HYPERBOLIC VOLUME OF N-MANIFOLDS WITH GEODESIC BOUNDARY AND ORTHOSPECTRA

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ABSTRACT. In this paper we describe a function $F_n : \mathbf{R}_+ \to \mathbf{R}_+$ such that for any hyperbolic n-manifold M with totally geodesic boundary $\partial M \neq \emptyset$, the volume of M is equal to the sum of the values of F_n on the *orthospectrum* of M. We derive an integral formula for F_n in terms of elementary functions. We use this to give a lower bound for the volume of a hyperbolic n-manifold with totally geodesic boundary in terms of the area of the boundary.

1. Introduction

We let M be a compact hyperbolic $n-$manifold with non-empty totally geodesic boundary. An orthogeodesic for M is a geodesic arc with endpoints in ∂M and perpendicular to ∂M at the endpoints. Then the *orthospectrum* Λ_M of M is the set (with multiplicities) of lengths of orthogeodesics. Let DM be the closed manifold obtained by doubling M along the boundary, then DM is a closed hyperbolic manifold and therefore the set of closed geodesics of DM is countable. As the orthogeodesics of M correspond to a subset of the closed geodesics of DM, the set of orthogeodesics of M is also countable and therefore Λ_M is also. By decomposing the unit tangent bundle of M we obtain the following theorem.

Theorem 1. *Given $n \geq 2$ there exists a continuous monotonically decreasing function $F_n : \mathbf{R}_+ \to \mathbf{R}_+$ such that if M is a compact hyperbolic $n-$manifold with non-empty totally geodesic boundary, then*

$$Vol(M) = \sum_{l \in \Lambda_M} F_n(l).$$

We give an integral formula for F_n over the unit interval of an elementary function and show show that F_n satisfies $\lim_{l\to 0^+} l^{n-2}F_n(l) = K_n > 0$. Therefore for $n \geq 3$, if a hyperbolic n-manifold with geodesic boundary has a short orthogeodesic, it has large volume. Conversely, if it doesn't have a short orthogeodesic, the boundary has a large embedded neighborhood and therefore large volume. Using this we prove the following theorem.

Theorem 2. *For $n \geq 3$, there exists a monotonically increasing function $H_n : \mathbf{R}_+ \to \mathbf{R}_+$ and a constant $C_n > 0$ such that if M is a hyperbolic n-manifold with totally geodesic boundary of area A then*

$$Vol_n(M) \geq H_n(A) \geq C_n.A^{\left(\frac{n-2}{n-1}\right)}$$

2. Decomposition via orthogeodesics

In this section we define F_n and show that it gives a decomposition of the volume of M as described in Theorem 1.

We let C_M be the set of orthogeodesics of M and write $C_M = \{\alpha_i\}_{i\in I}$. We further let $l_i = \text{Length}(\alpha_i)$.

Let $v \in T_1(M)$ and let α_v be the maximal length geodesic arc in M tangent to v. We define

$$T_1^f(M) = \{v \in T_1(M) \mid \partial\alpha_v \in \partial M\}\ .$$

If $v \in T_1^f(M)$ then α_v is a closed geodesic arc with endpoints in ∂M intersecting ∂M transversely. We define $\sim$ on $T_1^f(M)$ by letting $v \sim w$ if α_v is homotopic to α_w in M rel boundary ∂M. Let $v_i \in T_1(M)$ be such that v_i is tangent to the orthogeodesic α_i. Then obviously $\alpha_{v_i} = \alpha_i$ and we let $D_i = [v_i]$. We will show that the D_i are exactly the equivalence classes of $\sim$.

We consider the universal cover $\tilde{M}$ of M in $\mathbf{H}^n$. Then $\partial\tilde{M}$ is a collection of disjoint planes P_i bounding disjoint hyperbolic open half spaces H_i such that $\tilde{M} = \mathbf{H}^3 - (\cup_i H_i)$.

Given a $v \in T_1^f(M)$ we lift α_v to a geodesic arc $\tilde{\alpha}_v$ in $\tilde{M}$. Then $\tilde{\alpha}_v$ has endpoints in two disjoint components P_i, P_j of $\partial\tilde{M}$. As P_i, P_j are disjoint, there is a unique perpendicular $\tilde{\beta}$ between them in $\tilde{M}$. Then $\tilde{\alpha}_v$ is homotopic to $\tilde{\beta}$ in $\tilde{M}$ rel boundary. We let β be the geodesic arc obtained by projecting $\tilde{\beta}$ down to M. Then β is a geodesic arc with endpoints in ∂M and perpendicular to ∂M. Therefore β is an orthogeodesic of M and therefore $\beta = \alpha_k$ for some k. Also the homotopy between $\tilde{\alpha}_v$ and β in $\tilde{M}$ rel boundary, descends to a homotopy between α_v and α_k. Therefore if we take v_k be a tangent vector to α_k then $[v] = [v_k] = D_k$.

Now to show that $D_i \cap D_j = \emptyset$ for $i \neq j$, we note that if $v_i \sim v_j$ then α_i is homotopic to α_j rel boundary.We lift this homotopy to obtain a homotopy between lifts $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ in $\tilde{M}$ rel boundary. Let P_k, P_l be the components of $\partial\tilde{M}$ joined by $\tilde{\alpha}_i$. Then $\tilde{\alpha}_i$ is the unique perpendicular between P_k and P_l. As $\tilde{\alpha}_j$ is homotopic rel boundary to $\tilde{\alpha}_i$, it must also connect P_k, P_l and is also the unique perpendicular between P_k and P_l. Thus $\tilde{\alpha}_j = \tilde{\alpha}_j$ adn therefore $\alpha_i = \alpha_j$.

By the ergodicity of geodesic flow on the double DM, almost every α_v must have both endpoints in ∂M. Therefore $T_1^f(M)$ is of full measure in $T_1(M)$. Therefore integrating over the fibers, we have

$$\Omega_M(T_1(M)) = \text{Vol}(\mathbf{S}^{n-1}).\text{Vol}(M) = \sum_{i\in I}\Omega_M(D_i)$$

giving

$$\text{Vol}(M) = \frac{1}{\text{Vol}(\mathbf{S}^{n-1})}\sum_{i\in I}\Omega_M(D_i)$$

We now show that $\Omega_M(D_i)$ depends only on l_i. Let $p : T_1(\tilde{M}) \to T_1(M)$ be the covering map associated to the covering $\pi : \tilde{M} \to M$. We let Ω be the standard volume measure on $T_1(\mathbf{H}^n)$. Then p is a local isometry between Ω and Ω_M.

Given v_i a tangent vector to α_i, we lift to $\tilde{\alpha}_i$ in $\tilde{M} \subset \mathbf{H}^n$. Then $\tilde{\alpha}_i$ is the unique perpendicular between the planes P_j, P_k. We let $\tilde{D}_i$ be the set of tangent vectors in $T_1(\tilde{M})$ tangent to a geodesic arc with endpoints on P_j, P_k.

If $v \in D_i$ then α_v is homotopic to α_i and therefore lifting the homotopy, it lifts to a geodesic arc $\tilde{\alpha}_v$ with endpoints in P_j, P_k. Therefore we can lift any point of D_i to a point of $\tilde{D}_i$ and the lift is a local isometry between $T_1(M)$ and $T_1(\tilde{M})$. Also by projecting back down to M, every point of $\tilde{D}_i$ is a lift of a point of D_i. Therefore p restricts to a covering map from $\tilde{D}_i$ to D_i. To show it is a homeomorphism, if g is a covering transformation for the covering $\pi : \tilde{M} \to M$ that sends $\tilde{v} \in \tilde{D}_i$ to $\tilde{w} \in \tilde{D}_i$, then g must send the pair of boundary components P_j, P_k to themselves (possibly switching). Therefore g must preserve the perpendicular $\tilde{\alpha}_i$ and fix (at least) the center point of of $\tilde{\alpha}_i$. As covering transformations have no fixed points this is a contradiction. Therefore $\tilde{D}_i$ is an isometric lift of D_i. Therefore $\Omega_M(D_i) = \Omega(\tilde{D}_i)$.

