# From the boundary of the convex core to the conformal boundary

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### Abstract

If $N$ is a hyperbolic 3-manifold with finitely generated fundamental group, then the nearest point retraction is a proper homotopy equivalence from the conformal boundary of $N$ to the boundary of the convex core of $N$. We show that the nearest point retraction is Lipschitz and has a Lipschitz homotopy inverse and that one may bound the Lipschitz constants in terms of the length of the shortest compressible curve on the conformal boundary.

## 1 Introduction

Every hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$ has a natural "conformal boundary at infinity" $\partial_c N$, which is obtained by taking the quotient, by $\Gamma$, of the domain of discontinuity $\Omega(\Gamma)$ for $\Gamma$'s action on the boundary at infinity $S^2_\infty$ of hyperbolic 3-space. As $S^2_\infty$ may be identified with the Riemann sphere $\widehat{\mathbf{C}}$ and $\Gamma$ acts as a group of conformal automorphisms, we may think of $\partial_c N$ as a Riemann surface. If $\Gamma$ is nonabelian, $\partial_c N$ inherits a well-defined hyperbolic metric. Another fundamental object associated to a hyperbolic 3-manifold is its convex core $C(N)$, i.e. the smallest convex submanifold of $N$. It is well-known that (unless it is 2-dimensional) $C(N)$ is homeomorphic to $\widehat{N} = \partial_c N \cup N$ and that $\partial C(N)$ is a hyperbolic surface (in its intrinsic metric).

Since the conformal boundary and the boundary of the convex core are homeomorphic hyperbolic surfaces it is natural to ask about the relationship between the geometry of the two surfaces. Sullivan showed that there exists some uniform constant $K$ such that if $\partial_c N$ is incompressible in $\widehat{N}$, then there is a $K$-biLipschitz homeomorphism between the conformal boundary and the boundary of the convex core, see Epstein-Marden [6].

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One may produce sequences of examples of hyperbolic 3-manifolds where the minimal biLipschitz constant of a homeomorphism between the conformal boundary and the boundary of the convex core becomes arbitrarily large, see [6] or [5]. In these sequences, the length of the shortest compressible curve in the conformal boundary becomes arbitrarily small. It is thus natural to conjecture that there should be a biLipschitz homeomorphism between the conformal boundary and the boundary of the convex core, such that the biLipschitz constant is bounded above by a constant depending only on the length of the shortest compressible curve in the conformal boundary.

In this paper, we give a partial generalization of Sullivan's theorem to the setting of hyperbolic 3-manifolds with compressible conformal boundary. It is not difficult to combine the estimates in [5] and the techniques used in Epstein-Marden [6] to show that the nearest point retraction is a Lipschitz map from the conformal boundary to the boundary of the convex core and that there is a bound on the Lipschitz constant depending only on a lower bound for the injectivity radius of the domain of discontinuity. We adapt techniques from Bridgeman [3] to produce a homotopy inverse which is a Lipschitz map where again there is a bound on the Lipschitz constant depending only on a lower bound for the injectivity radius of the domain of discontinuity.

**Theorem 1:** *There exist functions* $J, L : (0,\infty) \to (0,\infty)$ *such that if* $N = \mathbf{H}^3/\Gamma$ *is a hyperbolic 3-manifold with finitely generated fundamental group and* $\rho_0$ *is a lower bound on the injectivity radius (in the Poincaré metric) of the domain of discontinuity* $\Omega(\Gamma)$*, then the nearest point retraction* $r : \partial_c N \to \partial C(N)$ *is* $J(\rho_0)$*-Lipschitz and has a* $L(\rho_0)$*-Lipschitz homotopy inverse.*

We will give explicit expressions for $J$ and $L$ later. For the moment, we simply note that as $\rho_0 \mapsto 0$, $J(\rho_0) = O(\frac{1}{\rho_0})$ and $L(\rho_0) = O(e^{\frac{C}{\rho_0}})$ for some constant $C > 0$. Although these expressions may seem to grow quite fast we will also see that their basic forms cannot be substantially improved.

In the case that the conformal boundary is incompressible, our techniques improve on the bounds obtained in Bridgeman [3].

**Theorem 2:** *If* $N = \mathbf{H}^3/\Gamma$ *is a hyperbolic 3-manifold with finitely generated fundamental group and each component of* $\Omega(\Gamma)$ *is simply connected, then the nearest point retraction* $r : \partial_c(N) \to \partial C(N)$ *is* 4*-Lipschitz and has a* $(1 + \frac{\pi}{\sinh^{-1}(1)})$*-Lipschitz homotopy inverse, where* $1 + \frac{\pi}{\sinh^{-1}(1)} \approx 4.56443$.

We note that the conformal boundary is incompressible if and only if each component of the domain of discontinuity is simply connected. A lower bound on the length of the shortest compressible curve in the conformal boundary is equivalent to a lower bound on the injectivity radius of the domain of discontinuity (in the Poincaré metric.) If $\Gamma$ is finitely generated, then Ahlfors' Finiteness theorem [1] implies that there is a lower bound on the injectivity radius of the domain of discontinuity.

One expects that the conclusions of Theorem 1 ought to guarantee the existence of a biLipschitz homeomorphism between the conformal boundary and the boundary of the convex core and uniform bounds on the biLipschitz constant. A realization of this expectation would produce a full generalization of Sullivan's theorem. In most cases, one uses Sullivan's theorem to assure that there is a biLipschitz equivalence of lengths of corresponding closed geodesics. Theorem 1 does produce this biLipschitz equivalence of lengths.

**Corollary 1:** *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated fundamental group and let $\rho_0$ be a lower bound for the injectivity radius in $\Omega(\Gamma)$. If $\alpha$ is a closed geodesic in $\partial_c N$ and $r(\alpha)^*$ denotes the closed geodesic in $\partial C(N)$ which is homotopic to $r(\alpha)$, then*

$$\frac{l_{\partial C(N)}(r(\alpha)^*)}{J(\rho_0)} \leq l_{\partial_c(N)}(\alpha) \leq L(\rho_0) l_{\partial C(N)}(r(\alpha)^*)$$

*where $l_{\partial C(N)}(r(\alpha)^*)$ denotes the length of $r(\alpha)^*$ in $\partial C(N)$ and $l_{\partial_c N}(\alpha)$ denotes the length of $\alpha$ in $\partial_c N$.*

We note that there is also a version of Theorem 1, where the bounds depend on the injectivity radius of the boundary of the convex hull $CH(L_\Gamma)$ of the limit set $L_\Gamma$ of $\Gamma$, see section 9. In fact, the bounds on the Lipschitz constant produced by generalizing the techniques of Bridgeman naturally give bounds which depend on the injectivity radius bounds on the boundary of the convex hull and it is necessary to prove that injectivity radius bounds on the boundary of the convex hull imply injectivity radius bounds on the domain of discontinuity (and vice versa.) We will also see that Theorem 1 holds more generally for analytically finite hyperbolic 3-manifolds and that Corollary 1 may be generalized to allow $\alpha$ to be any geodesic current on $\partial_c N$.

The key tool underlying the proofs of theorems 1 and 2 is an estimate on the average bending of a curve in the boundary of the convex core. Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold and $\alpha$ is a closed geodesic in $\partial C(N)$. We define the *average bending* $B(\alpha)$ of $\alpha$ to be

$$B(\alpha) = \frac{i(\alpha, \beta_N)}{l_{\partial C(N)}(\alpha)}$$

where $i(\alpha, \beta_N)$ is the total bending along $\alpha$ and $l_{\partial C(N)}(\alpha)$ is the hyperbolic length of $\alpha$ on $\partial C(N)$.

**Theorem 3:** *There exists a function $K : (0,\infty) \to (0,\infty)$ such that if $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated fundamental group and $\alpha$ is a closed geodesic on $\partial C(N)$, then*

1. *If* $\widehat{\rho}_\alpha$ *is a lower bound for* $inj_{\partial CH(L_\Gamma)}$ *at any point in the support of a lift* $\widetilde{\alpha}$ *of* $\alpha$*, then* $B(\alpha) \leq K(\widehat{\rho}_\alpha)$

2. *If* $\alpha$ *is contained in an incompressible component of* $\partial C(N)$*, then* $B(\alpha) \leq K_\infty$*, where* $K_\infty = \frac{\pi}{\sinh^{-1}(1)} \approx 3.56443$*.*

## 2 Background

An orientable hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$ is the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a discrete torsion-free subgroup of the group $Isom_+(\mathbf{H}^3)$ of orientation preserving isometries of $\mathbf{H}^3$. We may identify $Isom_+(\mathbf{H}^3)$ with the group $\mathrm{PSL}_2(\mathbf{C})$ of Mobius transformations of $\widehat{\mathbf{C}}$. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open set in $\widehat{\mathbf{C}}$ on which $\Gamma$ acts properly discontinuously, and the limit set $L_\Gamma$ is its complement. The conformal boundary $\partial_c N$ of $N$ is simply the quotient $\Omega(\Gamma)/\Gamma$. If $\Gamma$ is non-abelian, then $L_\Gamma$ is infinite and $\Omega(\Gamma)$ admits a canonical hyperbolic metric $\rho(z)dz$ called the Poincaré metric. We will assume throughout the paper that $\Gamma$ is nonabelian. The Kleinian group $\Gamma$ acts as a group of isometries of the Poincaré metric, so $\partial_c N$ is a hyperbolic surface. The hyperbolic 3-manifold $N$ is said to be *analytically finite* if $\partial_c N$ has finite area in this metric. Ahlfors' Finiteness Theorem asserts that $N$ is analytically finite if $\Gamma$ is finitely generated. We note that if $N$ is analytically finite then there is always a positive lower bound for the injectivity radius on $\Omega(\Gamma)$.

The *convex hull* $CH(L_\Gamma)$ of $L_\Gamma$ is the smallest convex subset of $\mathbf{H}^3$ so that all geodesics with both limit points in $L_\Gamma$ are contained in $CH(L_\Gamma)$. The *convex core* $C(N)$ of $N = \mathbf{H}^3/\Gamma$ is the quotient of $CH(L_\Gamma)$ by $\Gamma$. It is the smallest convex submanifold of $N$ so that the inclusion map is a homotopy equivalence. The boundary $\partial C(N)$ of the convex core is a pleated surface, i.e. there is a path-wise isometry $f : S \to \partial C(N)$ from a hyperbolic surface $S$ onto $N$ which is totally geodesic in the complement of a disjoint collection $\beta_N$ of geodesics which is called the pleating locus. The nearest point retraction $\widetilde{r} : \mathbf{H}^3 \to CH(L_\Gamma)$ is the map which takes a point to the (unique) nearest point in $CH(L_\Gamma)$. It extends continuously to a map $\widetilde{r} : \Omega(\Gamma) \cup \mathbf{H}^3 \to \partial CH(L_\Gamma)$, called the nearest point retraction, such that if $z \in \Omega(\Gamma)$, then $\widetilde{r}(z)$ is the (unique) first point of contact of an expanding family of horospheres based at $z$ with $\partial CH(L_\Gamma)$. This map descends to a map $r : \widehat{N} \to \partial C(N)$. We will often consider the restriction of $r$ to $\partial_c N$ (which we will simply call r) which gives a homotopy equivalence from $\partial_c N$ to $\partial C(N)$. For a complete description of the geometry of the convex hull see Epstein-Marden [6].

We have to modify the above description in the special case that $L_\Gamma$ lies in a circle in $S^2_\infty$. In this case, $CH(L_\Gamma)$ is a convex subset of a hyperbolic plane and $C(N)$ is a totally geodesic surface with boundary. In this case, we will consider $\partial C(N)$ to be the double of $C(N)$ (along its boundary considered as a hyperbolic surface) where we regard the 2 copies of $C(N)$ as having opposite normal vectors. One may still define $r : \partial_c N \to \partial C(N)$ in this setting and it remains a homotopy equivalence.

The pleating locus $\beta_N$ inherits a measure on arcs transverse to $\beta_N$ which records the total amount of bending along any transverse arc, so $\beta_N$ is a measured lamination. A *measured lamination* on a finite area hyperbolic surface $S$ consists of a closed subset $\lambda$ of $S$ which is the disjoint union of geodesics, together with an invariant measure (with respect to projection along $\lambda$) on arcs transverse to $\lambda$. Measured laminations whose support is a finite collection of simple closed geodesics are dense in the space $ML(S)$ of all measured laminations on $S$ (see [8]). If we lift $\lambda$ to the universal cover $\mathbf{H}^2$ of $S$, we obtain a $\pi_1(S)$-invariant subset of the space of geodesics on $G(\mathbf{H}^2)$. The transverse measure on $\lambda$ gives rise to a $\pi_1(S)$-invariant measure on $G(\mathbf{H}^2)$. More generally, a *geodesic current* is a $\pi_1(S)$-invariant measure on $G(\mathbf{H}^2)$. Bonahon [2] has extensively studied the space $\mathcal{C}(S)$ of geodesic currents on $S$. The support of a geodesic current projects to a closed union of geodesics and the set of geodesic currents supported on a finite collection of closed geodesics on $S$ is dense in $\mathcal{C}(S)$. The functions given by length and intersection number on closed geodesics extend in a natural way to continuous functions on $ML(S)$ and $\mathcal{C}(S)$.

## 3 Some basic facts from hyperbolic geometry

We begin by observing that among the triangles with a side of fixed length and opposite angle of fixed value, the isosceles triangle maximizes perimeter. We will omit the proof which is an elementary calculation involving hyperbolic trigonometry.

**Lemma 3.1** *Consider the set of all hyperbolic triangles with one side of fixed length $C$ and the opposite angle of fixed value $\gamma$ (where $0 < \gamma < \pi$.) Then the unique triangle in this set with maximal length perimeter is the isosceles triangle having the fixed side as base. The other sides have length*

$$\sinh^{-1}\left(\frac{\sinh(C/2)}{\sin(\theta/2)}\right).$$

We will also need an elementary observation about configurations of planes in $\mathbf{H}^3$. We will later use such configurations to enclose the convex hull.

