Bounds on Volume Increase Under Dehn Drilling Operations

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Abstract

In this paper we investigate how the volume of a hyperbolic manifold increases under the process of removing a curve, that is, Dehn drilling. If the curve we remove is a geodesic we show that for a certain family of manifolds the volume increase is bounded above by $\pi \cdot l$ where l is the length of the geodesic drilled. We further construct examples to show that there is no lower bound to the volume increase in terms of a monotonic unbounded function of length. In particular this shows that volume increase is not bounded below linearly in length.

1 Introduction

Let M and M' be three-dimensional manifolds such that M' is homeomorphic to $M - \alpha$, where α is a simple closed curve in M. Then we say that M' is obtained by *Dehn drilling* on M and α is the curve drilled.

If M and M' are complete hyperbolic manifolds of finite volume then Gromov ([Th]) showed that the volume of M' is greater than the volume of M. Therefore hyperbolic Dehn drilling always increases volume. In the case of drilling out a geodesic it is natural to ask how the volume increase is related to the length of the geodesic drilled.

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Question: 1 *Is the volume increase under hyperbolic Dehn drilling of a geodesic bounded above by a linear function of the length of the geodesic ?*

Question: 2 *Is the volume increase under hyperbolic Dehn drilling of a geodesic bounded below by a linear function of the length of the geodesic ?*

For a certain class of hyperbolic spaces called *basic polyhedra* , we can define an operation called *surgery* that corresponds to Dehn drilling of a geodesic. In this paper we study this Dehn drilling operation. We prove that volume increase under surgery is bounded above by π times the length of the geodesic drilled, thereby giving a partial answer to Question 1. We further derive a combinatorial bound on the volume increase under surgery. We use this combinatorial bound to construct examples which show that volume increase under Dehn drilling is not bounded below by any monotonic unbounded function of the length of the geodesic drilled. This answers Question 2 in general.

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2 Motivation

The motivation for relating the volume increase to the length of the geodesic drilled is the following. If M' is obtained by hyperbolic Dehn drilling of a geodesic α in M then we assume that there is a continuous one-parameter family of hyperbolic cone-manifolds $M_\theta, 0 < \theta \leq 2\pi$ interpolating between M and M'. By interpolating we mean that $M_{2\pi} = M$, $\lim_{\theta \to 0} M_\theta = M'$ and M_θ is homeomorphic to M with cone axis α and cone angle θ.

If we have a one-parameter family of cone-manifolds interpolating between M and M', we can apply a result of Craig Hodgson ([Ho]) which describes the derivative of volume of a smooth one-parameter family of cone manifolds. This is based on the following theorem of Scläfli.

Theorem 1 *(Scläfli) Let* X_t *be a smooth one-parameter family of polyhedra in a simply connected* n*-dimensional space of constant curvature* K*. Then the derivative of the volume of* X_t *satisfies the equation:*

$$(n-1)KdVol(X_t) \quad = \quad \sum_F V_{n-2}(F)d\phi_F$$

where the sum is over all co-dimension two faces of X_t*,* V_{n-2} *is the* $(n-2)$*-dimensional volume and* ϕ_F *is the dihedral angle at* F*.*

The proof in the case of compact polyhedra can be found in [V]. In the cone manifold case we have the following theorem.

Theorem 2 *(Hodgson[Ho]) Let* C_t *be a smooth family of (curvature* K*) cone-manifold structures on a manifold with fixed topological type of singular locus. Then the derivative of volume of* C_t *satisfies*

$$(n-1)KdVol(C_t) \quad = \quad \sum_\Sigma V_{n-2}(\Sigma)d\phi_\Sigma$$

where the sum is over all components Σ *of the singular locus of* C *and* θ_Σ *is the cone angle along* Σ*.*

Applying this result to the one-parameter family of cone manifolds M_θ, where the singular locus is a single cone axis α, we get

$$\begin{aligned} dV = & \quad -\frac{1}{2}l(\theta)d\theta \\ \Delta V = & \quad \frac{1}{2}\int_0^{2\pi} l(\theta)d\theta \end{aligned} \qquad (1)$$

where θ is the cone angle along α, $l(\theta)$ is the length of α in M_θ and ΔV is the volume increase under Dehn drilling.

The approach of the paper is to first show that surgery can be interpolated. We then prove that for this interpolation the length of the cone axis $l(\theta)$ is monotonically increasing. If $l(\theta)$ is monotonic then by equation 1 we have $\Delta V \leq \pi \cdot l$, where l is the length of the original geodesic.

To show that no lower linear bound for volume increase exists, we use the interpolation of surgery to find a combinatorial bound on the length of the cone axis $l(\theta)$. Then applying equation 1 we obtain a

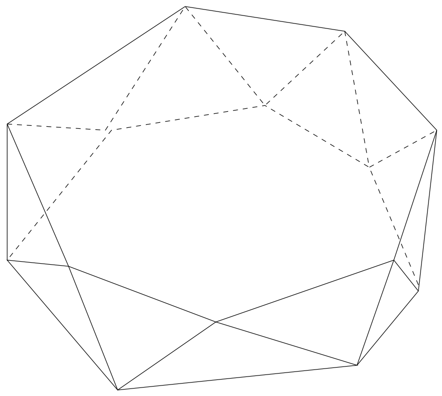


Figure 1: Ideal hyperbolic drum

combinatorial bound on volume increase that is independent of the length of the geodesic drilled. We then construct examples which are combinatorially bounded but whose geodesics have length tending to infinity.

3 Basic Polyhedra

The spaces we will consider are called *basic polyhedra*. A basic polyhedron is an ideal hyperbolic polyhedron P with all dihedral angles being $\pi/2$. Therefore all vertices have valence four. Let $\mathcal{B}$ be the set of all basic polyhedra. Examples of basic polyhedra are the ideal hyperbolic drums $\{T_n\}_{n\geq 3}$ which have two faces which are regular ideal n-gons, corresponding to the top and bottom of the drum and every other face is an ideal triangle (figure 1).

As all dihedral angles are $\pi/2$, basic polyhedra are orbifolds. In [B1] it was shown that if P is a basic polyhedron then it has a four-fold cover (in the orbifold sense) L_P where L_P is a hyperbolic link complement in S^3 having a boundary component for each vertex of P.

Surgery and Dehn drilling

If P is a basic polyhedron and F is a non-triangular face of P then we define *surgery* on P as follows. Choose two non-adjacent edges e_1, e_2 of F and pinch them together to form a new combinatorial polyhedron $\overline{P}$

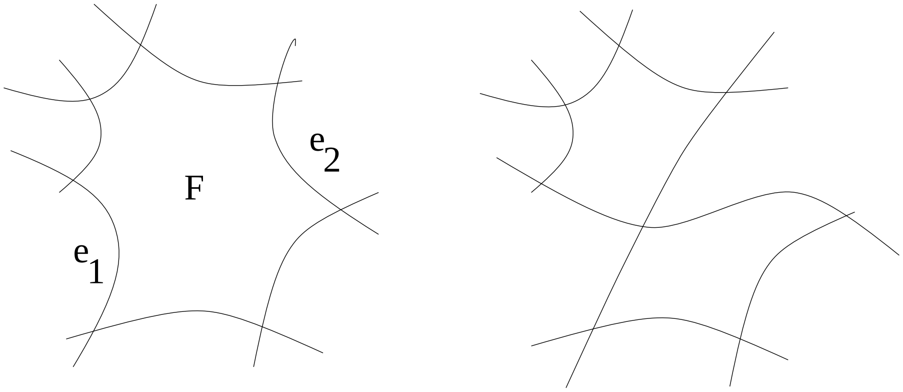


Figure 2: Surgery

(figure 2). By Andreev's theorem ([Th]), $\overline{P}$ has a realization P' which is an ideal hyperbolic polyhedron with all dihedral angles $\pi/2$ and therefore is also a basic polyhedron. We say P' is obtained by surgery on P. Therefore surgery is an operation defined on the set $\mathcal{B}$.