We now take the upper half space model for $\mathbf{H}^n$ and denote the planes P_i by the disks in $\mathbf{R}^{n-1}$ bounded by the half space H_i. We define

$$D = \{x \in \mathbf{R}^{n-1} \mid |x| \leq 1\} \text{ and } D_a = \{x \in \mathbf{R}^{n-1} \mid |x| \geq a\}$$

and the associated half-spaces by H, H_a respectively. For each l_i, we define $a_i = e^{l_i}$. Then by an isometry we can map the half-spaces H_j, H_k to half-spaces H, H_{a_i}. We let Q_a be the set of tangent vectors in $T_1(\mathbf{H}^n - (H \cup H_a))$ tangent to a geodesic arc with endpoints in D, D_a. Then we have $\Omega(Q_{a_i}) = \Omega(\tilde{D}_i)$ and we define

$$F_n(l) = \frac{\Omega(Q_{a_i})}{\mathrm{Vol}(\mathbf{S}^{n-1})}.$$

Then

$$\mathrm{Vol}(M) = \frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})} \sum_{i \in I} \Omega_M(D_i) = \frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})} \sum_{i \in I} \Omega(\tilde{D}_i) = \sum_{i \in I} F_n(l_i)$$

3. Integral formula

We consider the upper half space model of $\mathbf{H}^n$. As described above we have disks

$$D = \{x \in \mathbf{R}^{n-1} \mid |x| \leq 1\} \text{ and } D_a = \{x \in \mathbf{R}^{n-1} \mid |x| \geq a\}$$

bounding planes P, P_a respectively. Then Q_a is the set of tangent vectors tangent to a geodesic arc with endpoints on P_0, P_a.

We let Ω be the volume form on $T_1(\mathbf{H}^n)$. Let $v \in T_1(\mathbf{H}^n)$ and define g_v to be the directed geodesic with tangent vector v. Then g_v can be parameterized by the ordered pair of endpoints $(x, y) \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$. Furthermore if v has basepoint p then p is a unique signed hyperbolic distance from the highest point (in the upper half-space model) of g_v. We can therefore uniquely parameterize $T_1(\mathbf{H}^n)$ by triples $(x, y, t) \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}$ where (x, y) is the directed endpoints of g_v and t is the signed distance from the highest point of g_v.

In terms of this parametrization we have

$$d\Omega = \frac{2dV(x)dV(y)dt}{|x-y|^{2n-2}}$$

where $dV(x) = dx_1 dx_2 \dots dx_{n-1}$, for $x = (x_1, x_2, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$.

Now if $v \in Q_a$ then g_v has endpoints in $D \times D_a$ or $D_a \times D$. We define the map $L_a : (D \times D_a) \cup (D_a \times D) \to \mathbf{R}_+$ given by letting $L_a(x, y)$ equal the length of the segment of the geodesic with endpoints x, y between P_0 and P_a.

Then integrating along the t direction we have

$$F_n(l) = \frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})} \int_{Q_a} d\Omega = \frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})} \int_{Q_a} \frac{2dV(x)dV(y)dt}{|x-y|^{2n-2}}$$

Giving

$$F_n(l) = \frac{2}{\mathrm{Vol}(\mathbf{S}^{n-1})} \int_{(D \times D_a) \cup (D_a \times D)} \frac{L_a(x, y)dV(x)dV(y)}{|x-y|^{2n-2}}$$

As $L_a(x,y) = L_a(y,x)$ we have

$$F_n(l) = \frac{4}{\mathrm{Vol}(\mathbf{S}^{n-1})}\int_{D\times D_a}\frac{L_a(x,y)dV(x)dV(y)}{|x-y|^{2n-2}} = \frac{4}{\mathrm{Vol}(\mathbf{S}^{n-1})}\int_{|x|\leq 1}\int_{|y|\geq a}\frac{L_a(x,y)dV(x)dV(y)}{|x-y|^{2n-2}}$$

We first note the following fact from prior work.

Lemma 3. *(Bridgeman [1]) If $x,y \in \mathbf{R}$ then*

$$L_a(x,y) = \frac{1}{2}\log\left(\frac{(y^2-1)(x^2-a^2)}{(y^2-a^2)(x^2-1)}\right)$$

It follows that

$$F_2(l) = \frac{1}{\pi}\int_{-1}^{1}\int_{a}^{\infty}\frac{\log\left(\frac{(y^2-1)(x^2-a^2)}{(y^2-a^2)(x^2-1)}\right)dxdy}{(x-y)^2}.$$

The above integral can be explicitly calculated yielding the following;

Theorem 4. *(Bridgeman [1]) For $n = 2$*

$$F_2(l) = \frac{4}{\pi}\mathcal{L}\left(\frac{1}{\cosh^2\frac{l}{2}}\right)$$

where $\mathcal{L}$ is the Rogers $L-$function.

4. Higher dimensional case

We now consider the case of $n \geq 3$. Let $(x,y) \in D \times D_a$. We let $z = y - x$ with $u = z/|z|$. Also we define w to be the point on the line l joining x,y in $\mathbf{R}^{n-1}$ nearest to the origin in the Euclidean metric on $\mathbf{R}^{n-1}$. Then $w = r.v$ where $v \in \mathbf{S}^{n-2}$ and $r > 0$. Then $u,v \in \mathbf{S}^{n-2}$ are perpendicular and

$$x = s.u + r.v \qquad y = t.u + r.v$$

for unique $s,t \in \mathbf{R}$.

We reparametrise the integral in terms of r,s,t,u,v. We note that (u,v) parametrize the unit tangent bundle $T_1(\mathbf{S}^{n-2})$.

To calculate the Jacobian, we first change variables from (x,y) to (x,z) where $z = y-x$. Then the Jacobian is trivially equal to one. As $z = y-x = (t-s)u$, then from standard polar coordinates we have that

$$dV(z) = (t-s)^{n-2}d\Omega(u)d(t-s).$$

where $d\Omega(u)$ is the volume form on the unit sphere $\mathbf{S}^{n-2}$ in $\mathbf{R}^{n-1}$. Now for fixed u, then $x = s.u+r.v$ is in cylindrical polar coordinates with respect to v. We let $d\Omega(u,v)$ be the spherical volume form on the vectors perpendicular to u. Then as u,v we have in terms of spherical polar coordinates

$$dV(x) = r^{n-3}d\Omega(u,v).dr.ds$$

$$dV(x)dV(y) = (t-s)^{n-2}r^{n-3}d\Omega(u,v)d\Omega(u)dr.ds.d(t-s).$$

Therefore

$$dV(x)dV(y) = (t-s)^{n-2}r^{n-3}d\Omega(u,v)d\Omega(u)dr.ds.dt.$$

We consider the vertical hyperbolic plane P containing x,y. Then we see that P intersects both P_0,P_a in semicircles centered about w. By definition of $r = |w|$ we have that the semicircles have radii $r_1 = \sqrt{1-r^2}, r_2 = \sqrt{a^2-r^2}$ respectively. The geodesic with endpoints x,y is contained in

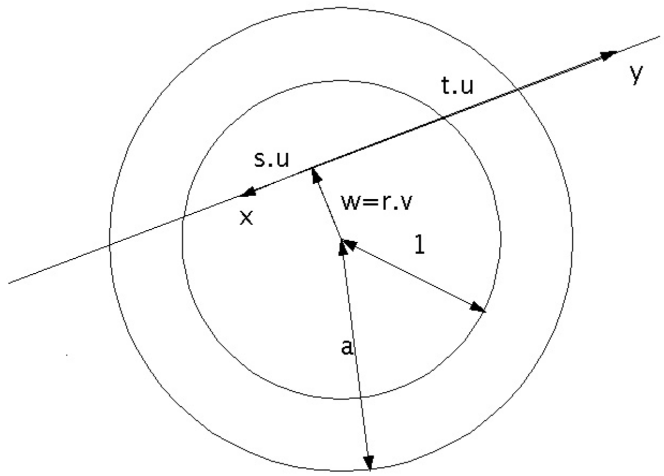

FIGURE 1. Parametrization by r, s, t, u, v

P. Therefore restricting to P and scaling by $1/r_1$ we have that the length $L_a(x,y)$ is given by the previous lemma

$$L_a(x,y) = L_{\frac{r_2}{r_1}}\left(\frac{s}{r_1}, \frac{t}{r_1}\right) = \frac{1}{2}\log\left(\frac{(t^2-r_1^2)(s^2-r_2^2)}{(t^2-r_2^2)(s^2-r_1^2)}\right).$$

