Length distortion and the Hausdorff dimension of limit sets

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Abstract

Let Γ be a quasi-Fuchsian Kleinian group. We define the distortion function along geodesic rays lying on the boundary of the convex hull of the limit set, where each ray is pointing in a randomly chosen direction. The distortion function measures the ratio of the intrinsic to extrinsic metrics, and is defined asymptotically as the length of the ray goes to infinity. Our main result is that the distortion function is both almost everywhere constant and bounded above by the Hausdorff dimension of the limit set of Γ. As a consequence, we are able to provide a geometric proof of the following result of Bowen: If the limit set of Γ is not a round circle, then the Hausdorff dimension of the limit set is strictly greater than one. The proofs are developed from results in Patterson-Sullivan theory and ergodic theory.

1 Introduction

The purpose of this paper is to relate the bending measure on the boundary of the convex hull to the Hausdorff dimension of the limit set. Let Γ be a quasi-Fuchsian Kleinian group, that is Γ is a discrete, torsion-free subgroup of $PSL(2, \mathbf{C})$ whose limit set is a Jordan curve on the sphere at infinity. We define a function that measures the ratio of the intrinsic to extrinsic metric on the boundary of the convex hull of the limit set of a quasi-Fuchsian group. Using the Patterson-Sullivan theory, it is shown that this function is constant and bounded above by the Hausdorff dimension of the limit set. By applying a standard argument from ergodic theory,

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we are then able to demonstrate a geometric proof of a result of Bowen: If Γ is quasi-Fuchsian Kleinian group whose limit set is not a round circle, then the Hausdorff dimension of the limit set is strictly greater then one.

The plan of this paper is as follows: Section 2 contains basic definitions from the theory of Kleinian groups and Section 3 has the statements of the theorems. In Sections 4 and 5 we develop the necessary technology from the Patterson-Sullivan theory, and then prove our results. The paper finishes by interpreting our results in terms of Bonahon's theory of Liouiville currents.

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2 Kleinian group basics

Let $\mathbf{H}^n$ be the n-dimensional ($n \geq 2$) ball (Poincarè) model of hyperbolic space; we denote the sphere at infinity by S^{n-1}_∞. Recall that a *Kleinian group* is a discrete subgroup of the group of orientation preserving isometries of $\mathbf{H}^n$. As such, Γ acts discontinuously on $\mathbf{H}^n$ so that, providing Γ has no elements of finite order, the quotient $\mathbf{H}^n/\Gamma$ is a Riemannian manifold of constant sectional curvature -1.

A Kleinian group also acts as a discrete group of conformal automorphisms on the Riemann sphere S^{n-1}_∞. The action of the group partitions S^{n-1}_∞ into two disjoint sets. The *regular set* Ω_Γ is the maximal open set upon which Γ acts discontinously, the *limit set* L_Γ is its complement.

The *convex hull* $CH(L_\Gamma)$ of L_Γ is the smallest convex set in $\mathbf{H}^n$, so that every hyperbolic geodesic with endpoints in L_Γ is in $CH(L_\Gamma)$. If one quotients $CH(L_\Gamma)$ by the action of the group, then the resulting submanifold $C(\Gamma) = CH(L_\Gamma)/\Gamma$ is the smallest convex submanifold in $N = \mathbf{H}^n/\Gamma$ so that the inclusion map is a homotopy equivalence. This submanifold is called the *convex core.* A finitely generated Kleinian group Γ is *geometrically finite* if $\operatorname{vol} C(\Gamma) < \infty$. This is but one of many equivalent definitions of the concept of geometric finiteness; see Bowditch [2] for a full discussion of the various different formulations.

The rest of this section focuses solely on discrete subgroups of orientation-preserving isometries of $\mathbf{H}^3$. This paper deals specifically with a certain class of geometrically finite Kleinian groups called *quasi-Fuchsian* groups. These are finitely generated, torsion-free Kleinian groups whose limit sets are Jordan curves. We also make the standing assumption that Γ contains no parabolic elements; as such the convex core $C(\Gamma)$ of a quasi-Fuchsian group Γ is a compact submanifold of N. If in fact $C(\Gamma)$ is a surface then Γ is a *Fuchsian group* (i.e the limit set is a round circle);

if Γ produces a convex core that is a 3-manifold then we refer to Γ as being *strictly quasi-Fuchsian.* Quasi-Fuchsian groups are a sub-class of *B-groups*, that is finitely generated Kleinian groups, each with an invariant simply connected component of the regular set. For the basics in the theory of Kleinian groups we refer the reader to Maskit [10].

We need finer information about the structure of the convex core of a quasi-Fuchsian manifold. Let Γ be a Fuchsian group whose limit set is the unit circle in the complex plane. Then $CH(L_\Gamma)$ is the complex plane in $\mathbf{H}^3$ that spans the unit circle. By contrast, suppose that Γ is strictly quasi-Fuchsian. Then the following facts hold.

Theorem 2.1 *(Epstein-Marden [6]) The convex hull of a strictly quasi-Fuchsian Kleinian group* Γ *has the following description.*

1. *The convex hull* $CH(L_\Gamma)$ *is a* 3*-dimensional submanifold of* $\mathbf{H}^3$*, so that the boundary of* $CH(L_\Gamma)$ *consists of two simply connected components* C^+ *and* C^-*, each homeomorphic to an open disk.*

2. *Both* C^+ *and* C^- *consists of pieces of hyperbolic planes meeting along infinite geodesic lines. The hyperbolic metric in* $\mathbf{H}^3$ *induces an intrinsic metric on each boundary component, so that both components are complete* 2*-dimensional hyperbolic spaces with respect to these intrinsic metrics.*

We denote the *intrinsic metric* on C^+ (respectively C^-) by ρ^+ (respectively ρ^-). The usual hyperbolic metric (the *extrinsic metric*) is denoted by ρ. Assume that Γ is strictly quasi-Fuchsian. Because each boundary component of the convex hull isometric to $\mathbf{H}^2$, then $\partial C(\Gamma) = C^+/\Gamma \cup C^-/\Gamma$ is the union of closed hyperbolic surfaces of genus $g \geq 2$. As an immersed surface in $\mathbf{H}^3/\Gamma$ each boundary component is only C^0.

