proposition 1 gives the required result. ■

As average bending can be defined for non-closed geodesics we can define a function $B : T_1(X_S) \to R_+$ on the unit tangent bundle of X_S by letting $B(v)$ be the average bending of the geodesic with tangent vector v. The function B is obviously invariant under geodesic flow and the following proposition shows that it is measurable. Therefore by the ergodicity of geodesic flow (see [Ho]), B is constant in almost every direction. Thus the average bending of a "random geodesic" is constant.

Proposition: 3 *The function $B : T_1(X_S) \to R_+$ is measurable and equal to a constant $B(S)$ almost everywhere. The constant $B(S) = 0$ if and only if $\beta_S = 0$*

Proof: Let S be a convex embedded pleated surface and $B : T_1(X_S) \to R_+$ be given by $B(v) = B(S, \alpha_v)$ where α_v is the geodesic, parameterized by arc length, with $\dot\alpha_v(0) = v$. Let $g_t : T_1(X_S) \to T_1(X_S)$ be the time t geodesic flow.

We define $B_1 : T_1(X_S) \to R_+$ by $B_1(v) = i(\alpha_v([0,1)), \beta_S)$. Therefore $B_1(v)$ is the intersection of β_S with the (half-open) unit length geodesic arc obtained by taking the geodesic flow starting at v.

The function B_1 is a measurable L^1 function and therefore by the Birkhoff-Khinchin Ergodic Theorem ([CFS]) there is a measurable function $\overline{B}$ such that

$$\overline{B}(v) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} B_1(g_k(v)) \text{ almost everywhere}$$

By Lemma 3 we can define B by

$$B(v) = \overline{\lim_{n\to\infty}} \frac{1}{2n+1} \sum_{k=-n}^{n} B_1(g_k(v))$$

Therefore $B = \overline{B}$ almost everywhere and is therefore measurable.

The function B is invariant under geodesic flow. By a result of Hopf ([Ho]), geodesic flow is ergodic. Therefore by the Ergodic Theorem B is equal to a constant $B(S)$ almost everywhere.

Also the Ergodic Theorem implies that for any ergodic flow the time average is equal the space average. Therefore as B is the time average of B_1, then

$$B(S) = \frac{1}{Vol_\Omega(T_1(X_S))} \int_{T_1(X_S)} B_1(w)\,d\Omega \qquad \text{where } \Omega \text{ is the volume form on } T_1(X_S)$$

If $\beta_S \neq 0$ then there is a tangent vector $\overline{v}$ such that $B_1(\overline{v}) > 0$. From the description of B_1, $\overline{v}$ must have a neighbourhood of positive measure in $T_1(X_S)$ on which B_1 is bounded away from zero. Thus the integral of B_1 is positive and therefore $B(S) > 0$. As $B(S)$ is obviously 0 when $\beta_S = 0$, $B(S) = 0$ if and only if $\beta_S = 0$.
■

In [Bon2], Bonahon gives an interpretation of the Liouville geodesic current based on Thurston's idea of "random geodesic" (see [Wol]) . In the following we use this interpretation to evaluate $B(S)$.

Proposition: 4 *If (X_S, β_S) are the pleating coordinates of S then*

$$B(S) = \frac{L_S(\beta_S)}{\pi^2 \cdot |\chi(S)|}$$

Proof: Let $B : T_1(X_S) \to R_+$ be defined as before by $B(v) = B(S, \alpha_v)$ where α_v is the geodesic, parameterized by arc length, with $\dot{\alpha}_v(0) = v$.

Let C be the diameter of X_S. Then for any $v \in T_1(X_S)$ we can construct a closed geodesic α_v^t by taking the closed geodesic in the homotopy class of the following curve. Take the geodesic segment of length $2t$ with midpoint having tangent vector v and then join the endpoints by a geodesic segment of length less than equal C. We divide α_v^t by its length to get a unit length geodesic current. Bonahon ([Bon2]) shows that for almost every v, the limit of these unit length geodesic currents as t tends to infinity is a constant times the Liouville geodesic current L for S. More precisely

$$\lim_{t\to\infty} \frac{\alpha_v^t}{L_S(\alpha_v^t)} = \frac{L}{\pi^2 \cdot |\chi(S)|} \quad \text{a.e.}$$

By continuity of the intersection number i, we can take the intersection with β_S to get

$$\lim_{t\to\infty} \frac{i(\alpha_v^t, \beta_S)}{L_S(\alpha_v^t)} = \frac{i(L, \beta_S)}{\pi^2 \cdot |\chi(S)|} = \frac{L_S(\beta_S)}{\pi^2 \cdot |\chi(S)|} \quad \text{a.e.}$$

From Proposition 3 we have that

$$B(v) = \lim_{t\to\infty} \frac{i(\alpha_v([-t,t]), \beta_S)}{2t} \quad \text{a.e.}$$

To find $L_S(\alpha_v^t)$ we lift $\alpha_v([-t,t])$ to H^2. We get a triangle with one side of length $2t$ and another of length $\leq C$. The third side maps to a closed curve $\overline{\alpha}_v^t$ homotopic to α_v^t. By the triangle inequality we have $2t - C \leq L_S(\overline{\alpha}_v^t) \leq 2t + C$. As the diameter of X_S is C and $\overline{\alpha}_v^t$ is homotopic to α_v^t, we have $2t - 3C \leq L_S(\alpha_v^t) \leq 2t + C$.