Then we have

$$F_n(l) = \frac{2}{V_{n-1}}\int_0^1 dr \int_{\mathbf{S}^{n-2}} d\Omega(u) \int_{\mathbf{S}^{n-3}} d\Omega(u,v) \int_{|s|<r_1} ds \int_{t>r_2} \frac{\log\left(\frac{(t^2-r_1^2)(s^2-r_2^2)}{(t^2-r_2^2)(s^2-r_1^2)}\right)}{(s-t)^{2n-2}}(t-s)^{n-2} r^{n-3} dt$$

Integrating over u, v we get

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}}\int_0^1 r^{n-3} dr \int_{-r_1}^{r_1} ds \int_{t>r_2} \frac{\log\left(\frac{(t^2-r_1^2)(s^2-r_2^2)}{(t^2-r_2^2)(s^2-r_1^2)}\right)}{(t-s)^n} dt$$

We let $u = s/r_1, v = t/r_1$ and $b = r_2/r_1$ then

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}}\int_0^1 r^{n-3} dr \left(\frac{1}{r_1^{n-2}}\int_{-1}^{1} du \int_b^\infty \frac{\log\left(\frac{(v^2-1)(u^2-b^2)}{(v^2-b^2)(u^2-1)}\right)}{(v-u)^n} dv\right)$$

We define

$$M_n(b) = \int_{-1}^{1} du \int_b^\infty \frac{\log\left(\frac{(v^2-1)(u^2-r^2)}{(v^2-b^2)(u^2-1)}\right)}{(v-u)^n} dv$$

Therefore

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}} \int_0^1 \frac{r^{n-3}}{\left(\sqrt{1-r^2}\right)^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) dr. \tag{1}$$

We obtain an alternate integral form for F_n by letting $x = \sqrt{\frac{a^2-r^2}{1-r^2}}$. then

$$x^2 = \frac{a^2-r^2}{1-r^2} \qquad r^2 = \frac{x^2-a^2}{x^2-1} \qquad 2rdr = \frac{(a^2-1)}{(x^2-1)^2}2xdx$$

Giving

$$r^{n-3}dr = r^{n-4}r.dr = \left(\frac{x^2-a^2}{x^2-1}\right)^{\frac{n}{2}-2}\left(\frac{a^2-1}{(x^2-1)^2}\right)xdx = \frac{x(a^2-1)(x^2-a^2)^{\frac{n}{2}-2}}{(x^2-1)^{\frac{n}{2}}}dx$$

$$1-r^2 = \frac{a^2-1}{x^2-1} \qquad \frac{1}{(\sqrt{1-r^2})^{n-2}} = \left(\frac{x^2-1}{a^2-1}\right)^{\frac{n}{2}-1}$$

Therefore

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}} \int_a^\infty \left(\frac{x^2-a^2}{a^2-1}\right)^{\frac{n}{2}-2}\left(\frac{xM_n(x)}{x^2-1}\right)dx \tag{2}$$

4.1. **Formula For M_n.** We give an explicit formula for M_n. in order to do so we first need to state some integral formulae. For $n \geq 1$ we define the polynomial function P_n by

$$P_n(z) = \sum_{k=1}^n \frac{x^k}{k}.$$

We also define $P_0(z) = 0$. We note that for $|x| < 1$, $P_n(x)$ is the first n terms of the Taylor series of $-\log(1-x)$. We therefore define the function $L_n(x)$ by $L_n(x) = \log|1-x| + P_n(x)$. For $|x| < 1$ we have

$$L_n(x) = -\sum_{k=n+1}^\infty \frac{x^k}{k}.$$

We note that $L_0(x) = \log|1-x|$. We also note that $P_n(1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, the nth Harmonic number.

Then we have the following integral formula.

Lemma 5. *If $a \neq b \in \mathbf{R}$ and n a positive integer then*

$$\int \frac{dx}{(x-a)(x-b)^n} = \frac{1}{(a-b)^n}\left(\log\left|\frac{x-a}{x-b}\right| + P_{n-1}\left(\frac{a-b}{x-b}\right)\right) = \frac{L_{n-1}\left(\frac{a-b}{x-b}\right)}{(a-b)^n}$$

and

$$\int \frac{dx}{(x-a)(b-x)^n} = \frac{1}{(b-a)^n}\left(\log\left|\frac{x-a}{b-x}\right| + P_{n-1}\left(\frac{b-a}{b-x}\right)\right) = \frac{L_{n-1}\left(\frac{b-a}{b-x}\right)}{(b-a)^n}$$

Proof: We note that

$$\frac{1}{x^n(x-c)} = \frac{1}{c^n}\left(\frac{1}{x-c} - \frac{x^n-c^n}{x^n(x-c)}\right) = \frac{1}{c^n}\left(\frac{1}{x-c} - \frac{Q_{n-1}(x)}{x^n}\right)$$

where

$$Q_{n-1}(x) = \frac{x^n - c^n}{x - c} = \sum_{k=1}^{n} x^{k-1}c^{n-k}$$

Therefore

$$\int \frac{1}{x^n(x-c)}dx = \frac{1}{c^n}\left(\int \frac{dx}{x-c} - \int \frac{Q_{n-1}(x)}{x^n}dx\right) = \frac{1}{c^n}\left(\log|x-c| - \int \frac{Q_{n-1}(x)}{x^n}dx\right)$$

As

$$\frac{Q_{n-1}(x)}{x^n} = \sum_{k=1}^{n} \frac{c^{n-k}}{x^{n-k+1}}$$

then

$$\int \frac{Q_{n-1}(x)}{x^n}dx = \log|x| - \sum_{k=1}^{n-1} \frac{c^{n-k}}{(n-k).x^{n-k}} = \log|x| - P_{n-1}\left(\frac{c}{x}\right)$$

Therefore

$$\int \frac{1}{x^n(x-c)}dx = \frac{1}{c^n}\left(\log\left|\frac{x-c}{x}\right| + P_{n-1}\left(\frac{c}{x}\right)\right) = \frac{L_{n-1}\left(\frac{c}{x}\right)}{c^n}$$

The result now follows by elementary substitution. ■

Integrating by parts we get,

Corollary 6. *For* $n \geq 2$

$$\int \frac{\log|x-a|}{(x-b)^n}dx = \frac{1}{n-1}\left(\frac{L_{n-2}\left(\frac{a-b}{x-b}\right)}{(a-b)^{n-1}} - \frac{\log|x-a|}{(x-b)^{n-1}}\right)$$

and

$$\int \frac{\log|x-a|}{(b-x)^n}dx = \frac{1}{n-1}\left(\frac{\log|x-a|}{(b-x)^{n-1}} - \frac{L_{n-2}\left(\frac{b-a}{b-x}\right)}{(b-a)^{n-1}}\right)$$

Furthermore for $k \geq 1$

$$\lim_{x\to a}\left(\frac{\log|x-a|}{(b-x)^k} - \frac{L_n\left(\frac{b-a}{b-x}\right)}{(b-a)^k}\right) = \frac{\log|b-a| - P_n(1)}{(b-a)^k}$$