We can define a developing map on each boundary component so that we can carry the geometric analysis on a boundary component of the convex hull over to $\mathbf{H}^2$. The developing maps induce holonomy representations of the fundamental groups of the surface into $PSL(2, \mathbf{R})$. The holonomy representations are Fuchsian groups, and we refer to them as either the $+$ *Fuchsian model* or the $-$ *Fuchsian model.* By abuse of notation we will refer to both the quasi-Fuchsian group and the Fuchsian models by Γ; we hope that the context will make it clear which group we are talking about. For a more detailed discussion of developing maps and holonomy see [4] and [16].

Let Γ be any Kleinian group acting on $\mathbf{H}^3$. The convex hull $CH(L_\Gamma)$ can be realized as the union of ideal tetrahedra [16], where each such tetrahedron has the property that all its vertices lie in the limit set. It is thus intuitively clear that the boundary of the convex hull consists of pieces of hyperbolic planes and infinite geodesic lines. The planar pieces of the convex hull are called *flat pieces*; the complement of the flat pieces are called *bending lines* and are of measure zero in any component of the boundary of the convex hull. Note that we can define a normal vector at any point on a flat piece. The *bending* of an arc tranverse to the bending lines is simply the total angle of turning of the normal vector along the arc. (The total angle is given by a regular, countably additive measure invariant under isotopies of the bending lines and defined on the collection of embedded intervals on $\partial CH(L_\Gamma)$; see [6] and [16].)

We will need the following elementary fact differentiating Fuchsian groups from strictly quasi-Fuchsian groups.

Proposition 2.2 *Let Γ be a strictly quasi-Fuchsian Kleinian group. Then for each component of the boundary of the convex hull there is an embedded arc transverse to the locus of bending lines that has positive total bending.*

3 Statement of results

Suppose Γ is a quasi-Fuchsian group; we pull the analysis on each boundary component of the convex hull back to the Poincarè model $\mathbf{H}^2$ of the hyperbolic plane isometric to that component. For ease of exposition we will only deal with the boundary component C^+ of the convex hull in all that follows. All results immediately extend to the full boundary of the convex hull.

We define R^+ to be the function

$$R^+(x,y) = \frac{\rho^+(x,y)}{\rho(x,y)}$$

for all $(x,y) \in \mathbf{H}^2 \times \mathbf{H}^2$ so that $x \neq y$. Let $T_1(\mathbf{H}^2)$ be the unit tangent bundle of $\mathbf{H}^2$. We represent a point $\eta \in T_1(\mathbf{H}^2)$ by a pair (x,v) where x is the *basepoint* of η and v is the unit tangent vector at x that gives the *direction* of η. It is a consequence of the Cartan-Hadamard Theorem that, for $x \in \mathbf{H}^2$, the exponential map $\exp_x : T_x(\mathbf{H}^2) \to \mathbf{H}^2$ is a diffeomorphism. Define for η the function

$$\mathcal{R}^+(x,v) = \limsup_{t\to\infty} R^+(x, \exp_x tv). \tag{1}$$

We call $\mathcal{R}^+$ the *distortion function* of Γ on C^+.

Denote by $D(L_\Gamma)$ the Hausdorff dimension of the limit set of Γ (see Section 4.) The Main Theorem is:

Main Theorem: *Let* Γ *be a quasi-Fuchsian group. There exists a constant* $1 \leq K_\Gamma^+ \leq D(L_\Gamma)$ *so that*

$$\mathcal{R}^+(x,v) = K_\Gamma^+$$

a.e. with respect to the 1*-dimensional Lebesgue measure on* S^1_∞*.*

The constant K_Γ^+ is called the *length distortion* of Γ on C^+. It is a result of Sullivan [14] that the Hausdorff dimension of the limit set of a geometrically finite Kleinian group with non-empty regular set is strictly less then 2.

We now provide a brief résumé of the proof of the Main Theorem. The idea is to show first that the distortion function at (x,v) is determined asymptotically by a sequence of orbit points of x under the group action. We then demonstrate that the set of "directions" at a point x, on which the length distortion function exceeds the desired bound, is of measure zero. This is accomplished using a sequence of technical results from the Patterson-Sullivan theory culminating with Proposition 5.2. The desired conclusion follows from observing that the geodesic flow on a closed hyperbolic surface is ergodic.

In the case that Γ is strictly quasi-Fuchsian we can refine the lower bound of the distortion function given in the Main Theorem. Let x, y be any two points on C^+. Then it clear that $\rho^+(x,y) \geq \rho(x,y)$, and so $\mathcal{R}^+(x,v) \geq 1$ for all $(x,v) \in T_1(\mathbf{H}^2)$. However, using a standard argument in ergodic theory we can improve this estimate and show:

Theorem 3.1 *If* Γ *is strictly quasi-Fuchsian, then* $K_\Gamma^+ > 1$*.*

As a direct corollary of this theorem and the Main Theorem we obtain a geometric proof (i.e. in terms of the geometry of the convex hull) of the following result of Bowen.