Choose C_1 so that every geodesic arc of length $\leq C$ has intersection number $\leq C_1$ with β_S. Then

$$i(\alpha_v([-t,t]), \beta_S) - 3C_1 \leq i(\alpha_v^t, \beta_S) \leq i(\alpha_v([-t,t]), \beta_S) + C_1$$

Thus

$$B(v) = \lim_{t\to\infty} \frac{i(\alpha_v([-t,t]), \beta_S)}{2t} = \lim_{t\to\infty} \frac{i(\alpha_v^t, \beta_S)}{L_S(\alpha_v^t)} = \frac{L_S(\beta_S)}{\pi^2 \cdot |\chi(S)|} \quad \text{a.e.}$$

Therefore

$$B(S) = \frac{L_S(\beta_S)}{\pi^2 \cdot |\chi(S)|}$$

■

Note that the bound on the length of the bending lamination β_S also follows from the evaluation of $B(S)$ above by using the fact that $B(S) \leq K$.

5 Lipschitz bounds and quasi-Fuchsian groups

As mentioned before the theory of convex pleated surfaces is closely related to quasi-Fuchsian groups. A conformal structure on a surface X is given by a smooth atlas with conformal transition maps. The space of conformal structures (up to isotopy) on a surface X is called *Teichmüller space* and denoted $T(X)$. By the Uniformization Theorem, given a conformal structure we can represent it by the quotient of the unit disk by a discrete group of Moebius transformations. Therefore every conformal structure on X uniquely determines a hyperbolic structure on X and $T(X)$ can be considered as the space of hyperbolic structures (up to isotopy) on X.

If Γ is a quasi-Fuchsian group and S_Γ a boundary component of the convex core then S_Γ is a convex pleated surface with pleating coordinates (X_Γ, β_Γ). Also, facing S_Γ at infinity is the conformal structure Y_Γ obtained by taking the quotient of one component Ω^1_Γ of the domain of discontinuity by the group Γ.

We now use average bending to compare the hyperbolic metric on Y_Γ with the hyperbolic metric on X_Γ.

Definition: 4 *Let $(X_1, d_1), (X_2, d_2)$ be two metric spaces then a map $f : (X_1, d_1) \to (X_2, d_2)$ is* K-Lipschitz *if $d_2(f(a), f(b)) \leq K d_1(a, b)$ for all $a, b \in X_1$. We call K the Lipschitz constant of f.*

If $[f]$ is a homotopy class of a homeomorphism from X_1 to X_2 then the minimum Lipschitz constant of $[f]$ is the infimum of Lipschitz constants for all Lipschitz homeomorphisms in $[f]$. In particular if $X_1, X_2 \in T(X)$ are two hyperbolic structures on X then we define the minimum Lipschitz constant between X_1 and X_2 to be the minimum Lipschitz constant of the homotopy class given by the identity map.

Proposition: 5 *Let Γ be a quasi-Fuchsian group. Then the minimum Lipschitz constant between X_Γ and Y_Γ is bounded by $1 + K$.*

Proof: Let α be a simple closed curve and let $L_{X_\Gamma}(\alpha), L_{Y_\Gamma}(\alpha)$ be the lengths of the geodesics homotopic to α in X_Γ, Y_Γ respectively. Then from [Mc] we have that

$$L_{Y_\Gamma}(\alpha) \leq L_{X_\Gamma}(\alpha) + i(\alpha, \beta_\Gamma)$$

Dividing by $L_{X_\Gamma}(\alpha)$ we get

$$\frac{L_{Y_\Gamma}(\alpha)}{L_{X_\Gamma}(\alpha)} \leq \frac{L_{X_\Gamma}(\alpha) + i(\alpha, \beta_\Gamma)}{L_{X_\Gamma}(\alpha)} \leq 1 + B(S_\Gamma, \alpha)$$

By the universal bound on $B(S_\Gamma, \alpha)$ we have

$$\frac{L_{Y_\Gamma}(\alpha)}{L_{X_\Gamma}(\alpha)} \leq 1 + K$$

Thurston proved ([T2]) that the minimum Lipschitz constant between two hyperbolic structures is the supremum of the ratio of lengths of corresponding simple closed geodesics. Therefore by the above, X_Γ and Y_Γ are at most $1 + K$ Lipschitz.
■

References

[B] Martin Bridgeman *Average curvature of convex curves in H^2*
To appear *Proceedings* of the A.M.S.

[Bon1] Francis Bonahon *Bouts des variétés hyperboliques de dimension 3*
Annals of Math. 124, 71-158 (1986)

[Bon2] Francis Bonahon *The geometry of Teichmüller space via geodesic currents*
Inventiones mathematicae 92, 139-162 (1988)

[CFS] I.P. Cornfeld, S.V. Fomin and Ya. G. Sinai *Ergodic Theory*
Grundlehren der matematischen Wissenschaften 245 Springer-Verlag, (1982)

[EM] D. B. A. Epstein and A. Marden. *Convex hulls in hyperbolic space, a theorem of Dennis Sullivan, and measured pleated surfaces*
Analytical and Geometric Aspects of Hyperbolic Space, Cambridge University Press, 1987

[G] David Gabai *On the Geometric and topological rigidity of hyperbolic 3-manifolds*
Bulletin of the A.M.S., vol. 31, no. 2, Oct. 1994

[Ho] E. Hopf *Statistik der geodatischen linien in mannigfaltigkeiten negativer krummung*
Ber. Verh. Sachs. Akad. Wiss. Leipzig 91, pp 261-304, 1939

[Mc] Curtis T. McMullen *Complex earthquakes and Teichmüller theory*
Preprint (1996)

[T1] William P. Thurston *Geometry and Topology of Three-Manifolds*
Princeton Lecture Notes, (1979)

[T2] William P. Thurston *Minimal stretch maps between hyperbolic surfaces*
Preprint

[Wol] Scott Wolpert *Thurston's Riemannian metric for Teichmüller space*
Journal of Differential Geometry 23, 143-174 (1986)