Lemma 7. *The function* $M_n : (1,\infty) \to \mathbf{R}_+$ *has the explicit form*

$$M_n(b) = \frac{1}{(n-1)(n-2)}\left(\frac{1}{(b-1)^{n-2}}\left(\log(\frac{(b+1)^2}{(4b)}) + 2P_{n-2}(1) - L_{n-3}(\frac{b-1}{b+1}) - (-1)^n L_{n-3}(\frac{-b+1}{b+1})\right) + \right.$$
$$\frac{1}{(b+1)^{n-2}}\left(-\log(\frac{(b-1)^2}{(4b)}) - 2P_{n-2}(1) + L_{n-3}(\frac{b+1}{b-1}) + (-1)^n L_{n-3}(\frac{-b-1}{b-1})\right) +$$
$$\left.\frac{1}{(2b)^{n-2}}\left(L_{n-3}(\frac{2b}{b+1}) - L_{n-3}(\frac{2b}{b-1})\right) + \frac{1}{2^{n-2}}\left(L_{n-3}(\frac{2}{b+1}) - (-1)^n L_{n-3}(\frac{-2}{b-1})\right)\right)$$

Furthermore M_n *satisfies*

$$\lim_{b\to 1^+}(b-1)^{n-2}M_n(b) = D_n = \frac{2P_{n-2}(1)}{(n-1)(n-2)} \qquad \text{and} \qquad \lim_{b\to\infty}\frac{b^{n-1}}{\log b}.M_n(b) = \frac{4}{n-1}.$$

Proof: We let

$$I(u,b) = \int_b^\infty \frac{\log\left(\frac{(v^2-1)(b^2-u^2)}{(v^2-b^2)(1-u^2)}\right)}{(v-u)^n}dv$$

then $M_n(b) = \int_0^1 I(u,b)du$. We split $I = I_1 + I_2$ where

$$I_1(u,b) = \int_b^\infty \frac{\log\left(\frac{b-u}{v-b}\right)}{(v-u)^n}dv \qquad I_2(u,b) = \int_b^\infty \frac{\log\left(\frac{(v^2-1)(b+u)}{(v+b)(1-u^2)}\right)}{(v-u)^n}dv.$$

By the above lemma,

$$I_1(u,b) = \frac{1}{n-1}\left(\frac{\log(\frac{v-b}{b-u})}{(v-u)^{n-1}} - \frac{L_{n-2}\left(\frac{b-u}{v-u}\right)}{(b-u)^{n-1}}\right)\Bigg|_b^\infty$$

Therefore by the above corollary

$$I_1(u,b) = \frac{1}{n-1}\left(\frac{P_{n-2}(1)}{(b-u)^{n-1}}\right)$$

$$I_2(u,b) = \left(\frac{-1}{n-1}\right)\frac{\log\left(\frac{(v^2-1)(b+u)}{(v+b)(1-u^2)}\right)}{(v-u)^{n-1}}\Bigg|_b^\infty + \frac{1}{n-1}\int_b^\infty \frac{1}{(v-u)^{n-1}}\left(\frac{1}{v-1}+\frac{1}{v+1}-\frac{1}{v+b}\right)dv$$

Evaluating we get

$$I_2(u,b) = \frac{1}{n-1}\left(\frac{\log\left(\frac{(b^2-1)(b+u)}{(2b)(1-u^2)}\right)}{(b-u)^{n-1}} + \int_b^\infty \frac{1}{(v-u)^{n-1}}\left(\frac{1}{v-1}+\frac{1}{v+1}-\frac{1}{v+b}\right)dv\right)$$

Therefore

$$M_n(b) = \frac{1}{n-1}\left(\int_{-1}^1 \frac{\log\left(\frac{(b^2-1)(b+u)}{(2b)(1-u^2)}\right) + P_{n-2}(1)}{(b-u)^{n-1}}du + \int_{-1}^1\int_b^\infty \frac{1}{(v-u)^{n-1}}\left(\frac{1}{v-1}+\frac{1}{v+1}-\frac{1}{v+b}\right)dvdu\right)$$

We once again split this integral as follows;

$$J_1(b) = \frac{1}{n-1}\left(\int_{-1}^1 \frac{\log\left(\frac{(b^2-1)}{(2b)}\right) + P_{n-2}(1)}{(b-u)^{n-1}}du\right)$$

$$J_2(b) = \frac{1}{n-1}\left(\int_{-1}^1 \frac{\log\left(\frac{b+u}{(1-u)(1+u)}\right)}{(b-u)^{n-1}}du\right)$$

$$J_3(b) = \frac{1}{n-1}\left(\int_{-1}^1\int_b^\infty \frac{1}{(v-u)^{n-1}}\left(\frac{1}{v-1}+\frac{1}{v+1}-\frac{1}{v+b}\right)dvdu\right)$$

We now evaluate each of J_1, J_2, J_3.

J_1**:** Integrating we get

$$J_1(b) = \frac{\log\left(\frac{(b^2-1)}{(2b)}\right) + P_{n-2}(1)}{(n-1)(n-2)}\left(\frac{1}{(b-1)^{n-2}} - \frac{1}{(b+1)^{n-2}}\right)$$

J_2**:** By the above lemma

$$J_2(b) = \frac{1}{(n-1)(n-2)}\left(\frac{\log\left(\frac{b+u}{(1-u)(1+u)}\right)}{(b-u)^{n-2}} - \frac{L_{n-3}(\frac{2b}{b-u})}{(2b)^{n-2}} + \frac{L_{n-3}(\frac{b-1}{b-u})}{(b-1)^{n-2}} + \frac{L_{n-3}(\frac{b+1}{b-u})}{(b+1)^{n-2}}\right)\Bigg|_{-1}^{1}$$

Therefore

$$(n-1)(n-2)J_2(b) = \left(\frac{\log(\frac{b+1}{2}) - L_{n-3}(\frac{b-1}{b+1})}{(b-1)^{n-2}}\right) + \left(\frac{L_{n-3}(\frac{2b}{b+1}) - L_{n-3}(\frac{2b}{b-1})}{(2b)^{n-2}}\right) + \left(\frac{-\log(\frac{b-1}{2}) + L_{n-3}(\frac{b+1}{b-1})}{(b+1)^{n-2}}\right) +$$

$$\lim_{u\to 1-}\left(-\frac{\log(1-u)}{(b-u)^{n-2}} + \frac{L_{n-3}(\frac{b-1}{b-u})}{(b-1)^{n-2}}\right) + \lim_{u\to -1^+}\left(\frac{\log(1+u)}{(b-u)^{n-2}} - \frac{L_{n-3}(\frac{b+1}{b-u})}{(b+1)^{n-2}}\right)$$

By the corollary above

$$\lim_{u\to 1-}\left(-\frac{\log(1-u)}{(b-u)^{n-2}} + \frac{L_{n-3}(\frac{b-1}{b-u})}{(b-1)^{n-2}}\right) = \frac{P_{n-3}(1) - \log(b-1)}{(b-1)^{n-2}}$$

$$\lim_{u\to -1^+}\left(\frac{\log(1+u)}{(b-u)^{n-2}} - \frac{L_{n-3}(\frac{b+1}{b-u})}{(b+1)^{n-1}}\right) = \frac{\log(b+1) - P_{n-3}(1)}{(b+1)^{n-2}}$$

Therefore

$$(n-1)(n-2)J_2(b) = \left(\frac{\log(\frac{b+1}{2(b-1)}) + P_{n-3}(1) - L_{n-3}(\frac{b-1}{b+1})}{(b-1)^{n-2}}\right) + \left(\frac{L_{n-3}(\frac{2b}{b+1}) - L_{n-3}(\frac{2b}{b-1})}{(2b)^{n-2}}\right)$$

$$+\left(\frac{-\log(\frac{b-1}{2(b+1)}) - P_{n-3}(1) + L_{n-3}(\frac{b+1}{b-1})}{(b+1)^{n-2}}\right)$$

J_3**:** Switching the order of integration we get

$$J_3(b) = \frac{1}{(n-1)(n-2)}\int_b^\infty \left(\frac{1}{v-1} + \frac{1}{v+1} - \frac{1}{v+b}\right)\left(\frac{1}{(v-1)^{n-2}} - \frac{1}{(v+1)^{n-2}}\right)dv$$