Theorem 3.2 *(Bowen [3]) Let* Γ *be a strictly quasi-Fuchsian group, then* $D(L_\Gamma) > 1$*.*

Remark: Bowen showed his celebrated result using Markov partions. First he demonstrated that if the Nielsen boundary correspondence between two Fuchsian groups is absolutely continuous then the groups are in fact conjugate in $PSL(2,\mathbf{R})$ (this fact is due originally to Mostow [11]), from whence one can conclude that the limit set of any *strictly* quasi-Fuchsian group is non-rectifiable. Bowen then proceeds to show that the s-dimensional Hausdorff measure (s being the unspecified Hausdorff dimension of the limit set) of the limit set of *any* quasi-Fuchsian group is finite and positive. Since the limit set is non-rectifiable, the dimension s must be greater than 1.

4 Poincarè series and limit sets

This section gives a description of the parts of the Patterson-Sullivan theory needed to prove the Main Theorem. For further background and proofs see Nicholls [12].

Let K be a countable set of orientation-preserving hyperbolic isometries of $\mathbf{H}^n$. The standard sets usually considered are discrete subgroups but we will need to consider more general sets of isometries. The *orbit counting function* for K at $x \in \mathbf{H}^n$ is

$$N(K, x, r) = \#\{k \in K | \rho(x, kx) \leq r\},$$

where ρ is the hyperbolic metric on $\mathbf{H}^n$. We define the *limit set of* K by $L_K = \overline{Kx} \cap S^{n-1}$, where Kx is the orbit of x by K. A limit point ξ is a *conical limit point* of K if it is the limit of a sequence of orbit points which stay a bounded distance from a geodesic (and therefore all geodesics) with endpoint ξ. We denote the conical limit set of K by CL_K. It is easy to show that both L_K and CL_K are independent of the choice of x.

We can characterize the conical limit set in terms of shadows (Sullivan [14]). Let $x, y \in \mathbf{H}^n$ so that $x \neq y$, and fix $c > 0$. We denote the closed ball of radius c about x by $B(x, c)$. If x is not a point of $B(y, c)$, then the *shadow* of $B(y, c)$ from x, denoted by $b(x : y, c)$, is the projection of $B(y, c)$ onto $\mathbf{S}^{n-1}_\infty$ by geodesic rays based at x. Note that the shadow of a ball in H^n is a disk in $\mathbf{S}^{n-1}_\infty$ based at a point $\xi \in \mathbf{S}^{n-1}_\infty$, where ξ is the endpoint of the infinite geodesic ray based at x through y.

The conical limit set can be realized in terms of shadows of orbit points.

Lemma 4.1 *(Sullivan [14]; see also [12] Theorem 2.4.6) Let K be as above. Then*

$$CL_K = \bigcup_{c>0} \bigcap_{n=1}^{\infty} \bigcup_{\substack{k \in K \\ \rho(x,kx)>n}} b(x : kx, c).$$

Because the expression

$$\bigcap_{n=1}^{\infty} \bigcup_{\substack{k \in K \\ \rho(x,kx)>n}} b(x : kx, c)$$

is non-decreasing as c increases, we may assume that c takes values in $\mathbf{Z}^+$.

Let the *radius of the shadow* $b(x : y, c)$ be given by $r(x : y, c)$, where the distance is measured in the spherical metric. The next elementary result relates the size of the shadow to the hyperbolic distance between the projecting point x and the center y of the ball $B(y, c)$.

Proposition 4.2 *Given $x \in \mathbf{H}^n$ and $c > 0$, then there exist constants $A, B, C > 0$ such that $Ae^{-\rho(x,y)} \leq r(x : y, c) \leq Be^{-\rho(x,y)}$ for all y such that $\rho(x, y) \geq C$.*

Proof: It can be shown that the area (in the spherical metric) of a shadow $b(x : y, c)$ is given asymptotically in as $|y| \to 1$ by the expression

$$\frac{M(\cosh^2 c - 1)^{\frac{n-1}{2}}}{n-1}(1 - |y|)^{n-1}$$

(Theorem 1.2.2 in [12]). The result now follows from elementary hyperbolic geometry. ■

We define μ_s, the *s-dimensional Hausdorff measure* on S^{n-1}_∞, as follows. If E is a Borel set in $\mathbf{S}^{n-1}_\infty$ then let

$$\mu^\epsilon_s(E) = \inf\{\sum_{j=1}^{\infty} c^s_j : E \subset \bigcup B(x_j, c_j); c_j \leq \epsilon\}.$$

As μ^ϵ_s is clearly non-decreasing as ϵ decreases, we can take the (possibly infinite) limit

$$\mu_s(E) = \lim_{\epsilon \to 0} \mu^\epsilon_s(E).$$

The *Hausdorff dimension* of a set $E \subset \mathbf{S}^{n-1}_\infty$, denoted by $D(E)$, is given by

$$D(E) = \inf\{s : \mu_s(E) = 0\} = \sup\{s : \mu_s(E) = \infty\}.$$

Fix $s \in \mathbf{R}^+$. We define the *Poincarè series* of K by

$$g_s(x, y) = \sum_{k \in K} e^{-s\rho(x,ky)}.$$

Suppose that K is in fact a discrete group. Let

$$\delta(\Gamma) = \inf\{s : \ g_s < \infty\};$$

then $\delta(\Gamma)$ is called the *exponent of convergence* of the Poincarè series. There is a remarkable connection between the internal geometry of a geometrically finite hyperbolic n-manifold and the conformal action of the holonomy representation of its fundamental group acting on $\mathbf{S}^{n-1}_\infty$. One facet of this relationship is:

Theorem 4.3 *(Patterson [13] and Sullivan [15]) Let* Γ *be a geometrically finite subgroup of the orientation preserving isometries of* $\mathbf{H}^n$*. Then* $\delta(\Gamma) = D(L_\Gamma)$.

Further, it is a result of Sullivan [15] and Tukia [18] that if Γ is a geometrically finite Kleinian group with non-empty regular set, then $\delta(\Gamma) < n-1$.

The following technical result is well-known, though we include a proof for completeness.