Considered as orbifolds, P' is obtained by Dehn drilling the unique geodesic perpendicular to both e_1 and e_2 in P. Furthermore, if we lift to the four-fold covers, we see that $L_{P'}$ is obtained by Dehn drilling a simple closed geodesic in L_P ([B2]). Thus surgery corresponds to a Dehn drilling operation.

Basic polyhedra have some interesting properties which are not needed in this paper but we will mention here. A partial ordering can be defined on $\mathcal{B}$ by saying $P \prec P'$ if P' is the result of repeated surgeries starting with P. In ([B2]) we show that this partial ordering has as initial objects the set $\{T_n\}_{n \geq 3}$ described above. Therefore we can enumerate all basic polyhedra by repeated surgery on these initial objects. Furthermore it is shown that this enumeration can be used to enumerate all link projections.

4 Interpolating Surgery

To consider how volume increases under surgery we first interpolate by a one-parameter family of cone manifolds. Let P be a basic polyhedron with F a non-triangular face of P and e_1, e_2 be two non-adjacent edges of F. Also let g be the unique geodesic in F perpendicular to e_1, e_2. To interpolate between P and

P' we need a continuous one-parameter family of polyhedra $P_\theta, 0 < \theta \leq \pi$, which have combinatorial type $P \cup \{g\}$, as well as the following properties. P_θ has dihedral angle θ along g, all other edges have the same dihedral angle as the corresponding edge of P, $P_\pi = P$, and $\lim_{\theta \to 0} P_\theta = P'$.

Intuitively, P_θ is obtained by bending P along g by an amount $\pi - \theta$. To find such a one-parameter family of polyhedra we use results of Rivin and Hodgson ([RH]) which characterize compact hyperbolic polyhedra in terms of their *spherical duals*.

Spherical Dual of a Polyhedron

If P is a hyperbolic polyhedron then its spherical dual P^* is a spherical complex combinatorially dual to the polyhedron P. P^* is constructed as follows. Each vertex v corresponds to a spherical polygon v^* with the length of the edges of v^* equal the exterior dihedral angles of the edges incident to v. Also the face angles of v^* are equal the complementary angles of the face angles at v (complement of $\alpha = \pi - \alpha$). Similarly an ideal vertex corresponds to a spherical hemisphere with vertices dividing the equator into arcs of length equal the exterior dihedral angles of the edges incident to the ideal vertex.

Let $S = P^*$. We define spherical complexes S_θ by cutting S open at e_1^*, e_2^* (dual edges to e_1, e_2), and attaching a spherical bigon B with angle $\pi - \theta$. B is further divided into two triangles T_1, T_2 by joining the two points on the equator by an edge g^*. If the interpolating polyhedron P_θ is realizable as a hyperbolic polyhedron then its spherical dual is S_θ with g^* dual to g. Note that each of e_1^*, e_2^* splits into two edges $e_i^*, \bar{e}_i^*, i = 1, 2$ (figure 3). Thus we have a continuous one-parameter family of spherical complexes $S_\theta, 0 \leq \theta \leq \pi$, which interpolates between the spherical duals of P and P'. Before we can interpolate between P and P' we need to know more about the map between hyperbolic polyhedra and spherical complexes.

The following theorem due to Igor Rivin and Craig Hodgson gives a characterization of compact hyperbolic polyhedra in terms of their spherical duals.

Theorem 3 *([RH]) A metric space (M, g) homeomorphic to S^2 can arise as the spherical dual P^* of a compact hyperbolic polyhedron P in H^3 if an only if the following hold.*

(a) The metric has constant curvature 1 away from a finite collection of cone points c_i.

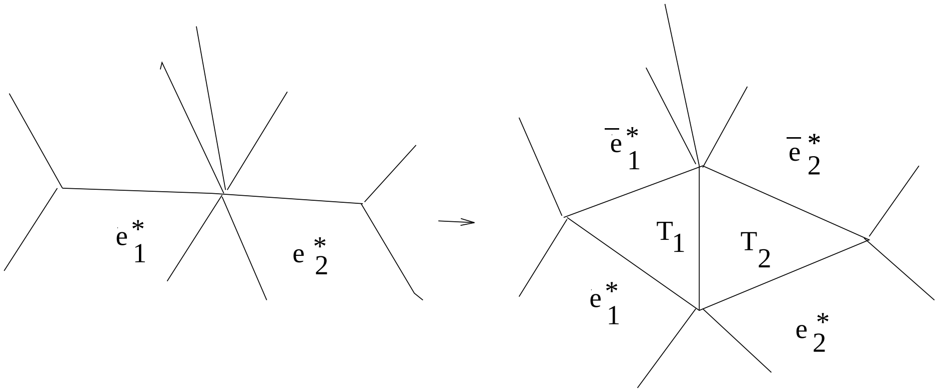


Figure 3: Cutting open S to make S_θ

(b) The cone angles at c_i are greater than 2π.

(c) The lengths of closed geodesics of (M, g) are all strictly greater than 2π.

Moreover P is uniquely determined.

Approximating by compact Polyhedra

Since the polyhedron P_θ is not compact we cannot apply the above result to show that it is realizable. We apply theorem 3 to find a family of compact hyperbolic polyhedra $P^t_\theta, t > 0$ approximating P_θ. This method of approximating is the method used by Rivin to approximate ideal hyperbolic polyhedra ([R]). First we define spherical complexes S^t_θ by modifying S_θ so that it is the dual of a compact hyperbolic polyhedron . To obtain S^t_θ, we increase the length of every edge of S_θ which does not belong to B by a factor of $(1 + t)$. Then we add a vertex v_i at the pole of each hemisphere H_i and join v_i by an edge to each vertex on the equator of H_i. Note that if we let $t = 0$ then $S^0_\theta = S_\theta$.

Lemma: 1 *Given $\theta, 0 < \theta \leq \pi$ then $\exists\ \delta > 0$ such that the spherical complex S^t_θ is the dual of a unique compact hyperbolic polyhedron P^t_θ for $0 < t < \delta$.*

Proof : Obviously S^t_θ is a spherical complex satisfying (a) and (b) of the characterization (theorem 3), and therefore we need only show that there are no geodesics of length $\leq 2\pi$. Let γ^t be a geodesic in S^t_θ. If

γ^t intersects the interior of a hemisphere H in a single arc then $l(\gamma^t|_{H^o}) = \pi$. Therefore if γ^t intersects more than two hemispheres in their interiors then $l(\gamma^t) \geq 3\pi$.

Case 1: γ^t doesn't intersect the interior of bigon B. Then we get a curve in S^t_π by removing the bigon B and stitching back the edges. This curve we also call γ^t and it is a geodesic in S^t_π.

If γ^t intersects the interior of two hemispheres H_1, H_2 and if γ^t doesn't contain an edge then H_1, H_2 intersect in points a distance $\geq \pi$ apart giving a contradiction. Thus γ^t must contain an edge and $l(\gamma^t) \geq 2\pi + \pi/2 > 2\pi$.

If γ^t intersects the interior of only one hemisphere H, then it must enter and leave H along an edge. We replace the arc in H^o by an edge path of the same length by going along the boundary of H. This gives us a new geodesic which we still call γ^t satisfying

$$l(\gamma^t) \quad \geq \quad \pi/2 + \pi/2 + \pi/2(1+t) + \pi/2(1+t) = 2\pi + t\pi > 2\pi$$

Therefore all geodesics in S^t_θ not intersecting the interior of B have length bounded away from 2π.

Case 2: γ^t intersects interior of B. There are a number of possibilities.

If γ^t enters B at either the apex of T_1 or T_2 then it has to leave at the other apex and $l(\gamma^t|_{B^o}) = \pi$. We replace this arc by edge path $e_1^* \cup e_2^*$ or $\overline{e}_1^* \cup \overline{e}_2^*$ depending on which gives a geodesic as a result (at least one must give a geodesic). Thus we have a geodesic which has the same length as γ^t and doesn't intersect the interior of B. Therefore γ^t has length $> 2\pi$ by case 1.