Therefore

$$J_3(b) = \frac{1}{(n-1)(n-2)}\left(\frac{\frac{1}{n-2}}{(b-1)^{n-2}} - \frac{\frac{1}{n-2}}{(b+1)^{n-2}} + \frac{L_{n-3}(\frac{2}{b+1})}{2^{n-2}} - \frac{L_{n-3}(\frac{-2}{b-1})}{(-2)^{n-2}} + \frac{L_{n-3}(\frac{-b-1}{b-1})}{(-b-1)^{n-2}} - \frac{L_{n-3}(\frac{-b+1}{b+1})}{(-b+1)^{n-2}}\right)$$

Combining J_1, J_2, J_3 we have

$$M_n(b) = \frac{1}{(n-1)(n-2)}\left(\left(\frac{\log\left(\frac{(b+1)^2}{4b}\right) + P_{n-2}(1) + P_{n-3}(1) - L_{n-3}(\frac{b-1}{b+1}) + \frac{1}{n-2} - (-1)^n L_{n-3}(\frac{-b+1}{b+1})}{(b-1)^{n-2}}\right) +\right.$$

$$\left(\frac{-\log\left(\frac{(b-1)^2}{(4b)}\right) - P_{n-2}(1) - P_{n-3}(1) + L_{n-3}(\frac{b+1}{b-1}) - \frac{1}{n-2} + (-1)^n L_{n-3}(\frac{-b-1}{b-1})}{(b+1)^{n-2}}\right) +$$

$$\left.\left(\frac{\frac{1}{b^{n-2}}(L_{n-3}(\frac{2b}{b+1}) - L_{n-3}(\frac{2b}{b-1})) + L_{n-3}(\frac{2}{b+1}) - (-1)^n L_{n-3}(\frac{-2}{b-1})}{2^{n-2}}\right)\right)$$

Noting that $P_{n-3}(1)+\frac{1}{n-2}=P_{n-2}(1)$ we simplify to get

$$M_n(b)=\frac{1}{(n-1)(n-2)}\left(\frac{1}{(b-1)^{n-2}}\left(\log(\frac{(b+1)^2}{(4b)})+2P_{n-2}(1)-L_{n-3}(\frac{b-1}{b+1})-(-1)^nL_{n-3}(\frac{-b+1}{b+1})\right)\right.$$
$$+\frac{1}{(b+1)^{n-2}}\left(-\log(\frac{(b-1)^2}{(4b)})-2P_{n-2}(1)+L_{n-3}(\frac{b+1}{b-1})+(-1)^nL_{n-3}(\frac{-b-1}{b-1})\right)$$
$$\left.+\frac{1}{(2b)^{n-2}}\left(L_{n-3}(\frac{2b}{b+1})-L_{n-3}(\frac{2b}{b-1})\right)+\frac{1}{2^{n-2}}\left(L_{n-3}(\frac{2}{b+1})-(-1)^nL_{n-3}(\frac{-2}{b-1})\right)\right)$$

It follows from the above directly that M_n satisfies

$$\lim_{b\to 1^+}(b-1)^{n-2}M_n(b)=\frac{2P_{n-2}(1)}{(n-1)(n-2)}.$$

We now consider the limit as b tends to infinity. As L_{n-3} is continuous except at $x=1$, we replace the L_{n-3} terms by the limiting value if the argument does not tend to 1 or is infinite. For other values we substitute $P_n(x)=\log|1-x|+P_n(x)$ and collect terms. Therefore

$$\lim_{b\to\infty}(b-1)^{n-2}M_n(b)=\frac{1}{(n-1)(n-2)}\lim_{b\to\infty}\left(\left(-\log(4)+2P_{n-2}(1)-\log(\frac{2}{b+1})-P_{n-3}(1)-(-1)^nL_{n-3}(-1)\right)\right.$$
$$+\left(\log(4)-2P_{n-2}(1)+\log(\frac{2}{b-1})+P_{n-3}(1)+(-1)^nL_{n-3}(-1)\right)$$
$$\left.+\frac{1}{2^{n-2}}\left(L_{n-3}(2)-L_{n-3}(2)\right)+\frac{(b-1)^{n-2}}{2^{n-2}}\left(L_{n-3}(\frac{2}{b+1})-(-1)^nL_{n-3}(\frac{-2}{b-1})\right)\right)$$

After cancellation we have

$$\lim_{b\to 1^+}(b-1)^{n-2}M_n(b)=\frac{1}{(n-1)(n-2)}\lim_{b\to\infty}\frac{(b-1)^{n-2}}{2^{n-2}}\left(L_{n-3}(\frac{2}{b+1})-(-1)^nL_{n-3}(\frac{-2}{b-1}\right).$$

For b large we can use the Taylor series expansion for L_{n-3} to get

$$L_{n-3}\left(\frac{2}{b+1}\right)=\frac{1}{n-2}\left(\frac{2}{b+1}\right)^{n-2}+\epsilon_1.\left(\frac{2}{b+1}\right)^{n-3}\qquad L_{n-3}\left(\frac{-2}{b-1}\right)=\frac{1}{n-2}\left(\frac{-2}{b-1}\right)^{n-2}+\epsilon_2.\left(\frac{-2}{b-1}\right)^{n-3}$$

Therefore

$$\lim_{b\to\infty}\frac{(b-1)^{n-2}}{2^{n-2}}\left(L_{n-3}(\frac{2}{b+1})-(-1)^nL_{n-3}(\frac{-2}{b-1})\right)=$$
$$\lim_{b\to\infty}\left(\frac{1}{n-2}+\epsilon_1\left(\frac{2}{b+1}\right)-(-1)^n\left(\frac{(-1)^{n-2}}{n-2}+\epsilon_2\frac{(-1)^{n-2}.-2}{b-1}\right)\right)=0$$

We now use this fact to show the exact asyptotic behavior of $M_n(b)$ as b tends to infinity. From the above we have

$$\lim_{b\to\infty}b^{n-2}M_n(b)=0.$$

We now expand $M_n(b)$ around infinity by writing as a Taylor series in $1/b$. From the form of M_n we must therefore have that for large b

$$M_n(b)=\frac{c_0+d_0\log b}{b^{n-1}}+\frac{c_1+d_1\log b}{b^n}+\dots$$

Thus we have that the leading terms is

$$M_n(b)\simeq\frac{d_0\log b}{b^{n-1}}\qquad\text{or}\qquad\lim_{b\to\infty}b^{n-1}\log b.M_n(b)=d_0.$$

To find d_0 we need only gather the $\log b$ terms. We can replace terms of the form $\log(sb+t)$ by $\log b$ and also can ignore terms $L_{n-3}(m(b))$ where $m(b)$ does not tend to 1 as b tends to infinity. Therefore

$$M_n(b) \simeq \frac{1}{(n-1)(n-2)}\left(\frac{2\log b}{(b-1)^{n-2}} - \frac{2\log b}{(b+1)^{n-2}}\right) \simeq \frac{1}{(n-1)(n-2)}\frac{2\log b}{b^{n-2}}\left(\frac{1}{(1-1/b)^{n-2}} - \frac{1}{(1+1/b)^{n-2}}\right)$$

Therefore

$$M_n(b) \simeq \frac{1}{(n-1)(n-2)}\frac{2\log b}{b^{n-2}}\left(1+(n-2)\frac{1}{b} - \left(1-(n-2)\frac{1}{b}\right)\right) = \left(\frac{4}{n-1}\right)\frac{\log b}{b^{n-1}}.$$

■

Corollary 8. *For n even there is a closed form for F_n.*

Proof: We note by equation 2, for all n we have

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}}\int_a^\infty \left(\frac{x^2-a^2}{a^2-1}\right)^{\frac{n}{2}-2}\left(\frac{xM_n(x)}{x^2-1}\right)dx.$$

By the above lemma, M_n is in terms of rational functions and rational functions times sums of linear log functions $\log(a+bx)$. Therefore for n even the function to be integrated in the above equation for F_n is a sum of rational functions and rational functions times linear log functions. As such functions have explicit formulae, we can therefore find formulae for F_n for n even. As the number of terms of M_n is linear in n we obtain approximately a quadratic number of terms for F_n. ■

4.2. **Three-dimensional case.** For $n = 3$ we have

$$F_3(l) = 2.\int_0^1 \frac{M_3\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)}{\sqrt{1-r^2}}.dr.$$

Using the above formula we have that

$$M_3(x) = \frac{2}{x^2-1}(1-\log(2)) - \frac{1}{2x}\left(\frac{x-1}{x+1}\right)\log(x-1) + \frac{1}{2x}\left(\frac{x+1}{x-1}\right)\log(x+1)$$

Also we have

$$M_3(x) \simeq \frac{1}{x-1}$$

for x close to 1.