Lemma 4.4 *(Sullivan [14]) Let* K *be a countable set of isometries of* H^n*, and suppose* $x \in H^n$ *and* $s \in \mathbf{R}^+$ *are fixed such that* $\sum_{k\in K} e^{-s\rho(x,kx)} < \infty$*. Then the following statements are true:*

1. *There exists a constant* A *such that* $N(K,x,r) \leq Ae^{sr}$*, and*
2. $D(CL_K) \leq s$.

Proof: Let $A = \sum_{k\in K} e^{-s\rho(x,kx)}$, and $B(x,r)$ the ball of hyperbolic radius r about x. Then

$$\sum_{k\in K\cap B(x,r)} e^{-s\rho(x,kx)} = \int_0^r e^{-st} dN(K,x,t) = N(K,x,t)e^{-st}|_0^r + s\int_0^r e^{-st}N(K,x,t)dt.$$

Therefore for any fixed r we have

$$A \geq N(K,x,t)e^{-st}|_0^r + s\int_0^r e^{-st}N(K,x,t)dt \geq N(K,x,r)e^{-sr},$$

and so $N(K,x,r) \leq Ae^{sr}$.

To prove that $D(CL_K) \leq s$, we recall that μ_s is the standard s-dimensional Hausdorff measure on the sphere S^{n-1}_∞. For $m \in \mathbf{Z}^+$ we define

$$C_m = \bigcap_{n=1}^{\infty} \bigcup_{\substack{k\in K \\ \rho(x,kx)>n}} b(x:kx,m).$$

From Lemma 4.1 we have that

$$CL_K = \bigcup_{m\in\mathbf{Z}^+} C_m.$$

For each $n > 0$

$$C_m \subset \bigcup_{\substack{k\in K \\ \rho(x,kx)>n}} b(x:kx,m).$$

If $\rho(x, kx) > n \geq C$ then by the Proposition 4.2 we $r(x : kx, m) \leq Be^{-n} = \epsilon_n$. Therefore

$$\mu_s^{\epsilon_n}(C_m) \leq \sum_{\substack{k \in K \\ \rho(x,kx)>n}} [r(x : kx, m)]^s \leq B^s \cdot \sum_{\substack{k \in K \\ \rho(x,kx)>n}} e^{-s\rho(x,kx)}.$$

Taking limits as n tends to infinity we get that

$$\mu_s(C_m) \leq B^s \cdot \lim_{n \to \infty} \sum_{\substack{k \in K \\ \rho(x,kx)>n}} e^{-s\rho(x,kx)} = 0,$$

and therefore $\mu_s(C_m) = 0$. As CL_K is a countable union of μ_s-measure zero sets, then $\mu_s(CL_K) = 0$ and so $D(CL_K) \leq s$. ■

This section is completed with a brief preparatory discusion of ergodic theory as it applies to discrete groups. We will make use of this in the proof of the Main Theorem to show that the distortion function of Γ (equation 1 in Section 3) is constant. Our description of the ergodic theory of geodesic flow on hyperbolic manifolds is brief and follows Nicholls [12] in all essential ways.

Let $\Omega(n) = \mathbf{H}^n \times S_\infty^{n-1}$; we think of $(x, \zeta) \in \Omega(n)$ as being a point in $\mathbf{H}^n$ with a given direction. As such, the point (x, ζ) defines an oriented geodesic through x with direction ζ appended. A Möbius transformation γ acts on $\Omega(n)$ by $\gamma(x, \eta) = (\gamma(x), \frac{\gamma'(x)}{|\gamma'(x)|})$. We can thus define a Möbius invariant measure $dM(x, \zeta) = dV(x)d\omega(\zeta)$, where V is the hyperbolic volume and ω is the $n-1$-dimensional Lebesgue measure on S_∞^{n-1}. Let η_- and η_+ be the beginning and end points of this geodesic, and let z be the Euclidean mid-point of the geodesic so that s is the directed hyperbolic distance from z to x. Then there is a correspondence $(x, \zeta) \longleftrightarrow (\eta_-, \eta_+, \mathbf{R})$, where $(\eta_-, \eta_+, \mathbf{R}) \in S_\infty^{n-1} \times S_\infty^{n-1}$ (minus the diagonal) $\times \mathbf{R}$. The *geodesic flow* g_t is a function on $\Omega(n) \times \mathbf{R}$ into $\Omega(n)$, and is defined by

$$g_t(\eta_-, \eta_+, s) = (\eta_-, \eta_+, s + t).$$

The action of g_t on (x, ζ) is to move it a directed distance t along the oriented geodesic defined by (x, ζ). See chapter 8 of Nicholls [12] for a detailed discussion.

When Γ is a Kleinian group g_t descends to a mapping on $\Omega(n)/\Gamma$ since it commutes with Möbius transformations. Recall that geodesic flow is *ergodic* if for any Borel measurable set $A \subset \Omega(n)/\Gamma$ that is invariant under the action of g_t, then either $M(A) = 0$ or $M(\Omega(n)/\Gamma - A) = 0$. The result we need is due to Sullivan [14] and Hopf [8].

Theorem 4.5 *Let Γ be a geometrically finite Kleinian group acting on $\mathbf{H}^n$. Then geodesic flow is ergodic on $\Omega(n)/\Gamma$ with respect to the measure differentially defined by dM.*

5 Proofs

In this section we provide proofs of the theorems listed in Section 3. We fix $x \in C^+$, and let $a_\gamma = e^{-\rho(x,\gamma x)}$, $b_\gamma = e^{-\rho^+(x,\gamma x)}$ be terms in the Poincarè series of the quasi-Fuchsian groups Γ and the $+$-Fuchsian model of Γ respectively.