Let $H_0$, $H_1$, and $H_2$ be three closed half-spaces in $\mathbf{H}^3$. Let $P_i$ denote the plane in $\mathbf{H}^3$ which bounds $H_i$ and let $D_i$ be the closed disk in $S^2_\infty$ which is the intersection of the closure of $H_i$ with $S^2_\infty$. Suppose that $D_0 \cap D_1 = \{a\}$ and $D_1 \cap D_2 = \{b\}$. Let $C$ be the closure of the complement of $H_1 \cup H_2 \cup H_3$.

Consider $\alpha$ to be a parametrized curve $\alpha : [0, 2] \to C$ such that $\alpha(i) \in P_i$, for $i = 0, 1, 2$. We denote the length of $\alpha$ by $l$. Therefore $\alpha$ is a curve with one endpoint on $P_0$, the other on $P_2$, and an interior point on $P_1$. We show that if $l$ is short enough, then $D_0$ and $D_2$ must intersect and that $l$ determines an upper bound for their angle of intersection. Recall that the angle of intersection of two half-spaces equals the angle of intersection of the associated disks on the sphere at infinity.

**Lemma 3.2** *If $l \leq 2\sinh^{-1}(1)$, then $D_0$ and $D_2$ intersect and their angle of intersection $\theta$ satisfies*

$$\theta \geq 2\cos^{-1}(\sinh(l/2))$$

*Proof of 3.2:* Let $\overline{\alpha}$ be the shortest curve in $C$ with one endpoint on $P_0$, the other on $P_2$, and an interior point on $P_1$. Let $H$ be the unique plane orthogonal to the three planes $P_i, i = 0, 1, 2$. We note that the circle on $S^2_\infty$ given by $H$ must pass through the two ideal points $a$ and $b$ described above. Thus, letting $L_i = P_i \cap H$, the line $L_1$ meets each of $L_0$ and $L_2$ in an ideal point. Furthermore, the disks $D_0$ and $D_2$ intersect if and only if the lines $L_0$ and $L_2$ intersect, and the angle of intersection of the lines is equal to the angle of intersection of the disks.

As orthogonal projection onto $H$ decreases distance, $\overline{\alpha}$ must be contained in the plane $H$. Thus $\overline{\alpha}$ is a planar curve. Using planar hyperbolic geometry, it can be shown that the curve $\overline{\alpha}$ consists of two equal length geodesic segments with a common endpoint $v$ on $L_1$ which are perpendicular to $L_0$ and $L_2$ respectively. Let $l_{min}$ be the length of $\overline{\alpha}$. If $L_0$ and $L_2$ intersect in an angle $\theta$, then we let $T$ be the triangle given by the three lines $L_0$, $L_1$ and $L_2$. Applying elementary formulae from hyperbolic trigonometry one sees that:

$$\sinh(l_{min}/2) = \cos(\theta/2)$$

Therefore, we can write $l_{min}$ as a function of $\theta$ as follows:

$$l_{min}(\theta) = 2\sinh^{-1}(\cos(\theta/2))$$

As $l \geq l_{min}$ we have that

$$l \geq 2\sinh^{-1}(\cos(\theta/2))$$

The function $f(x) = \sinh^{-1}(\cos(x/2))$ is decreasing on $[0, \pi]$ and therefore

$$\theta \geq 2\cos^{-1}(\sinh(l/2))$$

If $T$ is ideal then $\theta = 0$ and $l_{min}(0) = 2\sinh^{-1}(1)$ which we denote $l_0$. If $L_0$ and $L_2$ do not intersect then there is an ideal triangle with two ideal vertices equal to the ideal endpoints of $L_1$, whose other ideal vertex lies between the ideal endpoints of $L_0$ and $L_2$ which are not endpoints of $L_1$. The curve $\overline{\alpha}$ has length at least $l_0$ and $l_{min} > l_0$. Therefore $l > l_0$ and if $l \leq l_0$ then $D_0$ and $D_2$ must intersect and $\theta$ must satisfy the given inequality.

$\boxed{3.2}$

## 4 Local intersection number estimates

In this section we show that if a geodesic arc in the boundary of the convex hull is short enough then its "total bending" is at most $2\pi$. How short it is necessary to make the arc will be an explicit function of the injectivity radius of the convex hull at the starting point of the arc. This estimate, Lemma 4.3, underlies all the results in the paper.

We first need to recall some background material on convex hulls. For a full description of convex hulls see [6].

If $\Gamma$ is a Kleinian group with convex hull $CH(L_\Gamma)$ then a *support plane* to $CH(L_\Gamma)$ is a hyperbolic plane $P$ in $H^3$ which bounds a half-space $H_P$ such that $H_P \cap \partial CH(L_\Gamma) \subseteq P$. As the half-space $H_P$ is considered to be implicit, $P$ is naturally oriented by taking the normal to point toward the interior of $H_P$.

Thus a support plane $P$ to a convex hull $CH(L_\Gamma)$ does not pass through $\partial CH(L_\Gamma)$ but has a glancing intersection with it. In general the intersection of $P$ and $\partial CH(L_\Gamma)$ can either be a single geodesic, called a *bending line*, or a flat piece of the convex hull boundary bounded by a set of disjoint geodesics, called a *flat*. If $P_1$ and $P_2$ are support planes with $P_1 \cap P_2 \neq \emptyset$ and $P_1 \neq P_2$ then the line $r = P_1 \cap P_2$ is called a *ridge line*.

If $x \in \partial CH(L_\Gamma)$ then either $x$ lies in the interior of a flat or $x$ is on some bending line. If $x$ is in the interior of a flat then there is a unique support plane $P$ containing $x$. If $x \in b$, where $b$ is a bending line, let $\Sigma(b)$ be the set of support planes to $b$. The set of oriented planes $S(b)$ containing $b$ is a circle and $\Sigma(b) \subseteq S(b)$. As $\Sigma(b)$ is connected, it is either a closed arc or a point. We let $P_1$ and $P_2$ be the two extreme planes of $\Sigma(b)$. If $b$ is oriented then we can refer to the extreme planes as left and right extreme planes. The *bending angle* at $b$ is defined to be the angle between $P_1$ and $P_2$. Thus the bending angle is the exterior dihedral angle between the extreme planes at $b$. If $x$ is on a bending line $b$, we define $\beta(x)$ to be the bending angle at $b$. Otherwise we define $\beta(x) = 0$.

The union of the bending lines in $\partial CH(L_\Gamma)$ is denoted $\beta_\Gamma$ and is called the *bending lamination.* Thurston defined a *transverse measure* on $\beta_\Gamma$ which assigns to every arc $\alpha$ transverse to $\beta_\Gamma$ a value $i(\alpha, \beta_\Gamma)$ corresponding to the amount of bending along $\alpha$ (see [8]). Therefore $\beta_\Gamma$ is a measured lamination. In particular, the bending measure is a countable additive measure on the set of transverse arcs (see [6]), i.e. if $\alpha$ is subdivided into subarcs $\{\alpha_1, \ldots, \alpha_n\}$ transverse to $\beta_\Gamma$, then

$$i(\alpha, \beta_\Gamma) = \sum_{i=1}^{n} i(\alpha_i, \beta_\Gamma).$$

If the arc $\alpha$ is a closed arc with endpoints $x, y$ whose interior $\alpha^o$ is transverse to $\beta_\Gamma$, then, we define

$$i(\alpha, \beta_\Gamma) = \beta(x) + i(\alpha^o, \beta_\Gamma) + \beta(y).$$

The bending lamination $\beta_\Gamma$ on $\partial CH(L_\Gamma)$ projects to the pleating locus $\beta_N$ of $\partial C(N)$.

In order to allow the subarcs to have endpoints on $\beta_\Gamma$ it is necessary to modify the definition somewhat to keep track of the trajectory of support planes. Let $P$ and $Q$ be support planes at $x$ and $y$ respectively. If $\alpha$ intersects a bending line $b$ then an orientation on $\alpha$ gives an orientation on the bending line $b$. Thus we orient $\alpha$ from $x$ to $y$ and let $\overline{P}$ be the rightmost support plane at $x$ and let $\overline{Q}$ be the leftmost support plane at $y$. Let $\theta_P$ be the exterior dihedral angle between $P$ and $\overline{P}$ and let $\theta_Q$ be the exterior dihedral angle between the support planes $\overline{Q}$ and $Q$. Then we define

$$i(\alpha, \beta_\Gamma)_P^Q = \theta_P + i(\alpha^o, \beta_\Gamma) + \theta_Q.$$

Notice that if $\alpha$ has unique support planes at its endpoints then $i(\alpha, \beta_\Gamma)_P^Q = i(\alpha, \beta_\Gamma)$.

Let $\alpha : [0, 1] \to \partial CH(L_\Gamma)$ be a path whose interior is transverse to $\beta_\Gamma$ and let $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ be a subdivision of $[0, 1]$. Let $\alpha_i$ be the closed subarc obtained by restricting $\alpha$ to the interval $[t_{i-1}, t_i]$. Let $P_i$ be a support plane at $\alpha(t_i)$ with $P_0 = P$ and $P_n = Q$. Then it follows from the additivity of the standard intersection number that

$$i(\alpha, \beta_\Gamma)_P^Q = \sum_{i=1}^{n} i(\alpha_i, \beta_\Gamma)_{P_{i-1}}^{P_i}$$

This is the key additivity property for our modified intersection number.

Let $\alpha : [0, 1] \to \partial CH(L_\Gamma)$ be a path whose interior is transverse to $\beta_\Gamma$ and let $P$ and $Q$ be support planes to $\partial CH(L_\Gamma)$ at $\alpha(0)$ and $\alpha(1)$. We now obtain an explicit description of the continuous path of support planes to $\alpha$ joining $P$ to $Q$. If there is not a unique support plane at a point $\alpha(t)$, it must lie on a bending line with positive bending angle. As each bending line with positive angle covers one of finitely many closed geodesics in $\beta_N$, there are only finitely many points in the image of $\alpha$ at which there is not a unique support plane. Let $\{0 \leq t_1 \leq t_1 \leq \ldots \leq t_{n-1} \leq 1\}$ denote the finite set of points where there is not a unique support plane to the image of $\alpha$. Let $t_0 = 0$ and $t_n = 1$. Let $b_i$ denote the bending line at the point $\alpha(t_i)$. If $t_i$ is not either 0 or 1, then let $\theta_i = \beta(\alpha(t_i)) > 0$ and let $\{P_t^i | t \in [0, \theta_i]\}$ denote the one parameter family of all support planes contained in $b_i$, parameterised by the exterior angle the support plane makes with the first extreme support plane of $b_i$ that $\alpha$ intersects. If $t_1 = 0$, then we let $\theta_1$ be the angle between $P$ and the rightmost support plane at $\alpha(0)$ and we begin the parameterization $\{P_t^i | t \in [0, \theta_1]\}$ at $P$. Similarly, if $t_{n-1} = 1$, then we let $\theta_{n-1}$ be the angle between $Q$ and the leftmost support plane at $\alpha(1)$ and we end the parameterization $\{P_t^i | t \in [0, \theta_{n-1}]\}$ at $Q$.

Let $I_j = (t_{j-1}, t_j)$ for all $j = 1, \dots, n$. If $t \in I_j$, then let $Q_t^j$ be the unique support plane passing through $\alpha(t)$. Then $\{Q^j | t \in I_j\}$ is a continuous 1-parameter family of support planes along $I_j$. We obtain a continuous 1-parameter family of support planes along $\alpha$ between $P$ and $Q$ by concatenating 1-parameter families $P^i$ and $Q^j$, ordered by the orientation of $\alpha$ from $\alpha(0)$ to $\alpha(1)$. Explicitly we define

$$\Theta_i = \sum_{j=1}^{i} \theta_j \qquad \text{and} \qquad k = 1 + \Theta_{n-1}.$$

We let $X_i = [t_i + \Theta_{i-1}, t_i + \Theta_i]$ for $i = 1, \dots, n-1$. We let $Y_i = (t_{i-1} + \Theta_{i-1}, t_i + \Theta_{i-1})$ for $i = 2, \dots, n-1$. We let $Y_1 = [0, t_1)$ and $Y_n = (t_{n-1} + \Theta_{n-1}, k]$. The intervals $X_i$ and $Y_i$ give a partition of $[0, k]$ and we define a piecewise linear continuous function $s : [0, k] \to [0, 1]$ by

$$s(t) = \begin{cases} t_i & t \in X_i \\ t - \Theta_{i-1} & t \in Y_i \end{cases}$$

The function $s$ is a continuous monotonic function. We define the support planes $P_t$ by letting $P_0 = P$, $P_k = Q$ and if $t \in (0, k)$ setting

$$P_t = \begin{cases} P^i_{t - t_i - \Theta_{i-1}} & t \in X_i \\ Q^i_{s(t)} & t \in Y_i \end{cases}$$

The family $\{P_t | t \in [0, k]\}$ is called the *continuous 1-parameter family of support planes along $\alpha$ from $P$ to $Q$*. Notice that $P_t$ is a support plane to $\alpha(s(t))$ and that if $P_{t_1} = P_{t_2}$ and $s(t_1) = s(t_2)$, then $t_1 = t_2$.

We say that $(P, Q)$ is a *roof* over $\alpha$ if for all $t \in [0, k]$, $P \cap P_t \neq \emptyset$ and the interiors of the half spaces $H_P$ and $H_{P_t}$ also intersect. Furthermore we say $(P, Q)$ is a *$\pi$-roof* if $(P, P_t)$ is a roof over $\alpha([0, s(t)])$ for all $0 \leq t < k$ but $(P, Q)$ is not a roof over $\alpha$.

We now define monotonicity for geodesics in the hyperbolic plane. Let $\{g_t\}$ be a continuous family of geodesics in a hyperbolic plane which is indexed by an interval $J$. We say that the family is *monotonic* on $J$ if given $a, b \in J$ such that $a < b$ and $g_a \cap g_b \neq \emptyset$ then $g_t = g_a$ for all $t \in [a, b]$. Monotonicity has the following continuity property. If $\{g_t\}$ is monotonic over $[a, b)$ and continuous on $[a, b]$, then it is monotonic on $[a, b]$.

The following lemma allows us to estimate the intersection number along a geodesic on $\partial CH(L_\Gamma)$ by using support planes. Its proof is given in the appendix.