If γ^t intersects the interior of two hemispheres H_1, H_2 and both are not incident to B then γ^t must contain an edge of S^t_θ and therefore $l(\gamma^t) \geq 2\pi + \pi/2 > 2\pi$. If either H_1, H_2 are incident to B only at an apex then γ^t intersects B in an arc from one apex to the other and has been already shown to have length $> 2\pi$. Therefore either γ^t contains an edge and is longer than 2π or H_1, H_2 intersect at a point p along their boundaries where γ^t traverses from H_1 to H_2. But since H_1, H_2 correspond to hemispheres in S^t_π whose intersection is contained in $e_1^* \cup e_2^*$, H_1, H_2 cannot intersect at point p. Therefore any geodesic in S^t_θ intersecting the interior of two or more hemispheres has length bounded away from 2π.

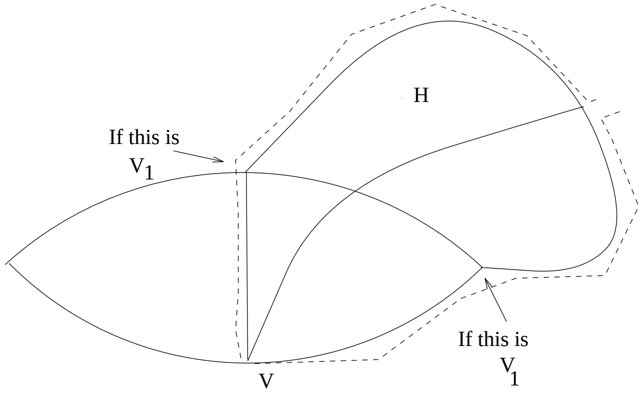


Figure 4: Replacing geodesic by edge geodesic

If γ^t only intersects the interior of one hemisphere H and it enters B along an edge and leaves along an edge then it must intersect B^o along the equatorial edge. Therefore there is a geodesic in S^t_π of length $l(\gamma^t) - (\pi - \theta)$ obtained by removing the bigon B as before and taking what is left of γ^t as the curve. This must be of length $> 2\pi$, therefore γ^t has length $\geq 2\pi + (\pi - \theta) > 2\pi$. If γ^t enters through H and leaves along an edge at vertex v of B (v is not an apex), then we first replace the interior arc in H by a boundary edge path b of H of the same length, we call this new curve γ^t also. Now γ^t contains two sides of a spherical triangle $\overline{T}$ in B. H intersects B in an edge e and let v_1 be the vertex of B contained in $b \cap e$. Now replace the part of γ^t consisting of two sides of the spherical triangle $\overline{T}$ by the edge of B joining v, v_1 (figure 4). This gives a geodesic of smaller length that is an edge path and therefore $l(\gamma^t) > 2\pi$.

If γ^t intersects no hemisphere in its interior then γ^t is an edge geodesic containing the equatorial edge of B. Removing the bigon we get a geodesic in S^t_π of smaller length which implies $l(\gamma^t) > 2\pi$.

Since no geodesic of S^t_θ has length $\leq 2\pi$, S^t_θ is the spherical dual of a unique compact hyperbolic polyhedron P^t_θ (by theorem 3). $\square$

To determine the combinatorial structure of P^t_θ, we need to find a cell division of S^t_θ into convex spherical polygons by geodesics of length less than π whose endpoints are cone points. This is how we described S^t_θ originally, but this may not be the only cell division possible. To show that it is, consider the apex v of one of the triangles T_1, T_2. The only other vertices a distance less than π from v are those vertices connected to v by an edge in the original cell division. Therefore the edges from v must be a subset of the original edges from v. Since the face angles at v must be less than π, the only edge of the original cell division that can be left out is one of the edges of B incident to v. But if the edge of B joining v, v_1 is left out then this forces a face aof at least π at v_1. Therefore all the edges of the original cell division incident to v are included. Similarly for the other apex $\overline{v}$. Now consider vertex v_1 on the equator of B. Vertex v_1 has an angle of π inside the bigon B. Therefore the equatorial edge of B must also be included in the division. Using the fact that any geodesic intersecting the interior of a hemisphere must have length $\geq \pi$ and that no vertex can have an angle of $\geq \pi$ subtended at it, it is easy to show that all other edges of the original division are necessary. There cannot be any more edges as any other edge from vertex to vertex would cross another edge in an interior point which can't happen. Therefore P_θ^t has dual given by the original cell division of S_θ^t.

A *generalized hyperbolic polyhedron* is when we allow vertices to be outside S_∞^2 in the Klein model. We call the vertices outside of S_∞^2 *hyper-ideal.* To each hyper-ideal vertex v we associate the plane perpendicular to it. This plane truncates the infinite volume end at v, meeting every plane it intersects orthogonally. In this way associated to a generalized hyperbolic polyhedron is a unique compact hyperbolic polyhedron obtained by truncating the hyper-ideal vertices.

Thus P_θ^t corresponds to a generalized hyperbolic polyhedron with combinatorial type $P \cup \{g\}$ and with the vertices corresponding to ideal vertices of P being hyper-ideal. Intuitively as t tends to 0 these hyper-ideal vertices tend to S_∞^2 and become ideal.

Continuous one-parameter family $X_{t,\delta}$

Lemma 1 shows that P_θ^t exists for $0 < t \leq k(\theta)$, $0 < \theta \leq \pi$ where $k(\theta) > 0$. We can choose $k(\theta) = c \cdot \theta$ where c is a positive constant. Rivin and Hodgson showed ([RH]) that the map from the space of compact hyperbolic polyhedra to their spherical duals is a homeomorphism and hence P_θ^t is a continuous two-parameter family of polyhedra in the above domain. When all dihedral angles of P_θ^t are acute, work of Alexandrof ([A]) shows that we obtain a unique limiting polyhedron as t tends to zero. Therefore

$$\lim_{t\to 0} P^t_\pi = P$$

Thus we extend P^t_θ by letting $P^0_\pi = P$. Similarly for $0 < \theta \leq \pi/2$, all dihedral angles are acute and we obtain

$$\lim_{(t,\theta)\to(0,0)} P^t_\theta = P'$$
$$\lim_{t\to 0} P^t_\theta = P_\theta$$

where P_θ is the unique hyperbolic polyhedron with combinatorial type $P \cup \{g\}$, dihedral angle θ along g and all other dihedral angles $\pi/2$.

We now choose a continuous one-parameter family of polyhedra $X_{t,\delta}$ interpolating between P and P' by taking the image of a piecewise linear path joining $(0,\pi)$ and $(0,0)$ in the domain of $P^{(\cdot)}_{(\cdot)}$. $X_{t,\delta}$ is the product of three continuous one-parameter families $X^i, i = 1,2,3$ obtained by taking the image of the path joining the four points $(0,\pi), (t,\pi), (t,\delta), (0,0)$ by straight lines.

The volume change ΔV from P to P' satisfies

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$$

where ΔV_i is the volume change over X^i.

By continuity we have

$$\begin{aligned} |\Delta V_1| &\leq \epsilon_1(t) \\ |\Delta V_3| &\leq \epsilon_3(t,\delta) \\ |l^t(\pi) - l^0(\pi)| &\leq \epsilon_g(t) \end{aligned}$$

where $l^t(\theta)$ is the length of the geodesic g in P^t_θ.

Therefore using the Scläfli formula we have that

$$\begin{aligned} \Delta V_2 = &\quad \tfrac{1}{2}\textstyle\int_\delta^\pi l^t(\theta)d\theta \\ \Rightarrow \Delta V = &\quad \tfrac{1}{2}\textstyle\int_\delta^\pi l^t(\theta)d\theta + \overline{\epsilon}(\delta,t) \qquad (2) \\ \Rightarrow \Delta V = &\quad \lim_{(t,\delta)\to(0,0)} \tfrac{1}{2}\textstyle\int_\delta^\pi l^t(\theta)d\theta \qquad (3) \end{aligned}$$

We will now investigate how the function l^t behaves and apply equation 3 to get bounds on the volume increase.

5 Linear Upper Bound on Volume Increase

In this section we will prove the following.