Let $\sigma = D(L_\Gamma)$. By work of Hopf, Patterson, and Sullivan we have both that

$$\sum_{\gamma \in \Gamma} a_\gamma^s < \infty \Leftrightarrow s > \sigma \qquad ,$$

and

$$\sum_{\gamma \in \Gamma} b_\gamma^s < \infty \Leftrightarrow s > 1.$$

For $x, y \in \mathbf{H}^3$ denote by $R^+(x,y)$ the ratio $\frac{\rho^+(x,y)}{\rho(x,y)}$; define $R_\gamma^+ = R^+(x,\gamma x)$ for $\gamma \in \Gamma$. Note that $b_\gamma = a_\gamma^{R_\gamma^+}$. If $\epsilon > 0$, then define the following subset of Γ by

$$\Gamma_\epsilon = \{\gamma \in \Gamma | R_\gamma^+ \geq \sigma + \epsilon\}.$$

Proposition 5.1 *Let Γ be a quasi-Fuchsian group, and fix $\epsilon > 0$. Then*

$$\sum_{\gamma \in \Gamma_\epsilon} b_\gamma^s < \infty \text{ for } s > \frac{\sigma}{\sigma + \epsilon}.$$

Proof: By Theorem 4.3 we have that if $k > \sigma$, then

$$\sum_{\gamma \in \Gamma} a_\gamma^k < \infty.$$

As $a_\gamma = b_\gamma^{1/R_\gamma^+}$, we observe that $\sum_{\gamma \in \Gamma} b_\gamma^{k/R_\gamma^+} < \infty$. Because $\Gamma_\epsilon \subseteq \Gamma$, we immediately see that

$$\sum_{\gamma \in \Gamma_\epsilon} b_\gamma^{k/R_\gamma^+} \leq \sum_{\gamma \in \Gamma} b_\gamma^{k/R_\gamma^+} < \infty.$$

If $\gamma \in \Gamma_\epsilon$ then by definition $R^+_\gamma \geq \sigma + \epsilon$ and therefore $\frac{k}{R^+_\gamma} \leq \frac{k}{\sigma+\epsilon}$ for any $k > 0$. Thus if $s > \frac{\sigma}{\sigma+\epsilon}$, then

$$\sum_{\gamma \in \Gamma_\epsilon} b^s_\gamma < \infty.$$

■

By applying Lemma 4.4.2 we immediately obtain:

Proposition 5.2 *Let Γ_ϵ be defined as above, and regarded as a subset of the + Fuchsian model Γ. Then*

$$D(CL_{\Gamma_\epsilon}) \leq \frac{\sigma}{\sigma + \epsilon}.$$

From the previous proposition it is immediate that CL_{Γ_ϵ} has Lebesgue measure zero. Thus the set

$$CL_0 = \bigcup_{n \in \mathbf{Z}^+} CL_{\Gamma_{\frac{1}{n}}}$$

also is of measure zero.
We are now ready to prove the Main Theorem.

Proof of Main Theorem:

Let Γ be a quasi-Fuchsian group and suppose $\mathcal{R}^+ : T_1(\mathbf{H}^2) \to \mathbf{R}$ is the distortion function for Γ. As $\mathcal{R}^+$ is the limit superior of a sequence of continuous functions it is measurable. We let $\xi(x, v) \in S^1_\infty$ be the $+\infty$ endpoint on S^1_∞ of the geodesic ray based at x in the direction v.

Let $(x, v), (y, w) \in T_1(\mathbf{H}^2)$ such that $\xi(x, v) = \xi(y, w) = \xi$. Let $L = \rho^+(x, y)$, then by the convexity of the distance function [4], and the fact that $\xi(x, v) = \xi(y, w)$, we have $\rho^+(\exp_x tv, \exp_y tw) \leq L$ for all $t \geq 0$. Also as $\rho \leq \rho^+$, we have

$$\rho(x, y) \leq \rho^+(x, y) \leq L$$

and

$$\rho(\exp_x tv, \exp_y tw) \leq \rho^+(\exp_x tv, \exp_y tw) \leq L.$$

Therefore

$$\rho(x, \exp_x tv) \leq \rho(x, y) + \rho(y, \exp_y tw) + \rho(\exp_y tw, \exp_x tv) \leq \rho(y, \exp_y tw) + 2L,$$

and similarly

$$\rho^+(x, \exp_x tv) \leq \rho^+(y, \exp_y tw) + 2L.$$

Taking ratios, we observe that

$$\frac{\rho^+(x, \exp_x tv) - 2L}{\rho(x, \exp_x tv) + 2L} \leq \frac{\rho^+(y, \exp_y tw)}{\rho(y, \exp_y tw)} \leq \frac{\rho^+(x, \exp_x tv) + 2L}{\rho(x, \exp_x tv) - 2L}.$$

Taking the limit superior as t tends to infinity, we get $\mathcal{R}^+(x, v) = \mathcal{R}^+(y, w)$. Therefore $\mathcal{R}^+$ only depends on the $+infinity$ endpoint of the geodesic ray from x in the direction v. In particular, as $\xi(g_t(x, v)) = \xi(x, v)$ then $\mathcal{R}^+(g_t(x, v)) = \mathcal{R}^+(x, v)$, and therefore $\mathcal{R}^+$ is invariant under geodesic flow on $\mathbf{H}^2$.

Let $\xi = \xi(x, v)$ and let $\{x_n = \gamma_n x\}$ be a sequence such that x_n is a nearest orbit point of x to $\exp_x nv$. Because $\rho(x_n, \exp_x nv) \leq K$, where K is the diameter of $\mathbf{H}^2/\Gamma$, we observe that the sequence $\{x_n\}$ converges conically to ξ.