**Lemma 4.1** *Let $\alpha : [0, 1] \to \partial CH(L_\Gamma)$ be a parameterized geodesic arc, let $(P, Q)$ be a roof over $\alpha$, and let $\{P_t | \ t \in [0, k]\}$ be a continuous one-parameter family of support planes over $\alpha$ joining $P$ to $Q$. Then*

1. $$i(\alpha,\beta_\Gamma)_P^Q \leq \theta \leq \pi.$$

   *where* $\theta$ *is the exterior dihedral angle between* $P$ *and* $Q$*, and*

2. *there is a* $\bar{t} \in [0,1]$ *such that* $P_t = P$ *if* $t \in [0,\bar{t}]$ *and the ridge lines* $\{r_t = P \cap P_t | t > \bar{t}\}$ *exist and form a monotonic family of geodesics on* $P$*.*

The following corollary follows immediately from Lemma 4.1 by continuity.

**Corollary 4.2** *If* $(P,Q)$ *is a* $\pi$*-roof over* $\alpha$ *then the interiors of the half spaces* $H_P$ *and* $H_Q$ *are disjoint and* $i(\alpha,\beta_\Gamma)_P^Q \leq \pi$*.*

The following functions come up naturally when we attempt to quantify how short we must make a geodesic in $\partial CH(L_\Gamma)$ in order to guarantee that its intersection with the bending measure is at most $2\pi$. We define the functions $F, G, K$ by

$$F(x) = \frac{x}{2} + \sinh^{-1}\left(\frac{\sinh(\frac{x}{2})}{\sqrt{1-\sinh^2(\frac{x}{2})}}\right) \qquad G(x) = F^{-1}(x) \qquad K(x) = \frac{2\pi}{G(x)}$$

From the equation it is easy to see that $F$ is monotonically increasing with domain $[0, 2\sinh^{-1}(1))$. The function $G(x)$ has asymptotic behaviour $G(x) \asymp x$ for $x \to 0$, and $G(x) \asymp 2\sinh^{-1} 1$ for $x \to \infty$. We further define $G_\infty = 2\sinh^{-1} 1 \approx 1.76275$ and $K_\infty = \frac{\pi}{\sinh^{-1}(1)} \approx 3.56443$.

The following lemma shows that if a short arc bends a lot, then it must begin at a point with small injectivity radius. In the next section, we will apply this local bound to obtain the global bound on average bending given in Theorem 3. Let $\widehat{\rho} : \partial CH(L_\Gamma) \to \mathbf{R}$ be the injectivity radius function.

**Lemma 4.3** *Let* $\alpha : [0,1] \to \partial CH(L_\Gamma)$ *be a parameterized geodesic arc with length* $l(\alpha)$ *and let* $P$ *be a support plane at* $\alpha(0)$*. If either*

1. $\alpha([0,1])$ *is contained in a simply connected component of* $\partial CH(L_\Gamma)$ *and* $l(\alpha) \leq G_\infty$*, or*

2. $l(\alpha) \leq G(\widehat{\rho}(\alpha(0)))$*,*

*then there is a support plane* $Q$ *at* $\alpha(1)$ *such that*

$$i(\alpha,\beta_\Gamma)_P^Q \leq 2\pi.$$

*Proof of 4.3:* Let $\alpha : [0,1] \to \partial CH(L_\Gamma)$ be a parameterized geodesic. Let $Q$ be the left-most support plane at $\alpha(1)$ and let $\{P_t | t \in [0,k]\}$ be the continuous one parameter family of support planes to $\alpha$ joining $P$ to $Q$. If $(P,Q)$ is a roof over $\alpha$, then, by Lemma 4.1, the exterior angle of intersection $\theta$ of $P$ and $Q$ is an upper bound for $i(\alpha,\beta_\Gamma)_P^Q$. Therefore $i(\alpha,\beta_\Gamma)_P^Q \leq \theta \leq \pi$.

Otherwise, we let $t_1$ be the smallest value of $t > 0$ such that $(P, P_t)$ is not a roof over $\alpha([0,s(t)])$. We let $s(t_1) = s_1$ and $\alpha_1 = \alpha|_{[0,s_1]}$. Then, $(P_0, P_{t_1})$ is a $\pi$-roof over $\alpha_1$ and so, by Corollary 4.2, $i(\alpha_1,\beta_\Gamma)_P^{P_{t_1}} \leq \pi$. If $(P_{t_1}, Q)$ is a roof over $\alpha([s_1,1])$, we let $\alpha_2 = \alpha|_{[s_1,1]}$. Therefore, the exterior angle of intersection $\theta_1$ of $P_{t_1}$ and $Q$ is an upper bound for $i(\alpha_2,\beta_\Gamma)_{P_{t_1}}^Q$. Thus we have

$$i(\alpha,\beta_\Gamma)_P^Q = i(\alpha_1,\beta_\Gamma)_P^{P_{t_1}} + i(\alpha_2,\beta_\Gamma)_{P_{t_1}}^Q \leq \pi + \theta_1 \leq 2\pi.$$

In the final case we let $t_2$ be the smallest value of $t \in [t_1,k]$ such that $(P_{t_1}, P_t)$ is not a roof over $\alpha([s_1,s(t)])$, and we let $s(t_2) = s_2$. If $s_2 = 1$, then $(P_{t_1}, Q)$ is a $\pi$-roof over $\alpha_2 = \alpha([s_1,1])$. Therefore $i(\alpha,\beta_\Gamma)_P^Q \leq 2\pi$ as above. Otherwise, let $\tilde{l} = l(\alpha([0,s_2]))$. Then $\tilde{l} < l(\alpha)$. As $G(x) < G_\infty = 2\sinh^{-1}(1)$, we have $\tilde{l} < 2\sinh^{-1} 1$.

The support planes $P_0$, $P_{t_1}$, and $P_{t_2}$ have the configuration described in Lemma 3.2. Also the curve $\alpha : [0,s_2] \to \partial CH(L_\Gamma)$ has one endpoint on $P_0$, the other on $P_{t_2}$, an interior point on $P_{t_1}$ and length $\tilde{l} < 2\sinh^{-1} 1$. Lemma 3.2 gives that $P_0$ and $P_{t_2}$ intersect and have angle of intersection $\theta$ satisfying

$$\theta \geq 2\cos^{-1}(\sinh(\tilde{l}/2)) > 0.$$

We join the endpoints $\alpha(0)$ and $\alpha(s_2)$ by the shortest curve on $P_0 \cup P_{t_2}$. This curve consists of two geodesic segments, the first on $P_0$ from $\alpha(0)$ to a point $V \in P_0 \cap P_{t_2}$, the second on $P_{t_2}$, from $V$ to $\alpha(s_2)$. We consider the triangle $T$ in $\mathbf{H}^3$ with vertices $\alpha(0)$, $\alpha(s_2)$, and $V$. The angle $\theta_V$ at $V$ satisfies $\theta_V \geq \theta$. Also, the opposite side joining $\alpha(0)$ to $\alpha(s_2)$ has length $l_V \leq \tilde{l}$. Therefore we have that the triangle $T$ has an angle bounded below by $\theta$ and opposite side bounded above by $\tilde{l}$. Let $\nu$ be the union of the 2 sides of $T$ lying in $P_0 \cup P_{t_2}$. Lemma 3.1 implies that

$$l(\nu) \leq 2\sinh^{-1}\left(\frac{\sinh(\tilde{l}/2)}{\sin(\theta/2)}\right).$$

Applying the bound for $\theta$ we obtain

$$l(\nu) \leq 2\sinh^{-1}\left(\frac{\sinh(\tilde{l}/2)}{\sqrt{1-\sinh^2(\tilde{l}/2)}}\right)$$

We obtain a closed curve $\eta$ by taking the geodesic segment $\alpha([0,s_2])$ along with $\nu$. Then,

$$l(\eta) \leq \tilde{l} + 2\sinh^{-1}\left(\frac{\sinh(\tilde{l}/2)}{\sqrt{1-\sinh^2(\tilde{l}/2)}}\right) = 2F(\tilde{l})$$

Let $\gamma = \widetilde{r}(\eta)$ where $\widetilde{r}$ is the nearest point retraction. Therefore $\gamma$ is the union of $\alpha([0,s_2])$ and $g = \widetilde{r}(\nu)$. In particular, $l(\gamma) \leq l(\eta) \leq 2F(\tilde{l})$.

If $\alpha$ is in a simply connected component of $\partial CH(L_\Gamma)$, then $\gamma$ is in a simply connected component, so $\gamma$ must be homotopically trivial.

If $\alpha$ is in a non-simply connected component, then $\tilde{l} < l(\alpha) \leq G(\widehat{\rho}(\alpha(0)))$. Therefore, by monotonicity of $F$, $F(\tilde{l}) < \widehat{\rho}(\alpha(0))$ and $l(\gamma) \leq 2F(\tilde{l}) < 2\widehat{\rho}(\alpha(0))$. As $\gamma$ contains the point $\alpha(0)$ and $l(\gamma) < 2\widehat{\rho}(\alpha(0))$, $\gamma$ is homotopically trivial in $\partial CH(L_\Gamma)$.

We now obtain a contradiction by showing that $\gamma$ is not homotopically trivial in $\partial CH(L_\Gamma)$. If $\gamma$ is homotopically trivial then it bounds a disk on $\partial CH(L_\Gamma)$. We now show that the curve $\gamma$ crosses a certain bending line exactly once. As bending lines are unbounded, they cannot have an end contained in a closed disk. This gives the desired contradiction.

We first consider a special case. Let either $P_0 = P_{t_1}$ or $P_{t_2} = P_{t_1}$. If $P_0 = P_{t_1}$, then $(P_0, P_{t_1})$ form a $\pi-$roof, so the associated half spaces $H_0$ and $H_{t_1}$ have disjoint interiors. Therefore the convex hull $CH(L_\Gamma)$ is contained in $P_{t_1}$ and we think of $\partial CH(L_\Gamma)$ as a two-sided planar region. As $(P_{t_1}, P_{t_2})$ also form a $\pi-$roof, the half spaces $H_{t_1}$ and $H_{t_2}$ have disjoint interiors, so $P_{t_2} = P_0$ and $H_{t_2} = H_0$. Therefore, $\gamma$ intersects exactly two bending lines once each and thus is homotopically non-trivial. The case $P_{t_2} = P_{t_1}$ is similar.

We now consider the case when neither $P_0$ nor $P_{t_2}$ is equal to $P_{t_1}$. Let $b_1$ be the first bending line on $P_{t_1}$ that the curve $\alpha([0,s_2])$ intersects and let $\alpha(\overline{s})$ be this first point of intersection. Since $\alpha([0,s_2])$ is a geodesic on $\partial CH(L_\Gamma)$ whose endpoints do not lie on $b_1$, it must intersect $b_1$ transversely at every point of intersection. We will call any transverse intersection a crossing. On the other hand, $g = \widetilde{r}(\nu)$ need not intersect $b_1$ transversely. We will say that $g$ crosses $b_1$ if there is an open subarc $g'$ of $g$ and a closed connected subarc $g_1$ of $g'$ such that $g_1 = g' \cap b_1$ and the two components of $g' - g_1$ lie on opposite sides of $b_1$. (Notice that this is well-defined if $g'$ lies in a small enough neighborhood of $b_1$.) We will show that $\alpha([\overline{s}, s_2])$ intersects $b_1$ exactly once and that $g$ does not cross $b_1$ at all, which will suffice to establish the contradiction.

Suppose that $\alpha([\overline{s}, s_2])$ has a second intersection point with $b_1$ at a point $x$. Since $P_{t_1} \cap P_{t_2} = \emptyset$, $x = \alpha(\tilde{s})$ where $\overline{s} < \tilde{s} < s_2$. Let $\tilde{t}$ be such that $s(\tilde{t}) = \tilde{s}$. Then by definition $\overline{t} < \tilde{t} < t_2$ and $x \in P_{\tilde{t}}$, so $(P_{t_1}, P_{\tilde{t}})$ form a roof over $\alpha([\overline{s}, \tilde{s}])$. If $P_{t_1} = P_{\tilde{t}}$ then, by Lemma 4.1, $P_t = P_{t_1}$ for all $t \in [t_1, \tilde{t}]$. Therefore, $\alpha([\overline{s}, \tilde{s}])$ is a geodesic arc contained in $P_{t_1}$ with two endpoints on the geodesic $b_1$. Thus, $\alpha([\overline{s}, \tilde{s}]) \subseteq b_1$ which contradicts the fact that $\alpha([\overline{s}, s_2])$ intersects $b_1$ transversely.

If $P_{t_1} \neq P_{\tilde{t}}$, then let $r_{\tilde{t}}$ be the ridge line $P_{t_1} \cap P_{\tilde{t}}$. By Lemma 1.9.2 in Epstein-Marden [6] (stated in the appendix as Lemma 10.2) if a ridge line intersects a bending line then they are equal. Since $x \in r_{\tilde{t}} \cap b_1$, we have $r_{\tilde{t}} = b_1$. By the monotonicity of the ridge lines we have that for each $t \in [t_1, \tilde{t}]$ either $P_t = P_{t_1}$ or $r_t = P_t \cap P_{t_1} = b_1$. Thus for all $t \in [t_1, \tilde{t}]$, $P_t$ is a support plane to $b_1$. If $b_1$ has a unique support plane, then $P_t = P_{t_1}$ for all $t \in [t_1, \tilde{t}]$ and this reduces to the above case. If $b_1$ has more than one support plane then we let $X$ and $Y$ be the extreme support planes at $b_1$. If $Z$ is another support plane for $b_1$, then $Z \cap \partial CH(L_\Gamma) = b_1$. As the only points of $\alpha([\overline{s}, \tilde{s}])$ in $b_1$ are the endpoints the only possible support plane for any point in the open arc $\alpha((\overline{s}, \tilde{s}))$ is either $X$ or $Y$. Since $\alpha((\overline{s}, \tilde{s}))$ is connected and $X \cap Y = b_1$, either $\alpha((\overline{s}, \tilde{s})) \subseteq X$ or $\alpha((\overline{s}, \tilde{s})) \subseteq Y$. We can assume $\alpha((\overline{s}, \tilde{s})) \subseteq X$. As the endpoints of $\alpha((\overline{s}, \tilde{s}))$ are in $b_1$, the geodesic arc $\alpha([\overline{s}, \tilde{s}])$ is in $X$ and intersects the geodesic $b_1$ at its endpoints. Therefore, $\alpha([\overline{s}, \tilde{s}]) \subseteq b_1$ which again contradicts the fact that $\alpha([\overline{s}, s_2])$ intersects $b_1$ transversely. Thus, we have established that $\alpha([\overline{s}, s_2])$ intersects $b_1$ exactly once.