Theorem 4 *Surgery increases volume by at most* $\pi/2 \cdot l$ *where* l *is the length of the geodesic drilled.*

The proof involves showing that the function l^t is monotonically increasing as a function of dihedral angle. Then by applying equation 3 we obtain the above result.

Monotonicity

The following theorem is an extension of Cauchy's topological lemma and is the step needed to prove monotonicity, but first a definition.

Definition: 1 *A* signed polyhedron P *is a polyhedron whose edges are labelled with either* $+$, $-$ *or* 0.

A vertex v of a signed polyhedron has *type* given by listing the signs on the edges incident to v in a clockwise or anti-clockwise manner (with the usual equivalence) and a face F is called a zero-face if all its edges are marked with zero.

Theorem 5 *Let* P *be a trivalent signed polyhedron such that, except for two disjoint faces* F_1, F_2*, every face is either a zero-face or has at least four sign changes. Then the space obtained by collapsing all zero-edges* $\tilde{P}$ *is topologically a bouquet of spheres where either*

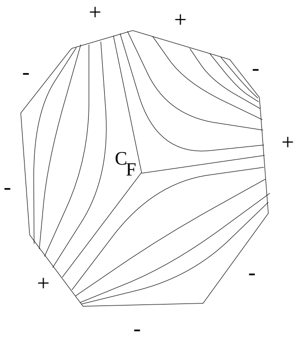
(1) Each sphere has the trivial cell division (a point and a disc).

or

(2) There is exactly one sphere $\tilde{S}$ in the bouquet with a non-trivial cell division. $\tilde{S}$ has two cells with no sign changes and every other cell has exactly four sign changes. Also all vertices of $\tilde{S}$ have type $(++-)$, $(--+)$ or $(+-+-)$ and therefore valence three or four.

Proof : Let Σ be the set of edges of P and $\Sigma = \Sigma_+ \cup \Sigma_- \cup \Sigma_0$ be the partition into $+$, $-$ and 0 edges. If $|\Sigma_0| = 0$ then construct a vector field on P as follows. In a neighborhood of Σ the vector field is transverse to Σ_+ and parallel to Σ_-. The index of a vertex v can easily be calculated from its type. One way is to read the type of a vertex from left to right, count $-1/4$ for sign changes, $-1/2$ for same sign and then add 1. This is never positive if the valence is at least three and if the valence of v is greater than four its index is always negative. In our case the vertices all have valence three but in general only vertices of type $(++-), (--+)$ and $(+-+-)$ (which have index zero), have non-negative indices.

The vector field is extended over a face F as follows. As we go round the edges of F the sign changes $2n$ times , which means that there are n disjoint $+$ segments joined together by n disjoint $-$ segments. Choose a center point c_F in the interior of F and for each $+$ segment we join an interior point of it to the center c_F by a prong and extend the vector field in a neighborhood of these n prongs in the obvious way. Cutting along the prongs divides F into n pieces over which the vector field can easily be extended. The index of the critical point c_F is $1 - n/2$ (figure 5). Note that if the face has no sign changes then the vector field can still be extended with c_F having index 1, in keeping with the formula. The interior of each face has at most one critical point and all other critical points are vertices of P. Each vertex contributes a non-positive index and each face $F \neq F_1, F_2$ has at least four sign changes, which implies c_F has non-positive index. In order to have the sum of the indices equal to two (the Euler number of P), the indexes of both c_{F_1}, c_{F_2} must be one and therefore they are of constant sign. Also all vertices and centers c_F , $F \neq F_1, F_2$ must have index zero. Therefore all vertices have type $(++-), (--+)$ and each F has exactly four sign changes.



3 pronged face F

C_F has index -1

Figure 5: Extend vector field over face

If $|\Sigma_0| \neq 0$ then we cannot find a vector field extending over Σ_0. We avoid this problem by collapsing these zero-edges. First consider $\overline{\Sigma}_0 \subseteq \Sigma_0$ to be the collection of edges belonging to loops in Σ_0. This decomposes P into regions $\{R_i\}$ which have disjoint interiors and are n_i-connected regions whose boundaries consist of zero-edges. Some regions correspond to the zero-faces of P. We collapse Σ_0 by first collapsing $\overline{\Sigma}_0$. Each R_i becomes a sphere S_i. If R_i is a zero-face then S_i has the trivial cell division of a point and a disc. If R_i is not a zero-face then each component β_{ij} of ∂R_i, collapses to a vertex v_{ij} on a sphere S_i. As R_i is not a face there is an edge path γ_{ij} splitting R_i into two regions R^1_{ij}, R^2_{ij} with endpoints on β_{ij}. If there are no other paths from β_{ij} to the interior then either R_i is the union of two faces or there are two faces which intersect in two separate edges (figure 6). Both give contradictions. Therefore collapsing β_{ij} gives vertex v_{ij} with valence at least three. Thus the vertices on S_i after collapsing $\overline{\Sigma}_0$, have valence at least three.

If we now collapse $\Sigma_0 - \overline{\Sigma}_0$ then each sphere S_i collapses to give another sphere $\tilde{S}_i$ in a bouquet of spheres $\tilde{P}$. The valence of a vertex in $\tilde{S}_i$ must be as big as any vertex that collapses onto it as the only zero-edges left on S_i form a collection of disjoint trees.

After collapsing Σ_0 we get a bouquet of spheres $\tilde{P}$. The zero-faces become spheres with the trivial cell division of a point and a disc and every other face F collapses to a cell $\tilde{F}$ of some $\tilde{S}_i$.

Therefore if $\tilde{S}_i$ does not correspond to a zero-face it has a decomposition into cells all but two of which have at least four sign changes and all vertices have valence at least three. Thus as before we can give $\tilde{S}_i$ a

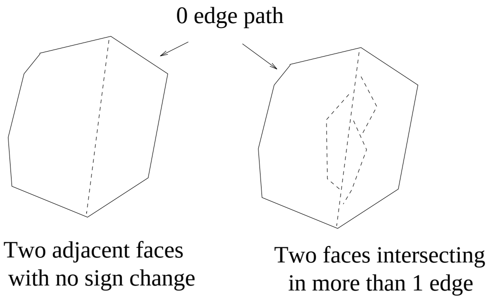


Figure 6: Valence after collapse is at least three

vector field which implies that there can be at most one sphere $\tilde{S}$ which has a non-trivial cell division. On $\tilde{S}$ the vertices all have type $(++-), (--+)$ or $(+-+-)$, the faces corresponding to F_1, F_2 give rise to cells $\tilde{F}_1, \tilde{F}_2$ of $\tilde{S}$ which have constant sign on their boundaries and all other cells have exactly four sign changes. □

Now we apply the above theorem. Let $\theta_1 < \theta_2$ and compare the corresponding faces of $P^t_{\theta_1}$ and $P^t_{\theta_2}$. Apart from the two faces F_1, F_2 which are incident and perpendicular to g , all other faces have identical angles. Let Q be the marked polyhedron combinatorially equivalent to both $P^t_{\theta_1}$, $P^t_{\theta_2}$ and marked with $+$, $-$ or 0 depending on whether the length of the edge in $P^t_{\theta_2}$ is bigger, smaller or equal in length to the corresponding edge of $P^t_{\theta_1}$. In order to show that Q satisfies the hypothesis of the above theorem we need the following lemma.

Lemma: 2 *([RH]) Let $E_1, E_2, \ldots, E_n$ be lines in H^2 defining a convex polygonal curve C. Let $A_i = E_i \cap E_{i+1} (i = 1, \ldots, n-1)$ be the vertices of C , let $\nu_i = E_i^{\perp} (i = 1, \ldots, n)$ be the outward unit normals to the Minkowski inner product $\langle \cdot, \cdot \rangle$, and let l_i be the length of $E_i \cap C = A_{i-1}A_i (i = 1, \ldots, n-2)$. Let $E'_1, E'_2, \ldots, E'_n$ be lines in H^2 defining another such curve C' with A'_i, ν'_i, l'_i defined as above. Assume that*

(i) $< A_i =< A'_i$ for $i = 2, \ldots, n-1$ and

(ii) $l'_i \geq l_i$ for $i = 2, \ldots, n-1$.