Therefore $d(x_{[t]}, \exp tv) \leq K + 1$ where $[t]$ is the greatest integer less than or equal to t. Hence

$$\rho(x, x_{[t]}) - K - 1 \leq \rho(x, \exp_x tv) \leq \rho(x, x_{[t]}) + K + 1,$$

and

$$\rho^+(x, x_{[t]}) - K - 1 \leq \rho^+(x, \exp_x tv) \leq \rho^+(x, x_{[t]}) + K + 1.$$

Taking ratios again, we get

$$\frac{\rho^+(x, x_{[t]}) - K - 1}{\rho(x, x_{[t]}) + K + 1} \leq \frac{\rho^+(x, \exp_x tv)}{\rho(x, \exp_x tv)} \leq \frac{\rho^+(x, x_{[t]}) + K + 1}{\rho(x, x_{[t]}) - K - 1}.$$

Taking the limit superior we see that $\mathcal{R}^+(x, v) = \limsup_{n\to\infty} R^+(x, x_n)$, where $R^+(x, x_n) = R^+_{\gamma_n}$.

If $\xi \notin CL_0$, then for each $\epsilon > 0$ there is an $N > 0$ so that $\gamma_n \notin \Gamma_\epsilon$ for $n > N$. Therefore $R^+(x, x_n) \leq \sigma + \epsilon$, for $n > N$. Thus

$$\limsup_{n\to\infty} R(x, x_n) \leq \sigma + \epsilon.$$

As this holds for each $\epsilon > 0$ we have that

$$\limsup_{n\to\infty} R^+(x, x_n) \leq \sigma.$$

Therefore $\mathcal{R}^+(x, v) \leq \sigma$ for $\xi(x, v) \notin CL_0$. As CL_0 is of measure zero, we have that $\mathcal{R}^+(x, v) \leq \sigma$ a.e.. Thus $\mathcal{R}^+$ is bounded measurable function which is invariant under geodesic flow. Also, by definition, $\mathcal{R}^+$ is invariant under the action of Γ on $\mathbf{H}^2$. Therefore by the ergodicity of geodesic flow on the closed surface $\mathbf{H}^2/\Gamma$ (Theorem

4.5) there exists a constant K_Γ^+ such that $\mathcal{R}^+ = K_\Gamma^+$ a.e.. As $\mathbf{R}^+(x,v) \le \sigma$ a.e., we have that $K_\Gamma^+ \le \sigma$ almost everywhere with respect to Lebesgue measure on S^1_∞. ■

We now define the *unit length distortion functions.* We define $R_1^+, S_1^+ : T_1(\mathbf{H}^2) \to \mathbf{R}$ of Γ on C^+ by letting $R_1^+(x,v) = R^+(x, \exp v)$ and $S_1^+ = 1/R_1^+$, its reciprocal. Both functions R_1^+, S_1^+ are obviously continuous.

Applying the Birkhoff ergodic theorem to the function S_1^+ we have that there is a measurable function $\overline{S_1^+}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} S_1^+(g_k(x,v)) = \overline{S_1^+}(x,v) \text{ a.e.}$$

By the ergodicity of the geodesic flow on the closed hyperbolic surface $\mathbf{H}^2/\Gamma$ this (time) average of S_1^+ is equal almost everywhere to a constant given by the (space) average of S_1^+ over $T_1(\mathbf{H}^2/\Gamma)$. Thus we have

$$\overline{S_1^+}(x,v) = \int_{T_1(\mathbf{H}^2/\Gamma)} S_1^+(x,v)\, dV = \overline{S_1^+}(\Gamma) \text{ a.e.}$$

The following is our proof of Bowen's result.

Theorem 5.3 *Suppose Γ is strictly quasi-Fuchsian, then $K_\Gamma^+ > 1$. In particular, the Hausdorff dimension of the limit set of a strictly quasi-Fuchsian group is greater than one.*

Proof: Since $\rho \le \rho^+$ we have that $S_1^+(x,v) \le 1$ for all (x,v). Therefore $\overline{S_1^+}(\Gamma) \le 1$. As Γ is strictly quasi-Fuchsian we can choose (x,v) such that the unit length geodesic arc from x to $\exp v$ has non-zero total bending. Therefore $\rho(x, \exp v) < \rho^+(x, \exp v)$ and $S_1^+(x,v) < 1$. As S_1^+ is a continuous function the space average of S_1^+ must be strictly less than 1. Therefore $\overline{S_1^+}(\Gamma) < 1$

Choose $(x,v) \in T_1(\mathbf{H}^2)$ and let $x_k = \exp kv$. Then we have that

$$\sum_{k=1}^{n} S_1^+(g_k(x,v)) = \sum_{k=1}^{n} \rho(x_{k-1}, x_k)$$

Therefore by the triangle inequality we have

$$\rho(x_0, x_n) = \rho(x, \exp nv) \le \sum_{k=1}^{n} S_1^+(g_k(x,v))$$

Note that $\rho^+(x, \exp nv) = n$. Therefore

$$\frac{1}{R^+(x, \exp nv)} = \frac{\rho(x, \exp nv)}{\rho^+(x, \exp nv)} = \frac{1}{n}\rho(x, \exp nv) \leq \frac{1}{n}\sum_{k=1}^{n} S_1^+(g_k(x, v))$$

Taking limits as n tends to infinity we get

$$\frac{1}{\mathcal{R}^+(x, v)} \leq \overline{S}_1^+(x, v)$$

Therefore

$$\frac{1}{K_\Gamma^+} \leq \overline{S}_1^+(\Gamma)$$

As $\overline{S}_1^+(\Gamma) < 1$ we have that $K_\Gamma^+ > 1$ as required ■

Remark: We note that Theorem 5.3 has an equivalent phrasing in terms of the spectral theory of the Laplacian acting on the quotient manifold as follows. *If* Γ *is strictly quasi-Fuchsian, then the lowest eigenvalue of the Laplacian acting on* $\mathbf{H}^3/\Gamma$ *is strictly less than one. Further, there exists a positive* L^2 *eigenfunction realizing the lowest eigenvalue.* See [**?**] for a full discussion of the deep relationships between Hausdorff dimension, the exponent of convergence of the Poincaré series, and the spectrum of the Laplacian.