We next argue that $g$ does not cross $b_1$. The following lemma will be the key technical tool in the argument.

**Lemma 4.4** *Let $c$ be a continuous curve in the complement of the convex hull $CH(L_\Gamma)$ such that $\widetilde{r}(c)$ crosses a bending line $b$. If $Z$ is any support plane to $b$, then there exists $z \in c$ such that the geodesic segment $z\widetilde{r}(z)$ is orthogonal to $Z$ and $z\widetilde{r}(z) \subset H_Z$.*

*Proof of 4.4:* First notice that if $x \in c$ and $\widetilde{r}(x)$ lies in the interior of a flat contained in a support plane $X$, then $x\widetilde{r}(x)$ is orthogonal to $X$ and $x\widetilde{r}(x) \subset H_X$. If $\widetilde{r}(x)$ lies in a bending line $b$ and $b$ has a unique support plane $X$ then, by continuity, $x\widetilde{r}(x)$ is orthogonal to $X$ and $x\widetilde{r}(x) \subset H_X$. This completes the proof of the lemma in this case.

If $\widetilde{r}(x)$ lies in a bending line $b$ and $b$ has more than one support plane then let $X$ and $Y$ be the extreme support planes at $b$. If $\widetilde{r}(c)$ crosses $b$ then $c$ contains an open subarc $c'$ and a closed connected subarc $c_1$ such that $\widetilde{r}^{-1}(b) \cap c' = c_1$ and the two components of $\widetilde{r}(c' - c_1)$ lie on opposite sides of $b$. By continuity, $c_1$ has endpoints $x$ and $y$ such that $x\widetilde{r}(x)$ is orthogonal to $X$, $x\widetilde{r}(x) \subset H_X$, $y\widetilde{r}(y)$ is orthogonal to $Y$, and $y\widetilde{r}(y) \subset H_Y$.

Let $\theta_1$ and $\theta_2$ be the exterior dihedral angles that $Z$ makes with the extreme support planes $X$ and $Y$. As $X$ and $Y$ are extreme support planes, $Z \subset H_X \cup H_Y$. Therefore $b$ splits $Z$ into pieces $Z_-$ and $Z_+$ where $Z_- \subset H_X$ and $Z_+ \subset H_Y$. If $z \in c_1$ then $z\widetilde{r}(z)$ is perpendicular to $b$ and we define $\theta(z)$ to be the angle $z\widetilde{r}(z)$ makes with $Z$ as measured from the side $Z_-$ with $\theta(z)$ positive for $z \in H_Z$. As $x\widetilde{r}(x)$ is orthogonal to $X$ and contained in $H_X$, $\theta(x) = \pi/2 - \theta_1$. Similarly, $\theta(y) = \pi/2 + \theta_2$. Thus, since $\pi/2 - \theta_1 \leq \pi/2 \leq \pi/2 + \theta_2$ and $\theta$ is continuous on $c_1$, there is a $z \in c_1$ such that

$\theta(z) = \pi/2$. Therefore $z\widetilde{r}(z)$ is orthogonal to $Z$ and $z\widetilde{r}(z) \subset H_Z$, so we have found the desired point.

$\boxed{4.4}$

Suppose that $g$ crosses $b_1$. Then, by Lemma 4.4, there is an $x \in \nu$ such that $x\widetilde{r}(x)$ is orthogonal to $P_{t_1}$ and $x\widetilde{r}(x) \subset H_{P_{t_1}}$. Either $x$ is in $P_0$ or $P_{t_2}$. Assume $x \in P_0$. Then $x \in H_{P_{t_1}} \cap P_0$. As $(P_0, P_{t_1})$ form a $\pi-$roof, $H_0$ and $H_{P_{t_1}}$ have disjoint interiors and either $P_0 = P_{t_1}$ or $P_0$ and $P_{t_1}$ are disjoint. We have already ruled out the case in which $P_{t_1} = P_0$, so $H_{t_1} \cap P_0 = \emptyset$. Thus we obtain a contradiction and it must be the case that no point of $g$ crosses $b_1$. The case where $x \in P_{t_2}$ is similar.

Therefore $\gamma$ crosses $b_1$ exactly once and is thus homotopically non-trivial. This gives the necessary contradiction.

$\boxed{4.3}$

## 5 Global intersection number estimates

The proofs of Theorems 1 and 2 rely heavily on the following global estimate on intersection numbers. Moreover, Theorem 3 is an immediate corollary.

**Proposition 5.1** *Suppose that $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold and $\alpha$ is a closed geodesic on $\partial C(N)$.*

1. *If $\widehat{\rho}_0$ is a lower bound for $inj_{\partial CH(L_\Gamma)}$ at any point in the support of a lift $\tilde{\alpha}$ of $\alpha$, then*
$$i(\alpha, \beta_N) \leq K(\widehat{\rho}_0) \cdot l_{\partial C(N)}(\alpha)$$
*where $l_{\partial C(N)}$ is the hyperbolic length of $\alpha$ on $\partial C(N)$.*

2. *If $\alpha$ is contained in an incompressible component of $\partial C(N)$, then*
$$i(\alpha, \beta_N) \leq K_\infty l_{\partial C(N)}(\alpha).$$

We recall that $K_\infty = \frac{\pi}{\sinh^{-1}(1)} \approx 3.56443$ and that $K(x) \asymp \frac{2\pi}{x}$ as $x \mapsto 0$.

*Proof of 5.1:* Let $\alpha : S^1 \to \partial C(N)$ be a closed geodesic on $\partial C(N)$. Let $\tilde{\alpha} : \mathbf{R} \to \partial CH(L_\Gamma)$ be a lift of $\alpha$ to $\partial CH(L_\Gamma)$ such that $\alpha(0)$ lies in a flat.

If we let $\tilde{\alpha}_n$ be the restriction of $\tilde{\alpha}$ to the interval $[0, n]$ then $i(\tilde{\alpha}_n, \beta_\Gamma) = n \cdot i(\alpha, \beta_N)$ and $l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n) = n \cdot l_{\partial C(N)}(\alpha)$.

Let $\tilde{\alpha}$ be in the connected component $C$ of $\partial CH(L_\Gamma)$. If $C$ is simply connected we let $G = G_\infty$. Otherwise we let $G = G(\widehat{\rho}_0)$. We subdivide $\tilde{\alpha}_n$ into $m$ subarcs of length less than or equal to $G$ where $m$ is given by

$$m = \left[\frac{l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n)}{G}\right]^+ \leq \frac{l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n)}{G} + 1$$

where $[x]^+$ is the least integer greater than or equal to $x$.

We let $\tilde{\alpha}_n^j$ be the subarcs, where $\tilde{\alpha}_n^j$ is restriction of $\tilde{\alpha}_n$ to the interval $[t_{j-1}, t_j]$ and $0 = t_0 < t_1 < \ldots < t_m = n$.

We define support planes $P_j$ at $\tilde{\alpha}_n(t_j)$ inductively. First, we let $P_0 = P$ where $P$ is the unique support plane to $\tilde{\alpha}_n(0)$. If $P_{j-1}$ is defined, then it is a support plane to $\tilde{\alpha}_n^j(t_{j-1}) = \tilde{\alpha}_n(t_{j-1})$. As the length of $\tilde{\alpha}_n^j$ is less than or equal to $G$, by Lemma 4.3, there is a support plane $P_j$ at $\tilde{\alpha}_n^j(t_j) = \tilde{\alpha}_n(t_j)$ such that $i(\tilde{\alpha}_n^j, \beta_\Gamma)_{P_{j-1}}^{P_j} \leq 2\pi$. As $\tilde{\alpha}_n(n)$ is in a flat, $P_m$ must be the unique support plane at $\tilde{\alpha}_n(n)$. Therefore, by additivity, we have

$$i(\tilde{\alpha}_n, \beta_\Gamma) = i(\tilde{\alpha}_n, \beta_\Gamma)_{P_0}^{P_m} = \sum_{j=1}^{m} i(\tilde{\alpha}_n^j, \beta_\Gamma)_{P_{j-1}}^{P_j}$$

As $i(\tilde{\alpha}_n^j, \beta_\Gamma)_{P_{j-1}}^{P_j} \leq 2\pi$ we obtain $i(\tilde{\alpha}_n, \beta_\Gamma) \leq 2\pi m$. Substituting the upper bound for $m$ we get

$$i(\tilde{\alpha}_n, \beta_\Gamma) \leq 2\pi \left(\frac{l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n)}{G} + 1\right)$$

Rewriting in terms of $\alpha$ we get

$$n\ i(\alpha, \beta_N) \leq \frac{2\pi n \cdot l_{\partial C(N)}(\alpha)}{G} + 2\pi$$

Dividing through by $n$ we get

$$i(\alpha, \beta_N) \leq \frac{2\pi l_{\partial C(N)}(\alpha)}{G} + \frac{2\pi}{n}$$

As this holds for all $n$ we have that

$$i(\alpha, \beta_N) \leq \frac{2\pi l_{\partial C(N)}(\alpha)}{G} = K\ l_{\partial C(N)}(\alpha)$$

where $K$ equals either $K_\infty$ or $K(\widehat{\rho}_0)$ depending on whether $\alpha$ is contained in an incompressible component of $\partial C(N)$ or not.

$\boxed{5.1}$

In the previous proposition the inequality obtained was linear. Then, since closed geodesics are dense in $\mathcal{C}(\partial C(N))$ and the length and intersection functions are continuous, we get the following generalized version of Theorem 3.

**Theorem 3:** *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $\alpha \in \mathcal{C}(\partial C(N))$ be a geodesic current in the boundary of the convex core of $N$.*

1. *If $\partial C(N)$ is incompressible, then $B(\alpha) \leq K_\infty$.*
2. *If $\partial C(N)$ is compressible and $\widehat{\rho}_\alpha$ is a lower bound for the injectivity radius of the boundary of the convex hull of the limit set at any point in the support of $\alpha$, then $B(\alpha) \leq K(\widehat{\rho}_\alpha)$.*

# 6 A homotopy inverse for the nearest point retraction

We may now combine Proposition 5.1 with earlier work of Thurston to obtain a Lipschitz homotopy inverse to the nearest point retraction. One should note that the bounds on the Lipschitz constant of the homotopy inverse depend on the injectivity radius of the boundary of the convex hull of the limit set. We will see later how to obtain a lower bound on the injectivity radius of $\partial CH(L_\Gamma)$ from a lower bound on the injectivity radius of $\Omega(\Gamma)$.

**Proposition 6.1** *Let $N$ be an analytically finite hyperbolic 3-manifold. If $\partial C(N)$ is compressible and $\widehat{\rho}_0$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$, then the nearest point retraction $r$ has a homotopy inverse that is $(1 + K(\widehat{\rho}_0))$ Lipschitz. If $\partial C(N)$ is incompressible, then the homotopy inverse is $(1 + K_\infty)$-Lipschitz.*

*Proof of 6.1:* Let $s : \partial C(N) \longrightarrow \partial_c(N)$ be a homotopy inverse to the nearest point retraction $r$. Let $K$ denote $K_\infty$ if $\partial C(N)$ is incompressible and $K(\widehat{\rho}_0)$ otherwise.

Let $\alpha$ be a simple closed geodesic in $\partial C(N)$ with length $l_{\partial C(N)}(\alpha)$ and let $l_{\partial_c N}(s(\alpha)^*)$ be the length of the geodesic representative of $s(\alpha)$ in $\partial_c N$. McMullen (Theorem 3.1 in [7]) showed that

$$l_{\partial_c N}(s(\alpha)^*) \leq l_{\partial C(N)}(\alpha) + i(\alpha, \beta_\Gamma)$$

Using Proposition 5.1 we get that

$$l_{\partial_c N}(s(\alpha)^*) \leq (1 + K) l_{\partial C(N)}(\alpha)$$

In [9], Thurston proved that if $f : X \rightarrow Y$ is a map between two finite area hyperbolic surfaces and

$$\frac{l_Y(f(\beta)^*)}{l_X(\beta)} \leq M$$

for any simple closed geodesic $\beta$ on $X$, then $f$ is homotopic to a $M$-Lipschitz map. Thus, we may conclude in our case that $s$ is homotopic to a $(1+K)$-Lipschitz map from $\partial C(N)$ to $\partial_c N$ as claimed.

$\boxed{6.1}$

The following proposition indicates that one cannot improve much on the bounds obtained in Proposition 6.1. Recall that $K(\widehat{\rho}_0) \asymp \frac{2\pi}{\widehat{\rho_0}}$ as $\widehat{\rho}_0 \mapsto 0$.

**Proposition 6.2** *There exist positive constants $C$ and $L$ such that if $N$ is a hyperbolic 3-manifold, there is a compressible closed geodesic $\gamma$ on $\partial C(N)$ with length $l_0 < L$ and $s : \partial C(N) \to \partial_c N$ is a $K$-Lipschitz homotopy inverse to the nearest point retraction, then*

$$K \geq \frac{C}{-l_0 \log(l_0)}.$$

*Proof of 6.2:* Let $l$ denote the length of $s(\gamma)^*$ in $\partial_c N$. Theorem 5.1 in [5] implies that if $l < 1$ then

$$l \geq \frac{\pi^2}{\sqrt{e}\log(\frac{4\pi e^{(.502)\pi}}{l_0})}.$$

If $\log(l_0) \leq -2\log(4\pi e^{(.502)\pi})$, then

$$l \geq \frac{2\pi^2}{-\sqrt{e}\log(l_0)}.$$

So choose $L = \frac{1}{(4\pi e^{(.502)\pi})^2}$ and $C = \frac{2\pi^2}{\sqrt{e}}$.