Then

$$\langle \nu'_1, \nu'_n \rangle \quad \leq \quad \langle \nu_1, \nu_n \rangle,$$

with equality if and only if $A_1A_2...A_{n-1}$ *is congruent to* $A'_1A'_2...A'_{n-1}$.

An easy consequence of this lemma is the following. Let F_a, F_b be two hyperbolic n-gons with face angles equal. If we label an n-gon F with $+$, 0 or $-$ by comparing lengths of corresponding sides of F_a, F_b, then either F has all sides marked with 0, in which case F_a, F_b are congruent, or, F has at least four sign changes as we go around the perimeter.

Thus Q satisfies the hypothesis of theorem 5 and therefore both faces F_1, F_2 have no sign changes.

To show that F_1, F_2 are marked with $-$, we need to relate the Minkowski inner product to face angles. If two edges of a compact hyperbolic polygon intersect in an angle $\theta < \pi$ then the inner product of their outward normals is $\cos(\theta_{ext})$ where θ_{ext} is the exterior angle of θ. Now the convex polygons p_1, p_2 corresponding to F_1 in $P^t_{\theta_1}, P^t_{\theta_2}$ respectively, have the same face angles except at the vertex incident to g, where p_i has the angle θ_i. Thus if the sides of p_2 are all bigger or equal in length to the sides of p_1 then we have by the above lemma that the inner product of the outward normals of the two sides intersecting at g in p_2 is less than or equal to the same inner product in p_1. Therefore $\cos(\pi - \theta_2) \leq \cos(\pi - \theta_1)$. But $\theta_1 < \theta_2$, and therefore the sides of p_2 must be less than or equal in length to the corresponding sides in p_1. This gives the needed step to show that both F_1, F_2 are of constant sign $-$.

To show that g is marked with $+$ note that if the edge g is marked with $-$ then we get two vertices of type $(- - -)$ and if the edge g is 0 then after collapsing we get a vertex of type $(- - - -)$, both giving contradictions.

Therefore g must be marked with $+$ and thus l^t is monotonically increasing in θ.

Theorem 5 gives more information about the pattern on a marked polyhedron P. The region R corresponding to $\tilde{S}$ is obtained by removing the zero-faces of P. When we collapse the boundary curves of R to get sphere S then any vertex of S has valence three or four. Any zero-edge path of more than one edge will give a vertex in $\tilde{S}$ with valence greater than four and therefore every zero-edge in S is isolated. These isolated zero-edges in S must join two trivalent vertices which have alternating signs as you go around a neighborhood of the zero-edge.

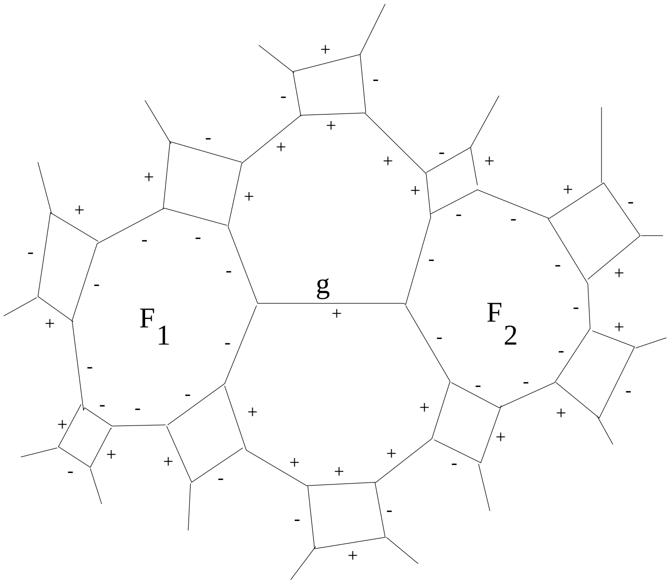


Figure 7: How edge lengths in P^t_θ vary with θ

In the case of the markings on Q we obtain figure 7 which shows that the length of many edges are monotonic.

Proof of Theorem 4: By the monotonicity of l^t and equation 2 we have that

$$\Delta V \leq \frac{1}{2}(\pi - \delta) l^t(\pi) + \epsilon(\delta, t)$$

where ΔV is the volume increase under surgery on P and $\epsilon(\delta, t)$ tends to 0 as both δ, t tend to 0 .

Also by continuity

$$|l^t(\pi) - l^0(\pi)| \leq \epsilon_g(t)$$

where $l^0(\pi) = l$ is the length of the geodesic drilled in P and $\epsilon_g(t)$ tends to 0 as t ends to 0. Thus taking the limit as δ, t both tend to 0 we get the bound

$$\Delta V \leq \frac{\pi}{2} \cdot l$$

□

6 Combinatorial Bound on Volume Increase

Here we will show that bending along a geodesic g on a face F increases volume at most linearly in the number of edges of F.

The idea is that as the dihedral angle θ at g varies, F is still intrinsically an ideal n-gon. If the length of g, $l(\theta)$ is long then this forces F to be narrow somewhere transverse to g. This implies that the two faces F_1, F_2 obtained by bending F along g (which meet with dihedral angle θ), have faces F_3, F_4 adjacent to F_1, F_2 respectively which are connected by a short path on the bent face F. As F_3, F_4 cannot intersect, this shows that $l(\theta)$ must decrease as we bend otherwise F_3, F_4 would be forced to intersect.

Long Geodesics give Thin Polygons

An ideal hyperbolic quadrilateral has two perpendiculars and it can be shown using elementary hyperbolic geometry that the hyperbolic sine's of the half lengths of these are reciprocal. The following lemma is a generalization of this fact.

Lemma: 3 *If F is an ideal hyperbolic n-gon and g a perpendicular between two edges of F then $\exists$ edges e_1, e_2 separated by g with distance d between them satisfying*

$$\sinh(d/2) \cdot \sinh(\frac{l}{2(n-3)}) \leq 1 \tag{4}$$

where l is the length of g.

Proof :

If e is one of the $n-2$ edges of F not intersecting g, orthogonally project e onto g to form an interval I_e. The endpoints of these $n-2$ intervals split g into at most $n-3$ subintervals. Therefore there is a subinterval I of length at least $l/(n-3)$ and there are two edges e_1, e_2 separated by g with $I = I_{e_1} \cap I_{e_2}$.

Let Q be the ideal quadrilateral with e_1, e_2 as opposite sides and call the other sides α_1, α_2. Also let Q_I be the ideal quadrilateral with opposite sides being β_1, β_2, the perpendiculars to g through the endpoints of I. We label the other sides of Q_I by $\overline{e_1}, \overline{e_2}$ where e_i separates $\overline{e_i}$ and g. Then any geodesic between α_1, α_2

intersects both β_1, β_2 and therefore the distance D between α_1, α_2 is greater than or equal to the distance between β_1, β_2. As the distance between β_1, β_2 is at least $l/(n-3)$, we have

$$D \geq \frac{l}{(n-3)}$$

Therefore if d is the distance between α_1, α_2 then

$$\begin{aligned} \sinh(d/2) \cdot \sinh(D/2) &= 1 \\ \Rightarrow \sinh(d/2) \cdot \sinh(\frac{l}{2(n-3)}) &\leq 1 \end{aligned}$$

□

Thin Bent Polygon Forces Intersection

Consider Q an ideal hyperbolic quadrilateral in H^2 with edges e_1, e_2, e_3, e_4 and g the geodesic perpendicular to edges e_3, e_4 (and therefore separating e_1, e_2). Furthermore let Q be embedded in H^3 bent along g with dihedral angle θ. To each edge e_1, e_2 associate a plane F_{e_1}, F_{e_2} which intersects Q orthogonally in the edge e_1, e_2 respectively.