Though we haven't stated it this way, what we have shown in the Main Theorem is that

$$\max(K_\Gamma^+, K_\Gamma^-) \leq D(L_\Gamma).$$

This naturally leads us to inquire:

Question 1: *If* Γ *is a quasi-Fuchsian group, then does* $K_\Gamma^+ = K_\Gamma^-$*?*

Of course, if Γ is Fuchsian then $CH(L_\Gamma)$ is a hyperbolic plane, and so Question 1 is vacuously true. Further, it is clear that if Γ is Fuchsian then $K_\Gamma = 1 = D(L_\Gamma)$. However, for strictly quasi-Fuchsian groups a most satisfying state of affairs would be an affirmative answer to the following question.

Question 2: *If* Γ *is a strictly quasi-Fuchsian group, then does* $\max(K_\Gamma^+, K_\Gamma^-) = D(L_\Gamma)$*?*

Note that there is no ambiguity in defining the length distortion function for a quasi-Fuchsian group since the regular set consists of two components, each kept invariant under the action of the full group. For Kleinian groups without an invariant simply connected component in their regular set there is some question as to how to define the length distortion function. Leaving that ambiguity aside for now, one still might hope that if Question 2 were true then this result would generalize to more complicated geometrically finite Kleinian groups uniformizing 3-manifolds with incompressible boundary. (Here "more complicated" means that Γ *is not* a B-group.) Unfortunately this seems unlikely, regardless of how one defines the length distortion function, as the example below demonstrates.

Example 5.4 Let M be a compact, orientable, irreducible 3-manifold with incompressible boundary, so that each component of the boundary is of genus $g \geq 2$. The manifold M is said to be *acylindrical* if every properly embedded incompressible annulus can be properly homotoped into the boundary of M. A celebrated result of Thurston [17] is that the space of marked hyperbolic 3-manifolds homotopically equivalent to M endowed and endowed with the algebraic topology is compact. Thurston further showed that there is a unique geometrically finite hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$ homeomorphic to M, so that the boundary of the convex core is totally geodesic. Thus regardless of which component subgroup G of Γ we consider, i.e. which component of the boundary of the convex core, then $K_G = 1$. But each such G is of infinite index in Γ (else Γ is either quasi-Fuchsian or a Z_2 extension of the Fuchsian group G and so Γ would not be acylindrical.) The group G is a finitely generated Fuchsian group and so is geometrically finite. Thus a result of Canary-Taylor [5] implies that $D(L_G) < D(L_\Gamma)$, and so the absence of bending on the boundary of the convex core *does not* imply that the Hausdorff dimension of the limit set of Γ is 1.

With a little more effort we can extract part of the argument in the example above allowing us to formulate a relative version of the Main Theorem for certain types of Kleinian groups that are not quasi-Fuchsian. We will assume that Γ is a purely loxodromic (torsion-free) Kleinian group with non-empty regular set, that Γ is non-elementary (i.e. Γ is not virtually abelian), and that

$$\overline{N_\Gamma} = (\mathbf{H}^3 \cup \Omega_\Gamma)/\Gamma$$

is a compact 3-manifold with incompressible boundary. In the following result, when we refer to "the average bending function restricted to a boundary component of the convex hull," we mean the average bending function defined asymptotically by the component subgroup of Γ acting precisely invariantly on the given boundary component.

Theorem 5.5 *Let Γ be as above, and let K be the maximum of the average bending function restricted to each boundary component of the convex hull of the limit set of Γ. Then $K \leq D(L_\Gamma)$; if $\overline{N_\Gamma}$ is not homeomorphic to $S \times [0,1]$ then $K < D(L_\Gamma)$.*

Proof: Part of the formulation we gave of Γ, i.e. that $\overline{N_\Gamma}$ is compact, is enough to assert that since Γ is purely loxodromic it is also convex co-compact ([9]); in particular Γ is geometrically finite. Let S be any component of $\partial\overline{N_\Gamma}$. Then the fact that S is incompressible implies that $\pi_1(S)$ is a surface group. As a finitely generated subgroup of a geometrically finite group, the component subgroup G of Γ representing S is in fact geometrically finite itself. Because G is a purely loxodromic geometrically finite surface group, it is quasi-Fuchsian (Theorem IX.D.21 in [10]).

A result in 3-dimensional topology (Hempel [7] Theorem 10.5) implies that either $G = \Gamma$, or G has infinite index in Γ. In the first case we are in the setting of the Main Theorem. If G is of infinite index in Γ, then we observe from a result of Canary-Taylor (Theorem 1 in [5]) that $D(L_G) < D(L_\Gamma)$. Again we run the Main Theorem to conclude that the length distortion function, restricted to the component of the boundary of the convex hull "facing" S, has a constant value $K_G^+ \leq D(L_G)$. As $\overline{N_\Gamma}$ is compact there are only finitely many such S, and the result follows. ■

6 Liouville Current and random geodesics

In this section we give a description of K_Γ^+ in terms of geodesic currents and random geodesics. Below is a brief discussion of geodesic currents but for complete reference see the paper by Bonahon [1].

Geodesic currents arise in the study of the collection of homotopy classes of closed geodesics on a closed hyperbolic surface.