Thus, if $l_0 \leq L$, then

$$K \geq \frac{l}{l_0} \geq \frac{C}{-l_0 \log(l_0)}.$$

(Notice that this claim is trivial in the remaining case that $l \geq 1$.)

$\boxed{6.2}$

In particular, notice that this shows that if $\partial C(N)$ contains arbitrarily short compressible curves, then there is no Lipschitz map from the convex core to the conformal boundary.

# 7 The nearest point retraction is Lipschitz

In this section we will show how to combine the techniques in section 2.3 of Epstein-Marden [6] and the results of [5] to show that the nearest point retraction is itself Lipschitz (and to produce bounds on the Lipschitz constant.) We remark that Epstein and Marden showed that the nearest point retraction is 4-Lipschitz if $\partial C(N)$ is incompressible.

**Proposition 7.1** *If* $N = \mathbf{H}^3/\Gamma$ *is an analytically finite hyperbolic 3-manifold and* $\rho_0$ *is a lower bound for the injectivity radius of* $\Omega(\Gamma)$*, then the nearest point retraction* $r : \partial_c N \to \partial C(N)$ *is* $J(\rho_0)$*-Lipschitz where*

$$J(\rho_0) = 2\sqrt{2}\left(k + \frac{\pi^2}{2\rho_0}\right)$$

*and* $k = 4 + \log(3 + 2\sqrt{2}) \approx 5.763$*.*

*Proof of 7.1:* Let $K = \sqrt{2}\left(k + \frac{\pi^2}{2\rho_0}\right)$. We will show that given any point $z \in \Omega(\Gamma)$ and any $\delta \in (0,1)$ there exists a neighborhood of $z$ on which $\tilde{r}$ is $2K\left(\frac{1+\delta}{1-\delta^2}\right)$-Lipschitz. It follows that $\tilde{r}$ is itself $2K\left(\frac{1+\delta}{1-\delta^2}\right)$-Lipschitz. Since $\delta$ can be chosen to be arbitrarily close to 0, it follows that $\tilde{r}$ (and hence $r$) is $2K$-Lipschitz as claimed.

Let $z \in \Omega(\Gamma)$ and let $P$ be the support plane to $\tilde{r}(z)$ which is orthogonal to $z\tilde{r}(z)$. We can always find a neighborhood $U$ of $z$ such that if $w \in U$ and $Q$ is the support plane to $\tilde{r}(w)$ which is orthogonal to $w\tilde{r}(w)$, then $P$ intersects $Q$. Given $\delta \in (0,1)$, we may further restrict $U$ so that it is is contained in the ball of radius $\frac{\delta}{4K}$ about $z$ in the Poincaré metric and that any point $w \in U$ may be joined to $z$ by a unique geodesic in $U$ of length $d_\Omega(z,w)$.

Let $w \in U$, let $Q$ be the support plane to $\tilde{r}(w)$ which is orthogonal to $w\tilde{r}(w)$, and let $g$ be the geodesic in $U$ joining $z$ to $w$. We may normalize so that $z = 0$, the unit circle is the boundary of the support plane $P$ and $\infty \in L_\Gamma$. It is shown, in the proof of Proposition 4.1 of [5], that if $p_\Omega(z)dz$ denotes the Poincaré metric on $\Omega(\Gamma)$ then $p_\Omega(z) \geq \frac{1}{Kd(z,L_\Gamma)}$ for all $z \in \Omega(\Gamma)$ where $d(z, L_\Gamma)$ denotes the Euclidean distance from $z$ to the limit set $L_\Gamma$.

Let $D$ be the unit disk and let $D_Q$ be the disk bounded by $\partial Q$. If we let $p_D(z)dz$ denote the Poincaré metric on $D$, then $\frac{p_D(z)}{p_\Omega(z)} \leq \frac{2Kd(z,L_\Gamma)}{1-|z|^2}$ for all $z \in D$.

Since $g$ has length at most $\frac{\delta}{4K}$, our inequality on the Poincaré metric implies that $g$ is contained in the ball of Euclidean radius $\delta$ about 0. In particular, if $z \in g$, then $\frac{p_D(z)}{p_\Omega(z)} \leq 2K\left(\frac{1+\delta}{1-\delta^2}\right)$. We divide $g$ up into 3 segments: $g_1 = g \cap (P - Q)$, $g_2 = g \cap (P \cap Q)$ and $g_3 = g \cap Q - P$. The inequality above implies that $l_D(g_1) \leq 2K\left(\frac{1+\delta}{1-\delta^2}\right) l_\Omega(g_1)$

where $l_D(g_1)$ denotes the length of $g_1$ in the Poincaré metric on $D$ and $l_\Omega(g_1)$ denotes the length of $g_1$ in the Poincaré metric on $\Omega(\Gamma)$. Similarly, $l_D(g_2) \leq 2K\left(\frac{1+\delta}{1-\delta^2}\right) l_\Omega(g_2)$ and $l_{D_Q}(g_3) \leq 2K\left(\frac{1+\delta}{1-\delta^2}\right) l_\Omega(g_3)$.

Let $\Omega' = D \cup D_Q$ and let $r' : \Omega' \to CH(\partial\Omega')$ be the nearest point retraction. Notice that $r'(0) = r(0)$ and $r'(w) = r(w)$. Let $r_D : D \to P$, $r_Q : D_Q \to Q$ and $r_L : P \cap Q \to L$ be the nearest point retractions. Then $r'|_{P-Q} = r_D$, $r'|_{Q-P} = r_Q$ and $r'|_{P\cap Q} = r_L$. Notice that $r_D$ and $r_Q$ are isometries with respect to the Poincaré metrics on $P$ and $Q$ and that $r_L$ is 1-Lipschitz with respect to the Poincaré metric on either $P$ or $Q$. It follows that

$$l_{\mathbf{H}^3}(r'(g)) \leq l_D(g_1) + l_D(g_2) + l_{D_Q}(g_3) \leq 2K\left(\frac{1+\delta}{1-\delta^2}\right) l_\Omega(g).$$

We recall that $\tilde{r} : \Omega(\Gamma) \to \partial CH(L_\Gamma)$ extends to $\tilde{r} : \mathbf{H}^3 \cup \Omega(\Gamma) \to CH(L_\Gamma)$. Then $\tilde{r}(r'(g))$ is a path joining $r(0)$ to $r(w)$ of length at most $2K\left(\frac{1+\delta}{1-\delta^2}\right) l_\Omega(g)$ (since $\tilde{r}$ is distance decreasing on $\mathbf{H}^3$). It follows that

$$d_{\partial CH(L_\Gamma)}(\tilde{r}(w), \tilde{r}(z)) \leq 2K\left(\frac{1+\delta}{1-\delta^2}\right) d_\Omega(z,w).$$

Hence, $\tilde{r}$ is $2K\left(\frac{1+\delta}{1-\delta^2}\right)$-Lipschitz on $U$ as required and we have completed the proof.

$\boxed{7.1}$

**Remarks:** (1) Epstein and Marden [6] showed that the nearest point retraction $r$ is 4-Lipschitz if $\partial C(N)$ is incompressible. In [5] it is shown that $r$ is homotopic to a $2\sqrt{2}$-Lipschitz map if $\partial C(N)$ is incompressible and to a $\sqrt{2}K$-Lipschitz map if not.

(2) In section 6 of [5], Canary constructs an infinite sequence of hyperbolic manifolds $\{N_n\}$ such that, for all large enough $n$, the shortest geodesic in $\partial_c N$ has length $\frac{1}{n}$ and the shortest geodesic in $\partial C(N)$ has length at most $\frac{4\pi}{e^{\pi(2n-1)}}$ and the nearest point retraction is not even homotopic to a map which is $\frac{5n}{2\log(5n)}$-Lipschitz. Hence, we cannot improve substantively on the form of the estimate obtained above.

## 8 The proof of Theorem 1

The only issue remaining in the proof of Theorem 1 is that the bound on the Lipschitz constant in Proposition 6.1 depends on an injectivity radius bound in the boundary of the convex hull, while the assumptions of Theorem 1 only give us an injectivity radius bound on the domain of discontinuity. The following lemma guarantees that injectivity radius bounds on the domain of discontinuity give us injectivity radius bound in the boundary of the convex hull.

**Lemma 8.1** *Let* $N = \mathbf{H}^3/\Gamma$ *be a hyperbolic 3-manifold and let* $\alpha$ *be a geodesic on* $\partial CH(L_\Gamma)$ *with length* $l(\alpha) < e^{-m} \approx .06798$ *(where* $m = \cosh^{-1}(e^2) \approx 2.68854$*), then*

$$l_{\partial_c N}(\widetilde{s}(\alpha)^*) \leq \frac{-\pi^2}{m + \log(l(\alpha))}$$

*where* $\widetilde{s} : \partial CH(L_\Gamma) \to \partial_c N$ *is a lift of a homotopy inverse to* $r : \partial_c N \to \partial C(N)$*.*

*Proof of 8.1:* There is a collar neighborhood $C$ of $\alpha$ on $\partial CH(L_\Gamma)$ which is isometric to $[-w, w] \times S^1$ with the metric $ds^2 = dr^2 + l(\alpha)^2 \cosh^2 r dt^2$ where $\alpha$ is identified with $\{0\} \times S^1$ and $w = \sinh^{-1}\left(\frac{1}{\sinh(l/2)}\right)$ (see Theorem 4.1.1 in [4].) Let $\alpha_1$ and $\alpha_2$ denote the boundary components of $C$. Then

$$l(\alpha_1) = l(\alpha_2) = l(\alpha)\cosh\left(\sinh^{-1}\left(\frac{1}{\sinh(l(\alpha)/2)}\right)\right) = l(\alpha)\coth\left(\frac{l(\alpha)}{2}\right) \leq 4.$$

(The last inequality follows since $l(\alpha)\coth\left(\frac{l(\alpha)}{2}\right)$ is an increasing function and $l(\alpha) < 1$.) We normalize the situation so that $\alpha$ passes through the origin, the origin lies on a bending line $L$ for $\partial CH(L_\gamma)$ and that $L$ is the $z$-axis in the Poincaré ball model for $\mathbf{H}^3$.

Let $\beta_1 = \widetilde{r}^{-1}(\alpha_1)$ and $\beta_2 = \widetilde{r}^{-1}(\alpha_2)$ be the set-theoretic pre-images of the curves $\alpha_1$ and $\alpha_2$ under $\widetilde{r}$. Then, $\beta_1$ and $\beta_2$ are homotopic simple closed curves in $\Omega(\Gamma)$. Our goal is to prove that $\beta_1$ and $\beta_2$ bound a "large" modulus annulus in $\Omega(\Gamma)$ and hence that the core curve of this annulus is "short." Since $r$ is a homotopy inverse to $s$ the core curve of the annulus is homotopic to $s(\alpha)$.

Notice that $L$ must pass through $C$ and intersects both $\alpha_1$ and $\alpha_2$ transversely at points, $x_1$ and $x_2$, and that

$$d(x_i, 0) \geq \sinh^{-1}\left(\frac{1}{\sinh(l(\alpha)/2))}\right) \geq \sinh^{-1}\left(\frac{1}{l(\alpha)}\right) \geq -\log(l(\alpha)).$$

(The middle inequality follows from the facts that $\sinh^{-1}$ is an increasing function and that $\sinh(x) \leq 2x$ if $x \leq 1$.)

Let $r_L : \Omega(\Gamma) \to L$ denote the nearest point projection onto $L$. One may calculate that if $x \in L$, $d(x, y) \leq 2$ and the family of horoballs about a point $z \in \Omega(\Gamma)$ hits $y$ before it hits $L$, then $d(x, r_L(z)) \leq \cosh^{-1}(e^2)$. Let $m = \cosh^{-1}(e^2)$. Let $L_0$ be the portion of $L$ joining $x_1$ to $x_2$ and let $L_m$ denote the portion of $L_0$ which is a distance more than $m$ from both $x_1$ and $x_2$. Let $A_m = \pi_L^{-1}(L_m)$. (Notice that since $l(\alpha) < e^{-m}$, $L_m$ and $A_m$ are non-empty.) Since $\beta_i = r^{-1}(\alpha_i)$ and $d(y, x_i) \leq 2$ for all $y \in \alpha_i$, $\beta_1$ and $\beta_2$ lie in opposite components of $\widehat{\mathbf{C}} - A_m$. Therefore, since $\beta_1$ and $\beta_2$ are homotopic in $\Omega(\Gamma)$, $A_m \subset \Omega(\Gamma)$.

It is an easy calculation to check that $\mathrm{mod}(A_m) \geq \frac{m+\log(l(\alpha))}{-\pi}$ where $\mathrm{mod}(A_m)$ is the conformal modulus of $A_m$. If $\alpha'$ is the core curve of $A_m$, then, see for example Theorem 2.6 in [6], $\alpha'$ has length at most $\frac{-\pi^2}{m+\log(l(\alpha))}$ in the Poincaré metric on $A_m$ and hence in the Poincaré metric on $\Omega(\Gamma)$. Since $\alpha'$ is homotopic to $s(\alpha)$, we see that

$$l_{\partial_c N}(s(\alpha)^*) \leq \frac{-\pi^2}{m+\log(l(\alpha))}.$$

$\boxed{8.1}$

In particular, lemma 9.1 guarantees that if $\rho_0$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$, then $g(\rho_0)$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$ where

$$g(\rho_0) = \frac{e^{-m}e^{\frac{-\pi^2}{2\rho_0}}}{2}.$$

If we define $L(\rho_0) = 1 + K(g(\rho_0))$, then we may combine Corollary 6.1 and Proposition 7.1 to obtain the following, slightly more general, version of Theorem 1:

**Theorem 1:** *If $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold and $\rho_0$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$, then the nearest point retraction $r : \partial_c(N) \rightarrow \partial C(N)$ is $J(\rho_0)$-Lipschitz and has a $L(\rho_0)$-Lipschitz homotopy inverse.*

The following slightly more general version of Corollary 1 is an almost immediate corollary of Theorem 1.