4.3. **Four-dimensional case.** For $n = 4$ we have

$$F_4(l) = 8.\int_0^1 \frac{r.M_4\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)}{1-r^2}.dr.$$

Using the above formula we have that

$$M_4(b) = \frac{1}{6}\left(\frac{3+2\log(\frac{(b+1)^2}{4b})}{(b-1)^2} - \frac{3+2\log(\frac{(b-1)^2}{4b})}{(b+1)^2} + \frac{\log(\frac{b-1}{b+1}) + \frac{b}{b+1} - \frac{b}{b-1}}{2b^2} + \frac{\log(\frac{b-1}{b+1}) + \frac{1}{b+1} + \frac{1}{b-1}}{2}\right)$$

Also we have

$$M_4(x) \simeq \frac{1}{2}\frac{1}{(x-1)^2}$$

for x close to 1.

5. Properties of F_n

We now describe the properties of F_n. In particular we complete the proof of Theorem 1.

Lemma 9. *The function F_n satisfies the following:*

(1) *There exists a $C_n > 0$ such that*

$$F_n(l) \leq \frac{C_n}{(e^l-1)^{n-2}}$$

(2) *F_n is continuous monotonically decreasing*

(3)

$$\lim_{l\to 0} l^{n-2}F_n(l) = K_n = \frac{2\pi^{\frac{n-3}{2}}P_{n-2}(1)\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2}-1)}{n.\Gamma(\frac{n+1}{2})\Gamma(n-1)}$$

(4)

$$\lim_{l\to\infty}\frac{e^{(n-1)l}}{l}.F_n(l) = \frac{(n-2)\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2}-1)}{(\Gamma(\frac{n+1}{2}))^2}$$

Proof:

(1) For $a = e^l$ we have

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}}\int_0^1 \frac{r^{n-3}}{\left(\sqrt{1-r^2}\right)^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)dr$$

with

$$M_n(b) = \int_{-1}^1 ds\int_b^\infty \frac{\log\left(\frac{(t^2-1)(s^2-b^2)}{(t^2-b^2)(s^2-1)}\right)}{(t-s)^n}dt.$$

By lemma 7 we have that there exists a $B_n > 0$ such that

$$M_n(x) \leq \frac{B_n}{(x-1)^{n-2}}.$$

Therefore

$$F_n(l) \leq \frac{2B_nV_{n-2}V_{n-3}}{V_{n-1}}\int_0^1 \frac{r^{n-3}}{\left(\sqrt{1-r^2}\right)^{n-2}.\left(\sqrt{\frac{a^2-r^2}{1-r^2}}-1\right)^{n-2}}dr.$$

$$F_n(l) \leq \frac{2B_nV_{n-2}V_{n-3}}{V_{n-1}}\int_0^1 \frac{r^{n-3}}{\left(\sqrt{a^2-r^2}-\sqrt{1-r^2}\right)^{n-2}}dr.$$

Rationalizing the denominator we get

$$F_n(l) \leq \frac{2B_nV_{n-2}V_{n-3}}{V_{n-1}}\int_0^1 \frac{r^{n-3}(\sqrt{a^2-r^2}+\sqrt{1-r^2})^{n-2}}{(a^2-1)^{n-2}}dr.$$

Therefore

$$(a-1)^{n-2}F_n(l) \leq \frac{2B_nV_{n-2}V_{n-3}}{V_{n-1}} \int_0^1 r^{n-3} \left(\frac{\sqrt{a^2-r^2}+\sqrt{1-r^2}}{a+1}\right)^{n-2} dr.$$

We have for $a \geq 0, 0 \leq r \leq 1$

$$0 \leq \frac{\sqrt{a^2-r^2}+\sqrt{1-r^2}}{a+1} \leq 1.$$

Therefore we have a $C_n > 0$ such that

$$(e^l-1)^{n-2}F_n(l) \leq C_n \qquad \text{giving} \qquad F_n(l) \leq \frac{C_n}{(e^l-1)^{n-2}}.$$

(2) We have for $a = e^l$ then by equation 2,

$$F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}} \int_a^\infty \left(\frac{x^2-a^2}{a^2-1}\right)^{\frac{n}{2}-2} \left(\frac{xM_n(x)}{x^2-1}\right) dx.$$

The function $\frac{x^2-a^2}{a^2-1}$ is monotonically decreasing in a. Therefore for $n \geq 4$ then $\frac{n}{2}-2 \geq 0$ and if $a < b$

$$\int_b^\infty \left(\frac{x^2-b^2}{b^2-1}\right)^{\frac{n}{2}-2} \left(\frac{xM_n(x)}{x^2-1}\right) dx \leq \int_b^\infty \left(\frac{x^2-a^2}{a^2-1}\right)^{\frac{n}{2}-2} \left(\frac{xM_n(x)}{x^2-1}\right) dx \leq \int_a^\infty \left(\frac{x^2-a^2}{a^2-1}\right)^{\frac{n}{2}-2} \left(\frac{xM_n(x)}{x^2-1}\right) dx$$

Thus F_n is monotonic for $n \geq 4$. The case for $n = 3$ we have an explicit form for M_3 given by

$$M_3(x) = \frac{2}{x^2-1}(1-\log(2)) - \frac{1}{2x}\left(\frac{x-1}{x+1}\right)\log(x-1) + \frac{1}{2x}\left(\frac{x+1}{x-1}\right)\log(x+1).$$

This function is monotonically decreasing. As

$$F_3(l) = 2.\int_0^1 \frac{M_3\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)}{\sqrt{1-r^2}}.dr$$

then it follows that F_3 is also monotonically decreasing.

(3) We now analyse the behavior of $F_n(l)$ as l tends to zero. By lemma 7 we have that

$$\lim_{b\to 1^+}(b-1)^{n-2}M_n(b) = D_n$$

. Let $\epsilon > 0$. then on $[0, 1-\epsilon]$ we have

$$\lim_{a\to 1^+}\left(\sqrt{\frac{a^2-r^2}{1-r^2}}-1\right)^{n-2} M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) = D_n \qquad \text{uniformly.}$$

Simplifying we have

$$\left(\sqrt{\frac{a^2-r^2}{1-r^2}}-1\right)^{n-2} = \frac{(a^2-1)^{n-2}}{(\sqrt{a^2-r^2}+\sqrt{1-r^2})(\sqrt{1-r^2})^{n-2}}$$

Therefore we have

$$\lim_{a\to 1^+}(a-1)^{n-2}M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) = D_n.(1-r^2)^{n-2} \qquad \text{uniformly on } r \in [1, 1-\epsilon].$$

Therefore

$$\lim_{a\to 1^+}(a-1)^{n-2}\int_0^{1-\epsilon}\frac{r^{n-3}}{\left(\sqrt{1-r^2}\right)^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)dr = D_n.\int_0^{1-\epsilon} r^{n-3}(\sqrt{1-r^2})^{n-2}dr$$

As $M_n(x) \leq \frac{B_n}{(x-1)^{n-2}}$ we have as above that

$$\lim_{a\to 1^+}(a-1)^{n-2}\int_{1-\epsilon}^{1}\frac{r^{n-3}}{\left(\sqrt{1-r^2}\right)^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)dr \leq B_n\int_{1-\epsilon}^{1} r^{n-3}dr \leq B_n.\epsilon.$$