Lemma: 4 *$F_{e_1} \cap F_{e_2} = \emptyset$ if and only if*

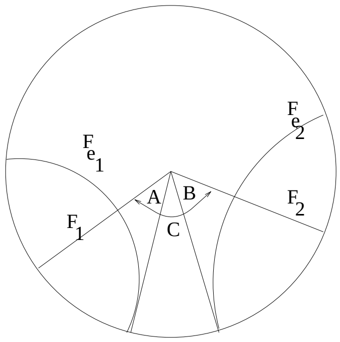
$$l \leq 2\sinh^{-1}(tan(\theta/2)) \tag{5}$$

where l is the length of g.

Proof :Q lies on the union of two planes F_1, F_2 in H^3 which intersect along g with dihedral angle θ. We first move the midpoint of g to the origin in the Poincare model of H^3 and consider the hyperbolic plane H through the origin perpendicular to Q. The planes F_{e_1}, F_{e_2} intersect H in two geodesics γ_1, γ_2 and F_1, F_2 intersect in two geodesics passing through the origin, perpendicular to γ_1, γ_2 respectively (figure 8).

From the origin in H, both γ_1, γ_2 have the same visual angle ϕ. Using planar hyperbolic geometry we obtain

$$\tan(\phi/2) = \sinh(l/2)$$



A + B < C

Figure 8: Sideview of the planes

where l is the length of the geodesic perpendicular to e_1, e_2 of Q. F_{e_1}, F_{e_2} are disjoint if and only if

$$\begin{aligned} \phi/2 &\leq \theta/2 \\ \Rightarrow \tan(\phi/2) &\leq \tan(\theta/2) \\ \Rightarrow \sinh(l/2) &\leq \tan(\theta/2) \\ \Rightarrow l &\leq 2\sinh^{-1}(tan(\theta/2)) \end{aligned}$$

□

Applying the above lemma we get

Theorem 6 *Let ΔV be the volume increase under surgery . Then*

$$\Delta V \leq 8(n-3) \tag{6}$$

where n is the number of edges of the pinched face.

The proof is contained in the next section. The idea of the proof is as follows. Let P' be obtained by surgery on P and assume that we can interpolate by a continuous one-parameter family P_θ. Considering the face F which is being bent in P_θ we note that the faces of P_θ incident to F must have disjoint interiors. Therefore we can apply the previous lemma to get a bound on the length function $l(\theta)$. Thus since $\sinh^{-1}(\tan(\theta/2))$ is integrable on $(0,\pi)$ with integral bounded by four, we obtain the bound on ΔV by integration.

The following proof involves modifying the previous lemmas to deal with the one-parameter family $X_{t,\delta}$ but the idea is still the same.

7 Proof of Combinatorial Bound

If P' is obtained by surgery on P the we can interpolate using the one-parameter family $X_{t,\delta}$ for some small t,δ. As before $X_{t,\delta}$ is the product of three one-parameter families X^i with volume increase ΔV_i. By continuity we have

$$\begin{aligned} \Delta V_2 = &\quad \tfrac{1}{2}\textstyle\int_\delta^\pi l^t(\theta)d\theta \\ \Delta V = &\quad \lim_{(t,\delta)\to(0,0)} \Delta V_2 \end{aligned}$$

where $l^t(\theta)$ is the length of the axis g in P^t_θ.

Therefore to find the combinatorial upper bound on volume we must consider the one-parameter family X^2. For simplicity of notation we will consider the one-parameter family $\overline{X}^2$ obtained by reversing the parametrization of X^2. Therefore $\overline{X}^2(\theta) = P^t_\theta$ where $\delta \leq \theta \leq \pi$. Let F be the ideal n-gon of P on which surgery is performed and $e_1,\ldots,e_n$ be its edges. Then $\overline{X}^2(\theta)$ has a right-angled 2n-gon with edges $e_1, f_1, \ldots, e_n, f_n$, where e_i correspond to the edges of F and f_i is the truncating edge between e_i and e_{i+1}. By continuity we have that

$$|L^t(f_i,\pi)| \leq \epsilon_{f_i}(t)$$

$$|l^t(\delta)| \geq \epsilon(\delta)$$

where $L^t(f_i,\theta)$ is the length of f_i in P^t_θ.

The following is a generalization of lemma 3.

Lemma: 5 *Let F be a right-angled $2n-$gon with edges $e_1, f_1, \dots, e_n, f_n$ and g a perpendicular to edges e_i, e_j of F then $\exists$ edges e_k, e_l separated by g with distance d between them satisfying*

$$\sinh(d/2) \cdot \sinh(\frac{l}{2(n-3)} - \epsilon) \leq 1 \tag{7}$$

where l is the length of g and $\epsilon = \max\{length(f_i)\}_{i=1}^{n}$.

Proof : For each edge f_i let v_i be the point on S^1_∞ obtained by continuing the unique perpendicular from g to f_i. Take v_i as the vertices of an ideal $n-$gon $\overline{F}$ and label the edges $\overline{e_i}$. The geodesic perpendicular to edges $\overline{e_i}, \overline{e_j}$ is of length l. Therefore applying lemma 3 we get edges $\overline{e_k}, \overline{e_l}$ whose projections onto g overlap in an interval of length D where $D \geq \frac{l}{(n-3)}$. Let $\epsilon = \max\{length(f_i)\}_{i=1}^{n}$, then since projection is distance decreasing the projections of e_k, e_l must overlap in an interval of length at least $D - 2\epsilon$ giving the above result. □

Proof of Theorem 6: We first apply the above lemma to the one-parameter family $\overline{X}^2$. As before note that $\overline{X}^2(\theta)$ has a right-angled $2n-$gon F with edges $e_1, f_1, \dots, e_n, f_n$. Also we note that by theorem 5 both $l^t(\theta)$ and $L^t(f_i, \theta)$ are monotonically increasing in θ. Given a δ choose t so that

$$\max\{L^t(f_i, \pi))\}_{1=1}^{n} < \epsilon_1 < \epsilon(\delta)/(n-3)$$

The monotonicity of $L^t(f_i, \theta)$ ensures that the lengths of the edges f_i will be less than ϵ_1 in $\overline{X}^2(\theta)$. F is embedded onto two planes F_1, F_2 in H^3 which meet in geodesic g with dihedral angle θ. From the above lemma there are two edges e_1, e_2 of F which are separated by g such that two perpendiculars to g with distance greater than $l^t(\theta)/(n-3) - 2\epsilon_1$ intersect both e_1, e_2. We obtain the geodesics $\overline{e_1}, \overline{e_2}$ on F_1, F_2 by joining the endpoints of the two perpendiculars to g.

In P^t_θ there are two faces F_{e_1}, F_{e_2} which intersect F along edges e_1, e_2 respectively in acute dihedral angles. We can associate to geodesics $\overline{e_1}, \overline{e_2}$ two planes $F_{\overline{e_1}}, F_{\overline{e_2}}$ intersecting F at right angles.

The faces F_{e_1}, F_{e_2} of P^t_θ are disjoint. But if $F_{\overline{e_1}}, F_{\overline{e_2}}$ intersect then so do F_{e_1}, F_{e_2}. Therefore a necessary condition on P^t_θ is that $F_{\overline{e_1}}, F_{\overline{e_2}}$ do not intersect. Thus we have that

$$\sinh(\frac{l^t(\theta)}{2(n-3)} - \epsilon_1) \leq \tan(\theta/2)$$

$$l^t(\theta) \leq 2(n-3)(\sinh^{-1}(\tan(\theta/2)) + \epsilon_1)$$

Therefore integrating from δ to π we see that

$$\Delta V_2 = \int_\delta^\pi l^t(\theta) d\theta \leq 2(n-3)\left(\int_\delta^\pi \sinh^{-1}(\tan(\theta/2)) d\theta + \epsilon_1\right)$$

The function $\sinh^{-1}(tan(x/2))$ is integrable over $(0, \pi)$ with a bound of 4. Therefore applying formula 3 we get

$$\Delta V \leq 8(n-3)$$

□

8 Non-existence of lower bound for volume increase

By a lower bound for volume increase we mean a lower bound given by some function f of the length of the geodesic drilled such that f is monotonically increasing to infinity. If the volume increase under Dehn drilling is bounded below by f then

$$\Delta V \geq f(l)$$

where l is the length of the geodesic drilled. We have shown in the previous section, that the volume increase by drilling out a geodesic belonging to an ideal n-gon of a basic polyhedron is bounded linearly in n. Therefore to prove that no lower bound exists it suffices to find a sequence of basic polyhedra each containing a face of bounded valence and such that the set of lengths of geodesics perpendicular to two sides is unbounded. Let $\{P_k\}_{k=0}^\infty$ be a sequence of basic polyhedra and N a number such that each P_k has a marked face F_k which is an n_k-gon, $n_k < N$. Also let l_k be the length of the longest geodesic α_k in F_k perpendicular to two edges of F_k. Then the volume increase ΔV_k, by drilling out α_k is bounded above by some constant K depending only on N. If the sequence l_k is not bounded then by the lower bound for volume increase $\Delta V_k \geq f(l_k^a)$. Therefore ΔV_k is unbounded giving a contradiction. We now construct the required family using circle packings.