**Corollary 1:** *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $\rho_0$ be a lower bound for the injectivity radius of $\Omega(\Gamma)$. If $\alpha$ is a geodesic current in $\partial_c N$ and $r(\alpha)^*$ denotes the geodesic current in $\partial C(N)$ which is homotopic to $r(\alpha)$, then*

$$\frac{l_{\partial C(N)}(r(\alpha)^*)}{J(\rho_0)} \leq l_{\partial_c(N)}(\alpha) \leq L(\rho_0) l_{\partial C(N)}(r(\alpha)^*)$$

*where $l_{\partial C(N)}(r(\alpha)^*)$ denotes the length of $r(\alpha)^*$ in $\partial C(N)$ and $l_{\partial_c(N)}(\alpha)$ denotes the length of $\alpha$ in $\partial_c(N)$.*

*Proof of Corollary 1:* We note that the bounds follow immediately from Theorem 1 when $\alpha$ is a closed geodesic. But we recall from Bonahon [2] that closed geodesics are dense in the space of geodesic currents, length is a continuous function on the space of currents on a surface, and that $r_* : \mathcal{C}(\partial_c(N)) \rightarrow \mathcal{C}(\partial C(N))$ is continuous. The general result then follows.

$\boxed{\text{Corollary 1}}$

Theorem 2 follows immediately from Proposition 6.1 and Epstein and Marden's result that the nearest point retraction is 4-Lipschitz when each component of $\Omega(\Gamma)$ is incompressible. It has the following immediate corollary in the spirit of Corollary 1.

**Corollary 2:** *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold such that $\partial_c N$ is incompressible in $\widehat{N} = N \cup \partial_c N$. If $\alpha$ is a geodesic current in $\partial_c N$ and $r(\alpha)^*$ denotes the geodesic current in $\partial C(N)$ which is homotopic to $r(\alpha)$, then*

$$\frac{l_{\partial C(N)}(r(\alpha)^*)}{4} \leq l_{\partial_c(N)}(\alpha) \leq \left(1 + \frac{\pi}{\sinh^{-1}(1)}\right) l_{\partial C(N)}(r(\alpha)^*).$$

**Remark:** Notice that $J(\rho_0) \asymp \frac{\sqrt{2}\pi^2}{\rho_0}$ and $L(\rho_0) \asymp 4\pi e^m e^{\frac{\pi^2}{2\rho_0}}$ as $\rho_0 \mapsto 0$. We observed in a previous remark that the form of $J(\rho_0)$ can not be substantially improved. It is an immediate consequence of Theorem 5.1 in [5] that if $\rho_0 < .5$ and $s$ is a $L$-Lipschitz homotopy inverse to $r$, then

$$L \geq \frac{\rho_0 e^{\frac{\pi^2}{2\sqrt{e}\rho_0}}}{2\pi e^{(.502)\pi}}$$

so again the form of $L(\rho_0)$ cannot be substantially improved.

## 9 An alternative version of Theorem 1

The following lemma allows us to translate injectivity radius bounds on the boundary of the convex core to injectivity radius bounds on the conformal boundary.

**Lemma 9.1** *Let $N$ be a hyperbolic 3-manifold and let $\widehat{\rho}_0$ be a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$. Then $f(\widehat{\rho}_0)$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$ where*

$$f(\widehat{\rho}_0) = \min\left\{\frac{1}{2}, \frac{\pi^2}{2\sqrt{e}\log\left(\frac{4\pi e^{(.502)\pi}}{2\widehat{\rho}_0}\right)}\right\}.$$

Notice that $f(\widehat{\rho}_0) \approx \frac{\pi^2}{-2\log\widehat{\rho}_0}$ as $\widehat{\rho}_0 \mapsto 0$.

*Proof of 9.1:* If not, then there exists a compressible curve $\alpha$ on $\partial_c N$ with length $L$ such that $L < 2f(\widehat{\rho}_0)$. Theorem 5.1 in [5] then implies that $r(\alpha)^*$ is a compressible geodesic on $\partial C(N)$ with length less that $2\widehat{\rho}_0$ which contradicts our assumptions.

$\boxed{9.1}$

Therefore, if we set $J'(\widehat{\rho}_0) = J(f(\widehat{\rho}_0))$ and let $L'(\widehat{\rho}_0) = 1 + K(\widehat{\rho}_0)$, then we obtain the following alternative formulation of Theorem 1:

**Theorem 1′:** *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $\widehat{\rho}_0$ be a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$. Then the nearest point-retraction is a $J'(\widehat{\rho}_0)$-Lipschitz map and has a homotopy inverse which is $L'(\widehat{\rho}_0)$-Lipschitz map.*

We also get the following alternative formulation of Corollary 1.

**Corollary 1′:** *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $s : \partial C(N) \to \partial_c N$ be a homotopy inverse to the nearest point retraction. If $\widehat{\rho}_0$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$ and $\alpha$ is a geodesic current on $\partial C(N)$, then*

$$\frac{l_{\partial_c N}(s(\alpha)^*)}{L'(\widehat{\rho}_0)} \le l_{\partial C(N)}(\alpha) \le J'(\widehat{\rho}_0) l_{\partial_c N}(s(\alpha)^*).$$

**Remark:** Notice that $J'(\widehat{\rho}_0) = O(\log(\frac{1}{\widehat{\rho}_0}))$ and $L'(\widehat{\rho}_0) = O(\frac{1}{\widehat{\rho}_0})$ as $\widehat{\rho}_0 \mapsto 0$. These asymptotics are much better than those in Theorem 1, since when $\Omega(\Gamma)$ has small injectivity radius, $\partial CH(L_\Gamma)$ has much smaller injectivity radius. Proposition 6.2 indicates that the form of $L'$ can not be substantially improved, while the examples discussed in the second remark in section 7 show that $J'(\widehat{\rho}_0)$ must grow at least as fast as $\frac{D\log(\frac{1}{\widehat{\rho}_0})}{\log\left(\log(\frac{1}{\widehat{\rho}_0})\right)}$ as $\widehat{\rho}_0 \mapsto 0$ (for some constant $D > 0$.)

## 10 Appendix: The proof of Lemma 4.1

In this section we review some of the theory of convex hulls of limit sets, as developed by Epstein and Marden [6]. We then give a proof of Lemma 4.1 which asserts that ridge lines are monotonic over the planes under a roof and that one can use the exterior dihedral angle of the roof to provide a bound on the bending measure.

If $L_\Gamma$ does not lie in a circle and $x \in CH(L_\Gamma)$, we will say that a neighborhood $U$ of $x$ in $\partial CH(L_\Gamma)$ is *adapted to $x$* if it has the following two properties:

1. $U$ is a spherical shell adapted to $x$, see Definition 1.5.3 in [6]. In particular, $U$ is simply connected and the intersection of any bending line or flat with $U$ is connected and convex.

2. If two bending lines $b_1$ and $b_2$ meet $U$, then any support plane to $b_1$ meets any support plane to $b_2$.

Lemma 1.8.3 in Epstein-Marden [6] guarantees that one can choose a set $U$ having property (2) above and also guarantees that ridge lines to support planes in a small enough neighborhood must lie close to one another.

**Lemma 10.1** (Epstein-Marden [6]) *If $x \in \partial CH(L_\Gamma)$ then there is an open neighborhood $U \subseteq \partial CH(L_\Gamma)$ of $x$ such that if two bending lines $b_1$ and $b_2$ meet $U$ then any support plane to $b_1$ intersects any support plane to $b_2$. Furthermore, if $b$ is a bending line containing $x$ and $N$ is a neighbourhood of $b$ in the space of geodesics, then, by taking $U$ small enough, we may assume that any ridge line, which is formed by the intersection of two distinct support planes at points of $U$ lies in $N$.*

Suppose that $x \in \partial CH(L_\Gamma)$ and $U$ is a neighborhood adapted to $x$. If $b_1$ and $b_2$ are distinct bending lines which intersect $U$ and lie in support planes $P_1$ and $P_2$, then $l_1$ and $l_2$ bound a strip in $U$. If $r = P_1 \cap P_2$, then we may define the corresponding *local roof* which is the union of the portion of $P_1$ between $b_1$ and $r$ and the portion of $P_2$ between $b_2$ and $r$. We say the open strip between $b_1$ and $b_2$ in $U$ is *under* this local roof.

We next recall the definition of the bending measure $\beta_\Gamma$. Let $\alpha : [0,1] \to CH(L_\Gamma)$ be a path which is transverse to the bending lamination and let $P$ and $Q$ be the unique support planes at $\alpha(0)$ and $\alpha(1)$. We say that a partition $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ of $[0,1]$ is *allowable* if each corresponding sub-arc $\alpha([t_{i-1}, t_i])$ is transverse to the bending lamination and lies under a local roof. Let $P_i$ be the support plane at $\alpha(t_i)$ and let $\theta_i$ be the exterior dihedral angle between $P_{i-1}$ and $P_i$. We define

$$i_{\mathcal{P}}(\alpha, \beta_\Gamma) = \sum_{i=1}^{n} \theta_i$$

and let

$$i(\alpha, \beta_\Gamma) = \inf_{\mathcal{P}} i_{\mathcal{P}}(\alpha, \beta_\Gamma)$$

where we take the infimum over all allowable partitions.

Notice that, by the definition of $i(\alpha, \beta_\Gamma)$, if $\alpha$ is under a local roof then we have that

$$i(\alpha, \beta_\Gamma) \leq \theta$$

where $\theta$ is the exterior dihedral angle between the support planes $P$ and $Q$ at the points $\alpha(0)$ and $\alpha(1)$.

We begin by showing that Lemma 4.1 is valid if the path remains under a local roof. We must first recall some basic facts about ridge lines and bending lines.

**Lemma 10.2** (Lemma 1.9.2 in Epstein-Marden [6]) *If any ridge line meets a bending line, then they are equal. If a bending line $b$ lies under the local roof formed by the* *support planes* $P_1$ *and* $P_2$ *and the bending lines* $b_1$ *and* $b_2$*, then* $b$ *is either equal to or disjoint from the ridge line* $r = P_1 \cap P_2$*. If* $P$ *is a support plane to* $b$ *then* $P$ *is either disjoint from the ridge line or else contains it.*

**Lemma 10.3** *If three distinct support planes* $P_1$*,* $P_2$*, and* $P_3$ *intersect in a common line* $l$*, then* $l$ *is a bending line with positive bending angle.*

*Proof of 10.3:* As support planes are oriented, consider the three normals $n_1$, $n_2$, and $n_3$ to the planes $P_1$, $P_2$, and $P_3$ at a common point $p \in l$. The normals divide the circle of planes $S(l)$ containing $l$ into three non-empty segments. At most one can be greater than or equal to $\pi$ in length. Choose the normal $n$ with segments of length less than $\pi$ on either side of it. Then the corresponding support plane $P$ is contained in the union of the half spaces of the other two. Therefore $P \cap \partial CH(L_\Gamma) \subseteq l$. As $P$ is a support plane, $l$ must be a bending line with positive bending angle.

$\boxed{10.3}$

We now prove the local version of Lemma 4.1. The restriction that the limit set does not lie in the circle is needed to avoid equal support planes with opposite orientation and bending lines with bending angle equal to $\pi$.

**Lemma 10.4** *Let* $\Gamma$ *be a Kleinian group whose limit set* $L_\Gamma$ *is not contained in a geometric circle. Let* $\alpha : [-1, 1] \to \partial CH(L_\Gamma)$ *be a geodesic arc whose interior is transverse to the bending lamination such that* $\alpha([-1, 1])$ *is contained in a neighborhood* $U$ *adapted to* $\alpha(0)$*. Let* $P$ *be a support plane at* $\alpha(0)$*, and let* $\{P_t|\ t \in [\bar{k}, k]\}$ *be a continuous one parameter family of support planes along* $\alpha$ *with* $P_0 = P$*. Then*

1. *If* $t_1 < t_2$ *and* $P_{t_1} = P_{t_2}$ *then* $P_t = P_{t_1}$ *for all* $t \in [t_1, t_2]$*.*
2. *there is a* $\bar{t} \in [0, 1]$ *such that* $P_t = P$ *if* $t \in [0, \bar{t}]$ *and the ridge lines* $\{r_t = P \cap P_t | t > \bar{t}\}$ *exist and form a monotonic family of geodesics on* $P$*.*

*Proof of 10.4:* Suppose that $t_1 < t_2$ and $P_{t_1} = P_{t_2}$ and let $s_1 = s(t_1)$ and $s_2 = s(t_2)$. Let $F = P_{t_1} \cap \partial CH(L_\Gamma)$. Since $F \cap U$ is simply connected and convex, $\alpha([s_1, s_2])$ is a geodesic arc in $F$. If $\alpha([s_1, s_2])$ is contained in a bending line $b$, then $\alpha$ intersects $b$ at a single point, so $s_1 = s_2$. Since $\alpha$ intersects $b$ transversely, the family $\{P_t|\ s(t) = s_1\}$ sweeps out an arc in $\Sigma(b)$. In this case, $P_{t_1} = P_{t_2}$ implies that $t_1 = t_2$. If $\alpha([s_1, s_2])$ is not contained in a bending line, then $F$ is a flat and $\alpha(s)$ is contained in the interior of the flat for all $s \in (s_1, s_2)$. Thus, if $s(t) \in (s_1, s_2)$, then $P_t = P_{t_1}$. If $\alpha(s_1)$ lies in boundary component $b$ of the flat, then again $\{P_t|\ s(t) = s_1\}$ sweeps out an arc in $\Sigma(b)$. This arc ends at $P_{t_1}$, since $\alpha$ is entering $F$ at this point. So, if

$t > t_1$, then $s(t) > s_1$. Similarly, if $t < t_2$, then $s(t) < s_2$. Therefore, if $t \in (t_1, t_2)$, then $s(t) \in (s_1, s_2)$, so $P_t = P_{t_1}$. This establishes claim (1).