If S is a closed hyperbolic surface then every homotopy class of closed curves corresponds to a unique multiple of a closed geodesic. Let $\tilde{S}$ be the universal cover of S and $G(\tilde{S})$ be the space of geodesics on $\tilde{S}$. We can identify a geodesic with its endpoints on S^1_∞ and therefore

$$G(\tilde{S}) \cong (\tilde{S}^1_\infty \times \tilde{S}^1_\infty - \text{ diagonal})/\mathbf{Z_2},$$

where $\mathbf{Z_2}$ acts by interchanging the endpoints.

We can lift a closed geodesic to get a discrete subset of $G(\tilde{S})$ which is $\pi_1(S)$ invariant. In this way we identify every homotopy class of closed curve on S with a $\pi_1(S)$ invariant discete subset of $G(\tilde{S})$ with a certain integral multiplicity. Equivelently we can identify the homotopy class with the Dirac measure it defines on $G(\tilde{S})$.

Definition: A *geodesic current* is a positive measure on $G(\tilde{S})$ that is invariant under the action of $\pi_1(S)$. The set $\mathcal{C}(S)$ of geodesic currents on S is endowed with the weak* uniform structure.

It can be shown that the length of a geodesic current is well-defined and a continuous function on $\mathcal{C}(S)$.

Associated with the hyperbolic structure on S is a natural geodesic current called the *Liouville current*, and denoted $\mathcal{L}$. It is the unique geodesic current (up to constant multiple) which is invariant under the action of the whole Möbius group on $G(\tilde{S})$. Also $\mathcal{L}$ is related to the measure M defined on $\Omega(2)$. From our discussion of geodesic flow (Section 4), we have that $\Omega(2) \cong (S^1_\infty \times S^1_\infty - \text{ diagonal}) \times \mathbf{R}$. Then we have that $dM = d\overline{\mathcal{L}} \times d\lambda$, where $\overline{\mathcal{L}}$ is the $\mathbf{Z_2}$ lift of $\mathcal{L}$ and λ is the Lebesgue measure on $\mathbf{R}$ (see Nicholls [12]).

If $(x, v) \in T_1(S)$, then we construct a closed geodesic $\alpha^t(x, v)$ by taking the unique closed geodesic in the homotopy class of the following closed curve. Letting K be the diameter of S, we take the geodesic arc of length t with initial tangent vector (x, v) and then join the endpoints by a geodesic segment of length less than or equal K. The following lemma informally says that a "random geodesic" on a closed hyperbolic surface is almost always equal (up to multiple) the Liouville current.

Lemma 6.1 *[1] If $\mathcal{L}$ is the Liouville current of a closed hyperbolic surface S then*

$$\lim_{t\to\infty} \frac{\alpha^t(x, v)}{t} = \frac{\mathcal{L}}{L(\mathcal{L})} \quad \text{a.e. w.r.t. measure } M$$

where $L(\mathcal{L})$ is the length of $\mathcal{L}$ in S.

Let Γ be a quasi-fuchsian group as above and let α be a non-trivial closed curve in C^+/Γ. Then we define $R^+(\alpha) = L^+(\alpha)/L(\alpha)$ where $L^+(\alpha)$ is the length of the closed geodesic homotopic to α in C^+/Γ and $L(\alpha)$ is the length of the closed geodesic homotopic to α in in H^3/Γ.

Lemma 6.2

$$\mathcal{R}^+(x, v) = \limsup_{t\to\infty} R^+(\alpha^t(x, v))$$

Proof: Let K be the diameter of C^+/Γ and $\overline{K}$ the diameter of the convex core $C(\Gamma)$ of Γ. Therefore by the triangle inequality

$$\rho^+(x, \exp tv) - K \leq L^+(\alpha^t(x, v)) \leq \rho^+(x, \exp tv) + K$$

In H^3/Γ we can join the ends with a geodesic arc of length less than or equal $\overline{K}$ and therefore

$$\rho(x, \exp tv) - \overline{K} \leq L(\alpha^t(x,v)) \leq \rho(x, \exp tv) + \overline{K}$$

Taking ratios we obtain

$$\frac{\rho^+(x, \exp tv) - K}{\rho(x, \exp tv) + \overline{K}} \leq \frac{L^+(\alpha^t(x,v))}{L(\alpha^t(x,v))} \leq \frac{\rho^+(x, \exp tv) + K}{\rho(x, \exp tv) - \overline{K}}$$

Taking the limit superior we get

$$\mathcal{R}^+(x,v) = \limsup_{t\to\infty} R^+(\alpha^t(x,v))$$

■

The function R^+ can be extended to a continuous function on the space of geodesic currents $\mathcal{C}(C^+/\Gamma)$ (see [1]). Therefore we have the following.

Corollary 6.3

$$\mathcal{R}^+(x,v) = \lim_{t\to\infty} R^+(x, exp_x(tv)) = R^+(\mathcal{L}) = K^+_\Gamma \ \ a.e.$$

Proof: As R^+ is continuous on the space of geodesic currents $\mathcal{C}(C^+/\Gamma)$ we have by the previous lemma

$$\lim_{t\to\infty} R^+(\frac{\alpha^t(x,v)}{t}) = R^+(\mathcal{L}) \ \text{ a.e}$$

As R^+ is a ratio $R^+(k\alpha) = R^+(\alpha)$ for any current α and therefore

$$\lim_{t\to\infty} R^+(\alpha^t(x,v)) = R^+(\mathcal{L}) \ \text{ a.e}$$

Thus by the above lemma we have

$$\mathcal{R}^+(x,v) = \limsup_{t\to\infty} R^+(\alpha^t(x,v)) = \lim_{t\to\infty} R^+(\alpha^t(x,v)) = R^+(\mathcal{L}) \ \text{ a.e}$$

Therefore $K^+_\Gamma = R^+(\mathcal{L})$ and by the existance of the limit almost everywhere we have $\mathcal{R}^+(x,v) = \lim_{t\to\infty} R^+(x, exp_x(tv))$ a.e.

■

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