Circle Patterns and Packings

If P is a hyperbolic polyhedron then in the upper half-space model each face F of P lies on a hemisphere H_F which intersects S^2_∞ in a circle C_F. Furthermore the angle at which two circles C_{F_1}, C_{F_2} intersect equals the dihedral angle between the faces F_1, F_2. Thus looking for a polyhedron of prescribed dihedral angles and combinatorics is equivalent to finding a *circle pattern* with prescribed angles of intersection and combinatorial pattern. We will use the terminology *circle packing* to refer to a collection of circles with intersections being tangential.

In the case of constructing a basic polyhedron we require that the circles meet in groups of four with opposite circles tangent and neighboring circles meeting orthogonally .

We start our circle pattern with a circle C_0 of radius one centered at the origin O. We further take circles C_1, C_1' of radius r such that both are orthogonal to C_0 and have centers on the positive, negative real axis respectively. Thus there are unique circles C_2, C_2' which are orthogonal to C_0, with centers on the positive, negative imaginary axis respectively and are tangent to both C_1, C_1'. This configuration of five circles depends only on our choice of r and therefore we have a one-parameter family $\mathcal{C}_r$ of circle patterns. By simple planar geometry the radius of both C_2, C_2' is $1/r$. This collection of five circles describes an ideal quadrilateral on the hyperbolic plane H_0 associated with circle C_0. Furthermore r is a coordinate system for the shape of the quadrilateral with it becoming narrow in the horizontal direction as r tends to infinity and becoming narrow in the vertical direction r tends to 0.

To extend $\mathcal{C}_r$ to the circle pattern of a basic polyhedron we perform a moebius transformation M on $\mathcal{C}_r$. M consists of a rotation of $\pi/2$ about the origin followed by an expansion by a real number k_r. The factor k_r is chosen so that $M(C_1)$ is tangent to the original C_1 and a simple calculation shows that $k_r = r^2 + \sqrt{r^4 - 1}$. Thus for r large these four tangent points are the only intersections between $\mathcal{C}_r$ and $M(\mathcal{C}_r)$ and so we extend circle pattern $\mathcal{C}_r$ by adding $M(\mathcal{C}_r)$ to form circle pattern $\overline{\mathcal{C}_r} = \mathcal{C}_r \cup M(\mathcal{C}_r)$ (figure 9).

Now take the collection of hemispheres $\mathcal{H}_r$ associated with $\mathcal{C}_r$ and let $\overline{\mathcal{H}_r} = \mathcal{H}_r \cup M(\mathcal{H}_r)$. Let N_r be the region between the two hemispheres $H_0, M(H_0)$ cut out by $\overline{\mathcal{H}_r}$. N_r is an infinite volume hyperbolic

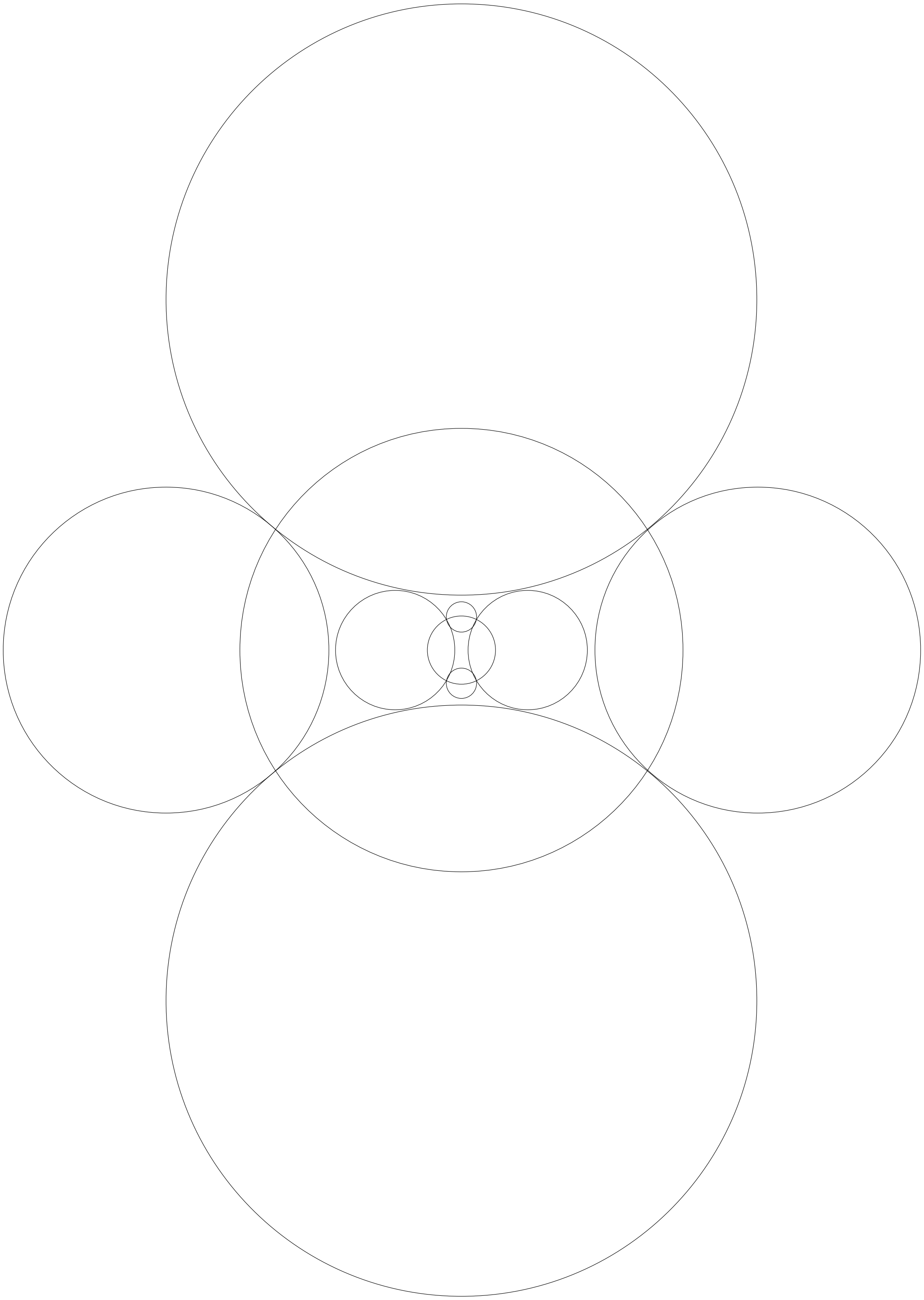


Figure 9: Circle pattern

orbifold with four infinite volume funnels. Each of these funnels meet S^2_∞ in a quadrilateral bounded by circles which are tangent at points of intersection. It can be easily shown that all four of these quadrilaterals are equivalent up to moebius transformation to a quadrilateral Q_r.

To form a basic polyhedron from N_r we must truncate the funnels of N_r. We do this by first finding a packing of Q_r by a finite number of circles such that the interstices are all triangular. Then having packed each of the four copies of Q_r, we are left with a finite number of triangular interstices which we truncate by adding the unique circle passing through the three vertices of each interstice. These circles are also perpendicular to the three sides of the interstices and therefore we have truncated the four infinite volume funnels of N_r to get a basic polyhedron P_r.