Let $\bar{t} = \sup\{t \in [0,k] | \, P_t = P_0\}$. By continuity, $P_{\bar{t}} = P_0$ and, by claim (1), $P_t = P_0$ for all $t \in [0, \bar{t}]$. By definition, if $t > \bar{t}$, then $P_t \neq P_0$ and the ridge line $r_t = P_t \cap P_0$ exists. In order to complete the proof of claim (2), it suffices to show that if $t_1 < t_2$ and $r_{t_1} \cap r_{t_2} \neq \emptyset$, then $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$. As $P_0$ and $P_{t_2}$ form a local roof, Lemma 10.2 implies that $P_{t_1}$ either contains $r_{t_2}$ or is disjoint from it. If $P_{t_1}$ is disjoint from $r_{t_2}$ then $r_{t_1} \cap r_{t_2} = \emptyset$. If $P_{t_1}$ contains $r_{t_2}$, then $r_{t_1} = r_{t_2}$. Thus the planes $P_0$, $P_{t_1}$, and $P_{t_2}$ are all support planes to $r_{t_2}$. If $P_{t_1} = P_{t_2}$, then $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$, which implies that $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$. If $P_{t_1} \neq P_{t_2}$, then the three planes $P_0$, $P_{t_1}$, and $P_{t_2}$ are distinct and Lemma 10.3 implies that $r_{t_1}$ is a bending line with positive bending angle. In this case, since $\alpha$ intersects $r_{t_1}$ once transversely, $P_t$ is a support plane to $r_{t_1}$ for all $t \in [t_1, t_2]$. Since $r_{t_1} \subset P_0$, it again follows that $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$. We have completed the proof of claim (2).

$\boxed{10.4}$

We next show that if the ridge lines are monotonic, then the exterior dihedral angle is monotonically increasing. We first recall some basic facts about angles of triples of planes in $\mathbf{H}^3$.

Suppose that $P_1$, $P_2$, and $P_3$ are three distinct planes bounding half spaces $H_1$, $H_2$, and $H_3$. We also suppose that, for all $i$ and $j$, $P_i$ and $P_j$ intersect transversely with exterior dihedral angle $\theta_{ij}$, and that there is no common point of intersection of the three planes. In this case, there is a plane or horoball $P$ perpendicular to all three and the intersection of the planes $P_1$, $P_2$, and $P_3$ with $P$ gives lines $l_1$, $l_2$, and $l_3$ that intersect to form a triangle $T$ with vertices $v_{ij} = l_i \cap l_j$. The angle of $T$ at $v_{ij}$ is the (exterior or interior) dihedral angle between the planes $P_i$ and $P_j$.

The following general fact is established in section 1.10 of [6].

**Lemma 10.5** *Let $P_1$, $P_2$, and $P_3$ be support planes to a convex set in $\mathbf{H}^3$. If the interior of the triangle $T$ is contained in the half space $H_2$ and is in the complement of $H_1$, then $T$ is also in the complement of $H_3$ and*

$$\theta_{12} + \theta_{23} \leq \theta_{13}.$$

*In particular, $\theta_{12} \leq \theta_{13}$*

### Configuration of planes

We notice that if $P_1$ and $P_3$ form a local roof with $P_2$ under it, then the hypotheses of Lemma 10.5 are satisfied.

**Lemma 10.6** *Let $P_1$, $P_2$, and $P_3$ be support planes to $\partial CH(L_\Gamma)$ with $b$ a bending line on $P_1$ and ridge lines $r_1 = P_1 \cap P_2$, $r_2 = P_1 \cap P_3$. If $r_1 \cap r_2 = \emptyset$ and $r_1$ separates $b$ and $r_2$, then $P_1$, $P_2$, and $P_3$ satisfy the assumptions of Lemma 10.5.*

*Proof of 10.6:* As $r_1$ separates $b$ and $r_2$ we have that $r_2$ is in the interior of $H_2$. Since $b$ and $r_1$ are on the same side of $P_3$, $r_1$ is in the interior of $H_3^c$. Since $r_1 \cap r_2 = \emptyset$, either $P_2 \cap P_3 = \emptyset$ or $P_2 \cap P_3 = r_3$ where $r_3$ is a ridge line. If $P_2 \cap P_3 = \emptyset$, then, since $r_2$ is in the interior of $H_2$, $P_3$ is in the interior of $H_2$. But this contradicts the fact that $P_3$ is a support plane. Therefore the ridge line $r_3 = P_2 \cap P_3$ exists and the planes $P_1$, $P_2$, and $P_3$ describe a triangle $T$ as above. As $r_2$ is in the interior of $H_2$, then $T$ is contained in $H_2$. Also, since $r_1$ is in the interior of $H_3^c$, $T$ is contained in the complement of $H_3$. Therefore, $P_1$, $P_2$, and $P_3$ satisfy the assumptions of Lemma 10.5.

$\boxed{10.6}$

We are now ready to analyze the situation when the ridge lines are monotonic.

**Lemma 10.7** *Let $\alpha : [0,1] \to \partial CH(L_\Gamma)$ be a parameterized geodesic arc whose interior is transverse to $\beta_\Gamma$ and let $\{P_t |\ t \in [0,k]\}$ be a continuous one-parameter family of support planes to $\alpha$. Suppose that the ridge lines $r_t = P_0 \cap P_t$ exist for all $t \in (0,k]$ and form a monotonic family of geodesics. Let $lim_{t\to 0} r_t = b$ where $b$ is a bending line on $P_0$. Let $t_1, t_2 \in (0,k]$ with $t_1 < t_2$. Then*

1. *If $r_{t_1}$ is a bending line then $r_t = b$ for $t \in (0,t_1]$*
2. *If $r_{t_1} = r_{t_2}$ then either*

   $$P_t = P_{t_1} \ \text{for all} \ t \in [t_1,t_2] \qquad \text{or} \qquad r_t = b \ \text{for all} \ t \in (0,t_2]$$
3. *If $P_{t_1} = P_{t_2}$ then $P_t = P_{t_1}$ for all $t \in [t_1,t_2]$*
4. *The exterior dihedral angle $\theta_t$ between $P_0$ and $P_t$ is monotonically increasing.*

*Proof of 10.7:* If $r_{t_1}$ is a bending line $b_0$ and $b_0 = b$ then by monotonicity $r_t = b$ for all $t \in (0,t_1]$ . If $b_0 \neq b$ then there must be a $t_3 \in (0,t_1)$ such that $r_{t_3}$ separates $b$ and $b_0$. Thus either $b$ or $b_0$ is in the interior of $H_{t_3}$, the half space corresponding to $P_{t_3}$. This contradicts the fact that both $b$ and $b_0$ are bending lines. Thus $b_0 = b$ and we have established claim (1).

Let $r_{t_1} = r_{t_2}$. Then, by monotonicity, $r_t = r_{t_1}$ for all $t \in [t_1,t_2]$. Either $P_t = P_{t_1}$ for all $t \in [t_1,t_2]$ or there is some $t_3 \in (t_1,t_2]$ such that the support plane $P_{t_3}$ is not equal to $P_{t_1}$. In this case, $r_{t_1}$ is contained in the three distinct support planes $P_0$, $P_{t_1}$, and $P_{t_3}$. Therefore, by Lemma 10.3, $r_{t_1}$ is a bending line $b_0$ on $P_0$. Thus, by claim (1), $b_0 = b$ and by monotonicity, $r_t = b$ for all $t \in (0, t_2]$. This establishes claim (2).

If $P_{t_1} = P_{t_2}$, then $r_{t_1} = r_{t_2}$. Therefore, by claim (2), either $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$ or $r_t = b$ for all $t \in (0, t_2]$. If $r_t = b$ for all $t \in (0, t_2]$, then let $X$ and $Y$ be the extreme planes at $b$ and let $s_2 = s(t_2)$. Since $\alpha([0, s_2]) \subset X \cup Y$, it intersects $b$ only once. So, $\{P_t | t \in [t_1, t_2]\}$ sweeps out an arc in $\Sigma(b)$ joining $P_{t_1}$ to $P_{t_2}$. Since $P_{t_1} = P_{t_2}$, $P_t$ must equal $P_{t_1}$ for all $t \in [t_1, t_2]$. Thus in either case $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$, which is claim (3).

We now show monotonicity of $\theta_t$. Let $t_1 \in (0, k]$, and let $s_1 = s(t_1)$. It suffices to show that there exists $\delta > 0$ such that $\theta_t \geq \theta_{t_1}$ for all $t \in [t_1, t_1 + \delta)$.

If $\alpha(s_1)$ is in the interior of a flat then there is some $\delta > 0$ so that $P_t = P_{t_1}$ for all $t \in [t_1, t_1 + \delta)$ and therefore $\theta_t = \theta_{t_1}$ for all $t \in [t_1, t_1 + \delta)$ which completes the proof in this case.

Now suppose that $\alpha(s_1)$ is contained in a bending line $b_1$ and $t_2 > t_1$. If $P_{t_1} = P_{t_2}$ then $\theta_{t_2} = \theta_{t_1}$. If $P_{t_1} \neq P_{t_2}$ and $r_{t_1} = r_{t_2}$ then, by claim (2), $r_t = b$ for all $t \in (0, t_2]$. Thus the support planes $\{P_t | \ t \in [0, t_2]\}$ sweep out an arc in $\Sigma(b)$ which begins at $P_0$, and again $\theta_{t_1} \leq \theta_{t_2}$. Therefore, $\theta_{t_1} \leq \theta_{t_2}$ if $P_{t_1} = P_{t_2}$ or $r_{t_1} = r_{t_2}$.

If $P_{t_1} \neq P_{t_2}$ and $r_{t_1} \neq r_{t_2}$, then, by monotonicity, either $r_{t_1}$ separates $b$ and $r_{t_2}$, or $r_{t_1} = b$. If $r_{t_1}$ separates $b$ and $r_{t_2}$, then we apply Lemma 10.6 to the support planes $P_0$, $P_{t_1}$, and $P_{t_2}$ to see that $\theta_{t_1} \leq \theta_{t_2}$. By combining the above, we see that if $r_{t_1} \neq b$, then $\theta_{t_1} \leq \theta_{t_2}$ for all $t_2 \in [t_1, k]$.

If $r_{t_1} = b$, then, by monotonicity, $r_t = b$ for all $t \in [0, t_1]$. As $P_{t_1} \neq P_0$, $b$ has positive bending angle. If $b$ is the bending line $b_1$ which contains $\alpha(s_1)$, then we may choose $\delta > 0$ such that if $t \in [t_1, t_1 + \delta)$, then $P_t$ is a support plane to $b$. This implies that if $t_2 \in [t_1, t_1 + \delta)$, then $r_{t_1} = r_{t_2} = b$. We saw above that this implies that $\theta_{t_1} \leq \theta_{t_2}$. If $b \neq b_1$, we choose a neighborhood $N$ of $b_1$ in the space of geodesics so that no geodesic in $N$ intersects $P_0$. Lemma 10.1 assures us that we can choose $\delta > 0$ such that if $t_2 \in [t_1, t_1 + \delta)$ and $P_{t_2} \neq P_{t_1}$, then $r_{t_1,t_2} = P_{t_1} \cap P_{t_2} \subset N$. If $P_{t_1} = P_{t_2}$ or $r_{t_1} = r_{t_2}$, then we have previously shown that $\theta_{t_2} \geq \theta_{t_1}$. If $P_{t_1} \neq P_{t_2}$ and $r_{t_1} \neq r_{t_2}$, then $r_{t_1,t_2} \subseteq N$, so $r_{t_1,t_2}$ is in the interior of $H_0^c$. Furthermore, $b$ does not lie in $P_{t_2}$, so $b$ is in the interior of $H_{t_2}^c$. In order to apply Lemma 10.5, we need to show that $r_{t_2}$ is in the interior of $H_{t_1}$. To do this we apply a simple continuity argument. As $r_{t_2} \neq r_{t_1} = b$, there is a $t_3 \in [t_1, t_2]$ such that $r_{t_3}$ separates $b$ and $r_{t_2}$. Thus $r_{t_2}$ is in the interior of $H_{t_3}$. Moreover, if $t \in [t_1, t_3]$, then $r_t \cap r_{t_2} = \emptyset$. So, for all $t \in [t_1, t_3]$, $P_t \cap r_{t_2} = \emptyset$. We consider the half spaces $H_t$ for all $t \in [t_1, t_3]$. As $r_{t_2}$ is in the interior of $H_{t_3}$ and $P_t \cap r_{t_2} = \emptyset$ for all $t \in [t_1, t_3]$, then by continuity $r_{t_2}$ is in the interior of $H_t$ for all $t \in [t_1, t_3]$. In particular, $r_{t_2}$ is in the interior of $H_{t_1}$. Lemma 10.5 then gives that $\theta_{t_1} \leq \theta_{t_2}$. So, if $r_{t_1} = b$, we have seen that there exists $\delta > 0$ such that $\theta_{t_1} \leq \theta_{t_2}$ for all $t_2 \in [t_1, t_1 + \delta)$. This completes the proof that $\theta_t$ is monotonic.

$\boxed{10.7}$

We are now ready to establish Lemma 4.1, which we restate here for reference.

**Lemma 4.1:** *Let $\alpha : [0,1] \to \partial CH(L_\Gamma)$ be a parameterized geodesic arc, let $(P,Q)$ be a roof over $\alpha$, and let $\{P_t|\ t \in [0,k]\}$ be a continuous one-parameter family of support planes over $\alpha$ joining $P$ to $Q$. Then*

1. $$i(\alpha, \beta_\Gamma)_P^Q \leq \theta \leq \pi.$$
   *where $\theta$ is the exterior dihedral angle between $P$ and $Q$, and*
2. *there is a $\bar{t} \in [0,1]$ such that $P_t = P$ if $t \in [0,\bar{t}]$ and the ridge lines $\{r_t = P \cap P_t | t > \bar{t}\}$ exist and form a monotonic family of geodesics on $P$.*

*Proof of 4.1:* First, we consider separately the case when $L_\Gamma$ lies in a circle. In this case, $CH(L_\Gamma) \subseteq \overline{P}$ where $\overline{P}$ is some plane and every bending line has bending angle $\pi$. As $(P,Q)$ is a roof, $\alpha$ cannot cross a bending line in its interior. Thus either $\alpha$ intersects no bending lines or $\alpha$ intersects bending lines only at its endpoints. If $\alpha$ intersects no bending line $b$ then $i(\alpha, \overline{\beta}_\Gamma)_P^Q = 0 = \theta$ and $\bar{t} = 1$. If $\alpha$ intersects a single bending line $b$ at one of its endpoints then, by definition, $i(\alpha, \overline{\beta}_\Gamma)_P^Q = \theta$ and $r_t = b$ for all $t > \bar{t}$. If $P$ and $Q$ are support planes to distinct bending lines, then one may apply Lemma 10.5 and the definition of the bending measure to conclude that $i(\alpha, \overline{\beta}_\Gamma)_P^Q \leq \theta$. It is again easy to check that the ridge lines are monotonic in this situation.