Also for $n \geq 3$

$$\int_{1-\epsilon}^{1} r^{n-3}(\sqrt{1-r^2})^{n-2}dr \leq \epsilon.$$

Therefore

$$\limsup_{a\to 1^+}\left|(a-1)^{n-2}\int_0^{1}\frac{r^{n-3}}{\left(\sqrt{1-r^2}\right)^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)dr - D_n.\int_0^{1} r^{n-3}(\sqrt{1-r^2})^{n-2}dr\right| \leq (B_n+D_n)\epsilon.$$

As ϵ is arbitrary we have

$$\lim_{l\to 0^+} l^{n-2}F_n(l) = \lim_{a\to a^+}(a-1)^{n-2}F_n(l) = \frac{2V_{n-2}V_{n-3}D_n}{V_{n-1}}.\int_0^1 r^{n-3}(\sqrt{1-r^2})^{n-2}dr.$$

Substituting $x = r^2$, and letting B be the Beta function then

$$\int_0^1 r^{n-3}(\sqrt{1-r^2})^{n-2}dr = \frac{1}{2}\int_0^1 x^{\frac{n}{2}-2}(1-x)^{\frac{n}{2}-1}dx = \frac{1}{2}B(\frac{n}{2}-1,\frac{n}{2}) = \frac{\Gamma(\frac{n}{2}-1).\Gamma(\frac{n}{2})}{2.\Gamma(n-1)}$$

Combining we get

$$\lim_{l\to 0^+} l^{n-2}F_n(l) = \frac{D_n.V_{n-2}V_{n-3}\Gamma(\frac{n}{2}-1).\Gamma(\frac{n}{2})}{V_{n-1}.\Gamma(n-1)}$$

As the volume formula for the $n-$sphere is

$$V_n = \frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})}$$

$$\lim_{l\to 0^+} l^{n-2}F_n(l) = K_n = \frac{2\pi^{\frac{n-3}{2}}P_{n-2}(1)\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2}-1)}{n.\Gamma(\frac{n+1}{2})\Gamma(n-1)}$$

We calculate K_n for some small values. Starting with $n = 3$ the first 10 values of K_n are

$$\frac{\pi}{2}, 1, \frac{11\pi^2}{192}, \frac{5\pi}{54}, \frac{137\pi^3}{30720}, \frac{7\pi^2}{1125}, \frac{121\pi^4}{458752}, \frac{761\pi^3}{2315250}, \frac{7129\pi^5}{566231040}, \frac{1342\pi^4}{93767625}, \ldots$$

(4) We now consider $F_n(l)$ for l large. Then for large a as $(a^2-r^2)/(1-r^2) > a^2-1$ for $0 < r < 1$, and we have that

$$\lim_{a\to\infty}\frac{\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)^{n-1}}{\log\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)}M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) = \frac{4}{n-1} \qquad \text{uniformly on } [0,1].$$

Therefore on the interval $[0, 1-\epsilon]$ we have

$$\lim_{a\to\infty} \frac{a^{n-1}}{\log a} M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) = \frac{4}{n-1}.(\sqrt{1-r^2})^{n-1} \qquad \text{uniformly on } [0,1-\epsilon].$$

Therefore

$$\lim_{a\to\infty} \frac{a^{n-1}}{\log a} \int_0^{1-\epsilon} \frac{r^{n-3}}{(\sqrt{1-r^2})^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) dr = \frac{4}{n-1}\int_0^{1-\epsilon} r^{n-3}\sqrt{1-r^2}dr$$

As this holds for all ϵ, the improper integral exists and we have

$$\lim_{a\to\infty} \frac{a^{n-1}}{\log a} \int_0^{1} \frac{r^{n-3}}{(\sqrt{1-r^2})^{n-2}}.M_n\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right) dr = \frac{4}{n-1}\int_0^{1} r^{n-3}\sqrt{1-r^2}dr = \frac{\sqrt{\pi}\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+1}{2}).(n-1)}$$

Therefore

$$\lim_{l\to\infty} \frac{e^{(n-1)l}}{l} F_n(l) = \frac{2V_{n-2}V_{n-3}}{V_{n-1}}.\frac{\sqrt{\pi}\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+1}{2}).(n-1)} = \frac{(n-2)\pi^{\frac{n-2}{2}}\Gamma(\frac{n}{2}-1)}{(\Gamma(\frac{n+1}{2}))^2}$$

■

The above lemma completes the proof of Theorem 1.

6. Volume Bounds

We now use the asymptotic behavior of F_n to get bounds on the volume of a hyperbolic n-manifold with geodesic boundary in terms of the area of the boundary.

We first describe the volume of an $r-$neighborhood of a boundary component in terms of r.

Lemma 10. *Let X be a finite volume subset of $\mathbf{H}^{n-1} \subseteq \mathbf{H}^n$. Let $N_r^+(X)$ be a one sided $r-$neighborhood of X in $\mathbf{H}^n$. Then*

$$Vol_n(N_r^+(X)) = Vol_{n-1}(X).S_n(r)$$

where

$$S_n(r) = \int_0^r \cosh^{n-1}(x)\ dx.$$

Proof: We consider X on the vertical hyperplane P in the upper half-space model of $\mathbf{H}^n$ given by $x_1 = 0$. Then by elementary hyperbolic geometry the boundary of $N_r^+(P)$ is the Euclidean plane $x_n = \tan(\theta_0).x_1$ for θ_0 satisfying $\sinh r = \tan\theta_0$.

We let $x_1 = r\sin\theta, x_n = r\cos\theta$. Then $(r,\theta,x_2,\ldots,x_{n-1}) \in N_r^+(X)$ if and only if $(0,x_2,x_3,\ldots,r) \in X$ and $0 < \theta < \theta_0$. Changing variables we get

$$\text{Vol}_n(N_r^+(X)) = \int_{N_r^+(X)} \frac{dx_1dx_2\ldots dx_n}{x_n^n} = \int_{N_r^+(X)} \frac{rd\theta drdx_2\ldots dx_{n-1}}{r^n\cos^n\theta} =$$

$$\int_X \frac{dx_2\ldots dx_{n-1}dr}{r^{n-1}}.\int_0^{\theta_0}\frac{d\theta}{\cos^n\theta}.$$

As $x_n = r$ on P we get

$$\text{Vol}_n(N_r^+(X)) = \int_X \frac{dx_2\ldots dx_{n-1}dx_n}{x_n^{n-1}}.\int_0^{\theta_0}\frac{d\theta}{\cos^n\theta} = \text{Vol}_{n-1}(X).\int_0^{\theta_0}\frac{d\theta}{\cos^n\theta}.$$

Therefore

$$\text{Vol}_n(N_r^+(X)) = \text{Vol}_{n-1}(X).S_n(r)$$

where

$$S_n(r) = \int_0^{\theta_0} \sec^n\theta\ d\theta$$

for $\sinh r = \tan\theta_0$. We let $\sinh x = \tan\theta$ then $\cosh x dx = \sec^2\theta d\theta$. As $\cosh x = \sec\theta$ we have $dx = \sec\theta d\theta$ and

$$S_n(r) = \int_0^{\theta_0} \sec^n\theta\ d\theta = \int_0^{\theta_0} \sec^{n-1}(\theta)\sec\theta d\theta = \int_0^r \cosh^{n-1}(x)dx.$$

■

We now prove Theorem 2. We first restate the theorem.