Unfortunately it is not always possible to find a finite packing of an arbitrary circular quadrilateral but using work of Robert Brooks([Br]) we will show that there is a sequence $\{r_n\}$ such that Q_{r_n} is packable and r_n monotonically tends to infinity.

We define the continued fraction $c(Q)$ of a quadrilateral Q by using the greedy algorithm. Assume that the four sides have been labelled left, right, top, bottom. We first pack Q from left to right by starting at the left side of Q and packing in the largest circle tangent to the left, top and bottom of Q. Now Q contains a smaller quadrilateral Q_1 which we again pack with the largest circle tangent to the left, top and bottom sides. After n_1 circles have been added the remaining quadrilateral Q_{n_1} is too narrow to fit any more circles which are tangent to the left, top and bottom so we now pack Q_{n_1} from top to bottom with circles tangent to the top, left and right sides. Again after n_2 circles have been packed into Q_{n_1} we are left with a quadrilateral Q_{n_1,n_2} too wide to pack with a circle tangent to the top, left and right. The packing algorithm is defined recursively by now applying the algorithm to the quadrilateral Q_{n_1,n_2} (figure 10). To each quadrilateral we get a sequence of integers $\{n_i\}$ and we define the function c on the space of quadrilaterals to be the real number with these as its continued fraction expansion, that is

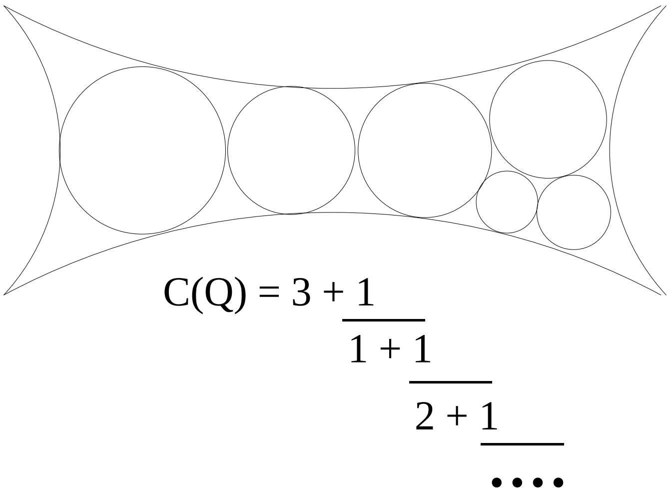


Figure 10: Greedy algorithm

$$c(Q) = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cdots}}}$$

Note that the greedy algorithm gives a finite packing of Q if and only if $c(Q)$ is rational. Now let $\mathcal{S}$ be the space of all configurations of four circles up to moebius transformation which have disjoint interiors and such that the complement of the interiors is two circular quadrilaterals. Then Robert Brooks showed ([Br]) that the map $\overline{c} : \mathcal{S} \to \Re^2$ defined by mapping a configuration of circles to the pair of continued fractions of its two circular quadrilateral regions, is not only continuous but a homeomorphism. This implies that c is continuous and therefore $c(Q_r)$ is a continuous function of r. It can be shown that as r tends to infinity the modulus of Q_r tends to infinity. Also $c(Q_r)$ tends to infinity as r tends to infinity and by continuity there exists a sequence $\{r_n\}$ such that $c(Q_{r_n}) = n$. Therefore Q_{r_n} can be packed with n circles from left to right.

Now basic polyhedron P_n is defined by truncating the funnels of N_{r_n} as described above. Figure 11 is a combinatorial picture of P_n. The polyhedra $\{P_n\}$ have the property of each containing an ideal quadrilateral face F_n whose modulus tends to infinity as a function in n. Thus F_n contains a geodesic α_n, perpendicular to two sides of F_n whose length monotonically increases to infinity as a function of n.

Thus this sequence of polyhedra is a counter-example to the existence of a lower bound to volume increase under Dehn drilling. We get a similar counter-example for the manifold case by looking at the four-fold covers of these polyhedra when viewed as orbifolds.

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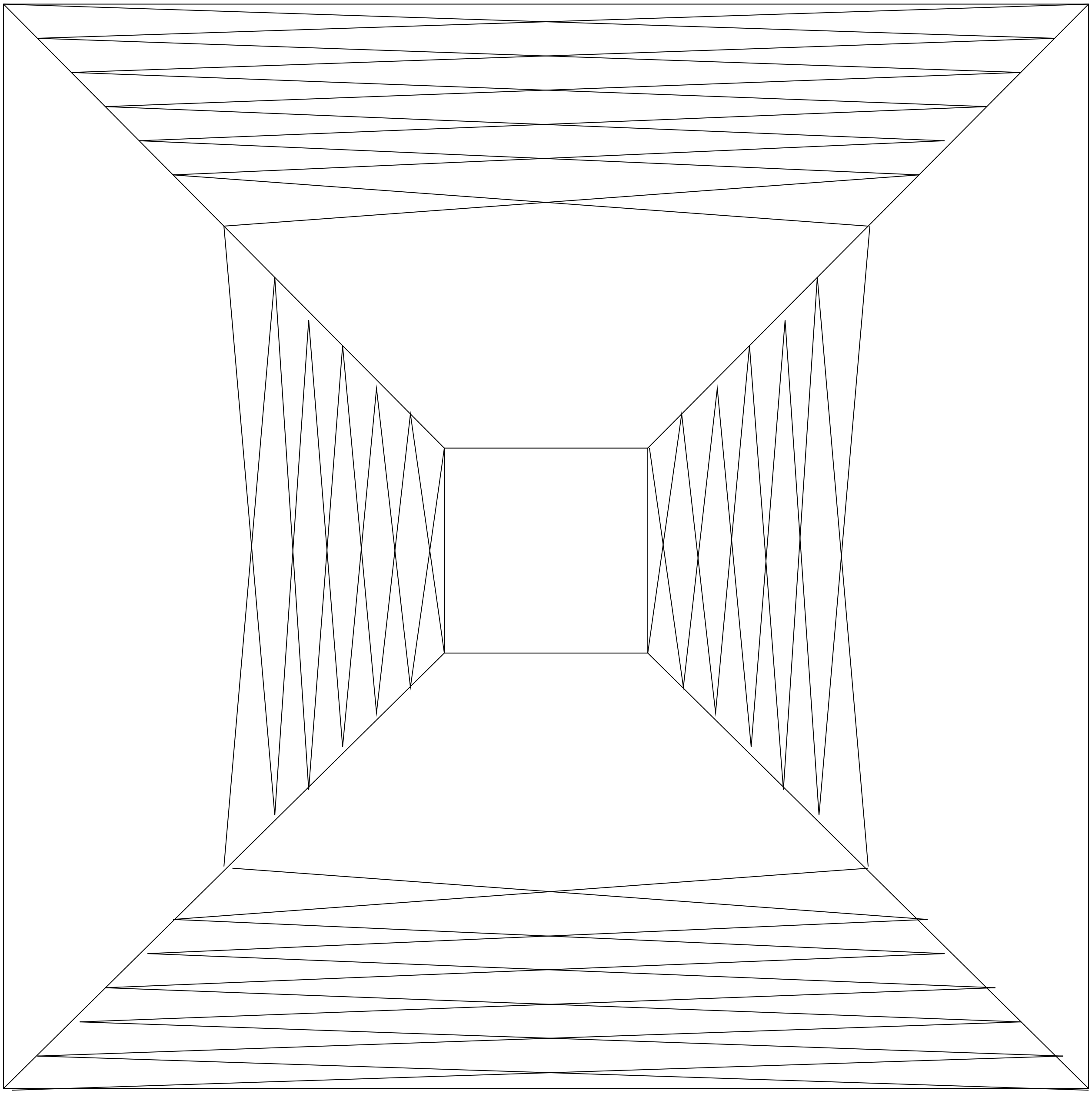

Figure 11: Polyhedron with thin ideal quadrilateral face