We may now assume that $L_\Gamma$ is not contained in a geometric circle and therefore has no bending angles of $\pi$.

We first prove claim (2), that the ridge lines are monotonic. Let $\{P_t|\ \in [0,k]\}$ be the continuous one-parameter family of support planes along $\alpha$ from $P$ to $Q$. We let $H_t$ be the half-space bounded by $P_t$ and let $D_t$ be the closed disk in $\widehat{\mathbf{C}}$ associated to $P_t$.

As $\alpha$ has a roof, then $P_0 \cap P_t \neq \emptyset$ for all $t \in [0,k]$. Let $\bar{t}$ be the maximum value such that $P_t = P_0$ for all $t \in [0,\bar{t}]$. If $\bar{t} = k$, then claim (2) is trivially true.

We consider the case when $\bar{t} < k$. Let $\bar{s} = s(\bar{t})$, then $\alpha([0,\bar{s}]) \subseteq P_0$. If $\alpha(\bar{s})$ is in the interior of the flat $F_0 = P_0 \cap \partial CH(L_\Gamma)$ then we obtain a contradiction to the maximality of $\bar{t}$. Therefore $\alpha(\bar{s})$ is on a bending line $b$. We let $U$ be adapted for $\alpha(\bar{s})$. By lemma 10.4, there is a $k_1 > \bar{t}$ such that the ridge lines $r_t$ for $t \in (\bar{t}, k_1]$ are well-defined and monotonic. Also by continuity $\lim_{t \to \bar{t}^+} r_t = b$. Therefore we define $r_{\bar{t}} = b$ and we obtain the monotonic family of geodesics $\{r_t\}$ for $t \in (\bar{t}, k_1]$.

As $(P,Q)$ is a roof over $\alpha$, if $P_0 \cap P_t$ is not a ridge line then $P_t = P_0$. Let $T$ be the maximum value such that the ridge lines $r_t$ exist and give a monotonic family of geodesics for $t \in (\bar{t}, T)$. As $P_T \cap P_0 \neq \emptyset$ then either $P_T = P_0$ or $r_T = P_T \cap P_0$ is a ridge line.

By lemma 10.7, the angle $\theta_t$ is an increasing function on $(\bar{t}, T)$. As $0 \leq \theta_t \leq \pi$, if $P_T = P_0$ then $\theta_T = \pi$ and $H_T$ has disjoint interior from $H_0$. This gives a contradiction to the assumption that $(P, Q)$ is a roof for $\alpha$.

Thus we can assume that ridge line $r_T$ exists. Then, by continuity, the family of geodesics $\{r_t |\ t \in (\bar{t}, T]\}$ is monotonic. If $T = k$, claim (2) holds. So assume that $T < k$.

Let $\overline{T}$ be the minimum value in $[\bar{t}, T]$ such that $P_{\overline{T}} = P_T$. Thus, since $r_t$ is monotonic on $(\bar{t}, T)$, Lemma 10.7 implies that $P_t = P_T$ for all $t \in [\overline{T}, T]$.

We now consider the ridge lines $r_t^T = P_t \cap P_T$. By the choice of $\overline{T}$ there is some $\delta_1 > 0$ such that $r_t^T$ is a ridge line for $t \in (\overline{T} - \delta_1, \overline{T})$. We define $b_T^- = lim_{t \to \overline{T}^-} r_t^T$. Similarly by choice of $T$ there is some $\delta_2 > 0$ such that $r_t^T$ is a ridge line for $t \in (T, T + \delta_2)$. We define $b_T^+ = lim_{t \to T^+} r_t^T$. Then $b_T^+$ and $b_T^-$ are both bending lines (possibly equal) on the support plane $P_T$. By definition of $T$, $\alpha(S) \in b_T^+$ where $S = s(T)$.

### Planes $P_0$ and $P_T$

As bending lines do not intersect, either $b_T^+ = b_T^-$ or they are disjoint geodesics on $P_T$. By Lemma 10.2, if a bending line intersects a ridge line they must be equal. Therefore neither $b_T^+$ nor $b_T^-$ transversely intersect $r_T$.

We will establish a contradiction by finding a $\delta > 0$ such that $r_t$ is monotonic on $[\bar{t}, T + \delta)$. We first show that there is a $\delta > 0$ so that $r_t$ is monotonic on $(T - \delta, T + \delta)$. If $b_T^+$ intersects $P_0$ then, since ridge lines are equal or disjoint, by Lemma 10.2, $r_T = b_T^+$. Therefore, by Lemma 10.7, $r_t = b$ for all $t \in [\bar{t}, T]$. As $P_T \neq P_0$, then $b$ has a positive bending angle. Therefore, there exists $\delta$ such that $P_t$ is a support plane to $b_T^+ = b$ for all $t \in (T - \delta, T + \delta)$. Therefore, $r_t = b$ for all $t \in (T - \delta, T + \delta)$ and is thus trivially monotonic on this region. If $b_T^+$ does not intersect $P_0$ then we may choose a neighbourhood $N$ of $b_T^+$ so that every geodesic in $N$ does not intersect $P_0$. We let $U$ be adapted for $\alpha(S)$ so that the ridge line associated to any two support planes to $U$ lies in $N$. Finally, we choose $\delta > 0$ so that the support planes $P_t$ for $t \in (T - \delta, T + \delta)$ are support planes to $U$. Since every geodesic in $N$ is disjoint from $P_0$, if $r_{t_1} \cap r_{t_2} \neq \emptyset$ and $t_1, t_2 \in (T - \delta, T + \delta)$, then $P_{t_1} = P_{t_2}$. Therefore, by Lemma 10.4, we have that $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$, so $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$. Since, $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$ whenever $r_{t_1} \cap r_{t_2} \neq \emptyset$ and $t_1, t_2 \in (T - \delta, T + \delta)$, $\{r_t\}$ is monotonic on $(T - \delta, T + \delta)$.

We now know that there exists $\delta > 0$ such that $\{r_t\}$ is monotonic on $(\bar{t}, T]$ and on $(T - \delta, T + \delta)$. If $\{r_t\}$ is non-constant on $(T - \delta, T]$, then $\{r_t\}$ is monotonic on $[\bar{t}, T + \delta)$ and we have completed the proof of claim (2). Otherwise, by Lemma 10.7, either $r_t = b$ for all $t \in [\bar{t}, T]$ or $P_t = P_T$ for all $t \in (T - \delta, T]$. If $r_t = b$ for all $t \in [\bar{t}, T]$, then $\{r_t\}$ is clearly monotonic on $[\bar{t}, T + \delta)$ and we are again done.

If $P_t = P_T$ for all $t \in (T-\delta, T]$, then $T \neq \bar{T}$ and $b_T^+$ and $b_T^-$ must be disjoint. We may then choose neighbourhoods $N^+$ and $N^-$ of $b_T^+$ and $b_T^-$, such that no geodesic in $N^+$ intersects any geodesic in $N^-$ and no geodesic in $N^+$ or $N^-$ intersects $P_0$. We choose $\delta_1 > 0$ so that if $t_1, t_2 \in (\overline{T} - \delta_1, \overline{T}]$ then $P_{t_1}$ and $P_{t_2}$ are either equal or their intersection is in $N^-$. Also we choose $\delta_2 > 0$ so that if $t_1, t_2 \in [T, T+\delta_2)$, then $P_{t_1}$ and $P_{t_2}$ are either equal or their intersection is in $N^+$. Let $\delta_0 = min(\delta_1, \delta_2, \delta)$.

We first show that $b_T^-$ separates $r_T$ from $b_T^+$. By the definition of $\overline{T}$, $r_t \neq r_T$ for any $t \in (\overline{T} - \delta_0, \overline{T})$. Thus $r_t$ separates $b$ and $r_T$ in $P_0$. So $r_T$ is in the interior of $H_t$ for any $t \in (\overline{T} - \delta_0, \overline{T})$. As $b_T^+$ and $b_T^-$ are bending lines they are on the same side of $r_t^T$ in $P_T$. Thus $b_T^+$ is in the interior of $H_t^c$. Therefore $r_t^T$ separates $r_T$ and $b_T^+$ in $P_T$. As $r_t^T$ tends to $b_T^-$ as $t \to \overline{T}^-$, by continuity, $b_T^-$ separates $r_T$ and $b_T^+$.

If $r_{t_1} = r_T$ for some $t_1 \in (T, T+\delta_0)$, then, by the monotonicity of $\{r_t\}$ on $(T-\delta_0, T+\delta_0)$, $r_t = r_T$ for all $t \in [T, t_1)$ which would imply that $r_t$ is monotonic on $[\bar{t}, t_1)$, which would contradict the maximality of $T$. Suppose that $t \in (T, T+\delta_0)$. Since $b_T^-$ separates $r_T$ and $b_T^+$ in $P_T$, and $r_t^T$ lies in $N^+$, $b_T^-$ separates $r_T$ and $r_t^T$ in $P_T$. Thus, $r_T$ is in the interior of $H_t^c$. If $r_t$ separates $b$ from $r_T$ then $b$ is in the interior of $H_t$. This contradicts the fact that $b$ is a bending line. Thus, for all $t \in (T, T+\delta_0)$, $r_T$ separates $b$ and $r_t$. Therefore, $\{r_t\}$ is monotonic on $(\bar{t}, T+\delta_0)$. This completes the proof of claim (2).

We now prove claim (1) by induction on the number of local roofs. If $(P, Q)$ is a local roof over $\alpha$ then $i(\alpha, \beta_\Gamma)_P^Q \leq \theta \leq \pi$ by the definition of intersection number. Assume now that we have established claim (1) for any arc which is covered by $n-1$ local roofs and that $\alpha$ is covered by $n$ local roofs with the $i^{th}$ having boundary support planes $P_{t_{i-1}}$ and $P_{t_i}$, so that $P_{t_0} = P_0$ and $P_{t_n} = P_k$. Let $\theta_{i,j}$ be the exterior dihedral angle between $P_{t_i}$ and $P_{t_j}$ and $r_{t_i} = P_0 \cap P_{t_i}$. It follows from the definition of the bending measure and our inductive assumption, that

$$i(\alpha, \beta_\Gamma)_P^Q \leq \theta_{0,n-1} + \theta_{n-1,n}.$$

If $P_{t_n} = P_{t_{n-1}}$ then $\theta_{n-1,n} = 0$ and so

$$i(\alpha, \beta_\Gamma)_P^Q = i(\alpha, \beta_\Gamma)_P^{P_{n-1}} \leq \theta_{0,n-1} = \theta$$

If $P_{t_n} \neq P_{t_{n-1}}$ then we consider the ridge lines $r_{t_{n-1}}$ and $r_{t_n}$. If $r_{t_{n-1}} = r_{t_n}$ then, as $P_{t_n} \neq P_{t_{n-1}}$, Lemma 10.7 implies that $r_t = b$ for all $t \in (\bar{t}, k]$. Thus, $\alpha$ intersects $\beta_\Gamma$ at a single point of the bending line $b$ with extreme support planes $P$ and $Q$. Therefore,

$$i(\alpha, \beta_\Gamma)_P^Q = \theta.$$

If $r_{t_{n-1}} \neq r_{t_n}$ then either $r_{t_{n-1}}$ separates $b$ and $r_{t_n}$ or $r_{t_{n-1}} = b$. If $r_{t_{n-1}}$ separates $b$ and $r_{t_n}$ then, by Lemma 10.6, the half-spaces $H_0$, $H_{t_{n-1}}$, and $H_{t_n}$ satisfy Lemma

10.5, so $\theta_{0,n-1} + \theta_{n-1,n} \leq \theta_{0,n}$ and therefore

$$i(\alpha, \beta_\Gamma)_P^Q \leq \theta_{0,n} = \theta$$

Now consider the case with $r_{t_{n-1}} \neq r_{t_n}$ and $r_{t_{n-1}} = b$. Let $r = P_{t_{n-1}} \cap P_{t_n}$. To apply lemma 10.5, we need to show that $b$ is in the interior of $H^c_{t_n}$ and $r_{t_n}$ is in the interior of $H_{t_{n-1}}$. As $b$ is a bending line which does not meet $H_{t_n}$, $b$ lies in the interior of $H^c_{t_n}$. Since $r_{t_{n-1}} \neq r_{t_n}$, Lemma 10.7 implies that $r_{t_n}$ is not a bending line. Choose $t_a \in [t_{n-1}, t_n]$, such that $r_{t_a}$ separates $b$ and $r_{t_n}$. Thus, $r_{t_n}$ is in the interior of $H_{t_a}$ and for all $t \in [t_{n-1}, t_a]$, $r_t = P_t \cap P_0$ is between $b$ and $r_{t_a}$. Therefore $P_t \cap r_{t_n} = \emptyset$. Considering the half spaces $H_t$ for $t \in [t_{n-1}, t_a]$, we note that $r_{t_n}$ is in the interior of $H_{t_a}$ and $P_t \cap r_{t_n} = \emptyset$ for all $t \in [t_{n-1}, t_a]$. Therefore, by continuity, $r_{t_n}$ is in the interior of $H_t$ for all $t \in [t_{n-1}, t_a]$. In particular, $r_{t_n}$ is in the interior of $H_{t_{n-1}}$. Applying lemma 10.5 we have that $\theta_{0,n-1} + \theta_{n-1,n} \leq \theta_{0,n}$ and therefore, again,

$$i(\alpha, \beta_\Gamma)_P^Q \leq \theta_{0,n} = \theta.$$

$\boxed{4.1}$

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