Theorem 2. *For $n \geq 3$, there exists a monotonically increasing function $H_n : \mathbf{R}_+ \to \mathbf{R}_+$ and a constant $C_n > 0$ such that if M is a hyperbolic n-manifold with totally geodesic boundary of area A then*

$$Vol_n(M) \geq H_n(A) \geq C_n.A^{\left(\frac{n-2}{n-1}\right)}$$

Proof: We let $A = \mathrm{Vol}_{n-1}(\partial M)$. We consider taking δ neighborhoods of ∂M in M denoted $N_\delta(\partial M)$. We let $\delta_0 > 0$ be the largest values such that the interior of $N_\delta(\partial M)$ is embedded. Then M has an orthogeodesic α of length $d \leq 2\delta_0$. Therefore there are two contributions to the volume, one part V_1 from the volume of the region associated with the orthogeodesic α, and the other V_2 from the neighborhood of the boundary with $\mathrm{Vol}_n(M) \geq \max(V_1, V_2)$. Then we have $V_1 = F_n(d)$ and as F_n is monotonically decreasing $V_1 \geq F_n(2\delta_0)$. As V_2 is the (one-sided) δ_0 nieighborhood of ∂M, by lemma 10 we have

$$V_2 = A.S_n(\delta_0).$$

We note that for small x, we have the simple approximation $S_n(x) \simeq x$, with $S_n(x) \geq \sinh x$ for all $x > 0$. Also by the formula for S_n, we have that S_n is convex.

By lemma 9, we have that F_n is a continuous monotonically decreasing positive function that tends to zero at infinity and tends to infinity at 0. We consider the continuous positive function $G_n(r) = \max(F_n(2r), A.S_n(r))$. Then G_n has a positive minimum value depending only on n and A denoted $H_n(A)$. As F_n is monotonically decreasing and S_n monotonically increasing, we have that $H_n = J^{-1}$, where J is the function

$$J(x) = \frac{F_n(2x)}{S_n(x)}.$$

We also note that $H_n(A)$ is the common value of the functions $F_n(2r), A.S_n(r)$ at their unique intersection point.

Therefore we have $\mathrm{Vol}_n(M) \geq G_n(\delta_0) \geq H_n(A)$. Obviously as V_2 is monotonic in volume, the function $H_n(A)$ is monotonic increasing in A (see figure 2).

By lemma 9, there is a $K_n > 0$ such that

$$\lim_{l\to 0^+} l^{n-2}F_n(l) = K_n$$

. Therefore there is a k_n such that for $l < 1$

$$F_n(r) \geq \frac{k_n}{r^{n-2}} = f_n(r).$$

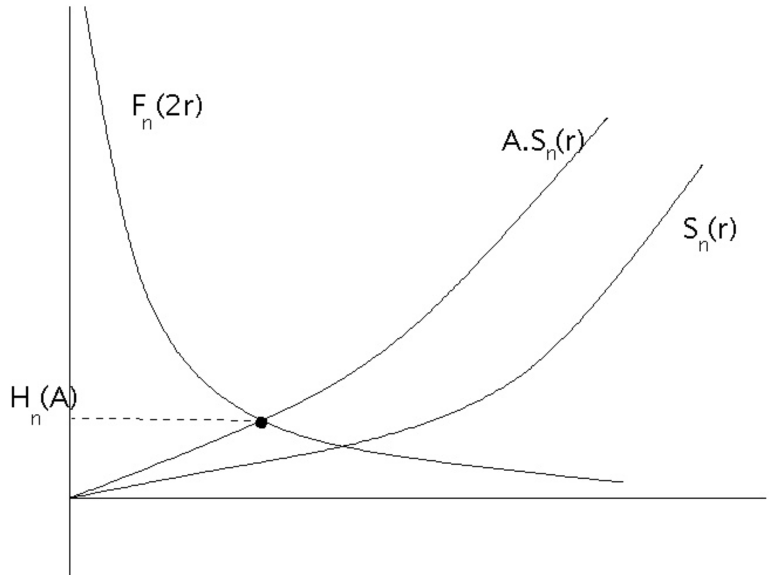

FIGURE 2. Graph of $G_n(r)$

Also we have that $S_n(r) \geq A.r = s_n(r)$ for all r. We consider the equation $f_n(2x) = s_n(x)$. This has a unique solution

$$\frac{k_n}{2^{n-2}.x^{n-2}} = A.x \qquad x = \left(\frac{k_n}{2^{n-2}A}\right)^{1/(n-1)} = \frac{k'_n}{A^{1/(n-1)}}$$

If $x < .5$ then we have for $r \leq x$ then $F_n(2r) \geq f_n(2r) \geq f_n(2x)$. Therefore $G_n(r) \geq f_n(2x)$ for $r \leq x$. Also for $r > x$ then $S_n(r) \geq s_n(r) \geq s_n(x)$. Therefore $G_n(r) \geq s_n(x)$ for $r > x$. Thus as $f_n(2x) = s_n(x)$ then $G_n(r) \geq s_n(x)$. Thus $H_n(A) \geq s_n(x)$ giving

$$H_n(A) \geq A.x = A.\frac{k'_n}{A^{1/(n-1)}} = k'_n A^{\frac{n-2}{n-1}}$$

If $x \geq .5$ then we have by monotonicity of f_n, s_n, that $f_n(1) \geq s_n(.5)$. Therefore for $r \geq .5$ $S_n(r) \geq s_n(r) \geq s_n(.5)$. Therefore $G_n(r) \geq s_n(.5)$ for $r \geq .5$. If $r < .5$ then $F_n(2r) \geq f_n(2r) \geq f_n(1) \geq s_n(.5)$. Therefore $G_n(r) \geq s_n(.5)$ for $r \leq .5$. Thus $G_n(r) \geq s_n(.5)$ for all r. Thus $H_n(A) \geq s_n(.5) = A/2$ giving

$$H_n(A) \geq \frac{1}{2}.A = C_n.A$$

■

We note that using the approximation $F_n(x) = \frac{K_n}{x^{n-2}}$ for x small we get the approximate bound of

$$H_n(A) \overset{\sim}{\geq} \left(\frac{K_n.A}{2}\right)^{\frac{n-2}{n-1}}.$$

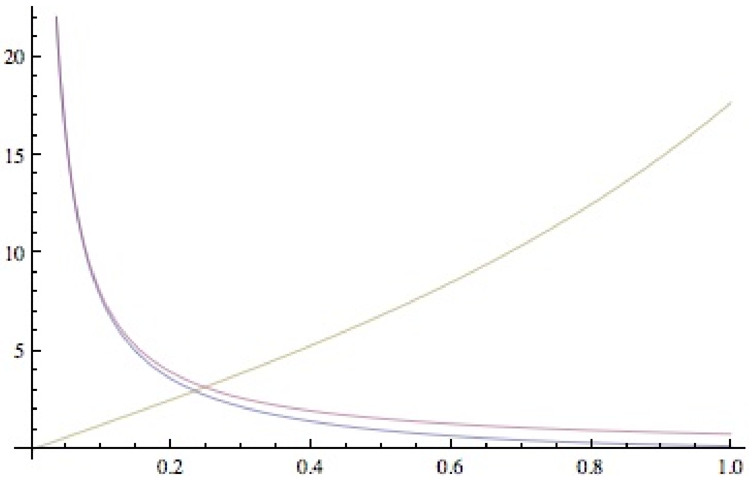

FIGURE 3. Graph of $F_3(2x)$, $\frac{\pi}{4x}$ and $4\pi S_3(x)$

7. General lower bound Examples

As the function $H_n(A)$ is monotonically increasing in A, if we have a general lower bound on the volume of a closed hyperbolic $n-$manifold then we obtain a lower bound for a hyperbolic $(n+1)-$manifold with geodesic boundary.

Hyperbolic 3-manifolds

In [5], Kojima and Miyamoto showed that the lowest volume hyperbolic three manifold with totally geodesic boundary has boundary a genus two surface and volume 6.452. Therefore 6.452 is the best general lower bound.

By the above lemma 9, $F_3(x) \simeq \frac{\pi}{2x}$ for small x. Graphing F_3 we see that this approximation is very close (see figure 3).

For comparison with Kojima and Miyamoto, we note that as a closed hyperbolic surface has area at least 4π, our methods give a lower bound of $H_3(4\pi)$ for the volume of a hyperbolic three-manifold with totally geodesic boundary. Plotting $F_3(2x)$ and $4\pi.S_3(x)$ we see that $H_3(4\pi) \approx 2.986$ (see figure 3).

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