Orthospectra and Dilogarithm Identities on Moduli Space

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Abstract

We show that the orthospectrum of a finite area hyperbolic surface S with non-empty totally geodesic boundary satisfies an infinite summation formula for the Rogers L-function. This formula gives an identity on the Moduli space for the underlying surface.

1 Orthospectrum of a hyperbolic surface

Let S be a finite area, hyperbolic surface with non-empty totally geodesic boundary ∂S. We let Ω be the volume measure on the unit tangent bundle $T_1(S)$. We let $\alpha(v)$ be the longest geodesic arc containing v as a tangent vector in S. Generically $\alpha(v)$ will be a geodesic arc with endpoints on ∂S.

We define the function $L : T_1(S) \rightarrow \overline{\mathbf{R}}$ by letting $L(v) = \text{Length}(\alpha(v))$. We note that $L(v)$ is measurable but can be infinite. We define measure M_S on the real line by $M_S = L_*\Omega$. Then M_S is a measure describing the distribution of the lengths of $\alpha(v)$.

As ∂S is totally geodesic, the components of ∂S are either finite length closed geodesics or bi-infinite geodesics. A *cusp* of S is an ideal vertex of a component of ∂S. We let N_S be the number of cusps of S. We denote by $\{\alpha_i\}$ the geodesic arcs in S which have endpoints perpendicular to ∂S and denote the length of α_i by l_i. We note that if S is an ideal $k-$gon then there are a finite number of geodesics α_i. Otherwise there are an infinite number. We call the set $\{l_i\}$ (with multiplicities) the *orthospectrum* of S. By doubling S we see that the orthospectrum of S corresponds to a subset of the closed geodesics of a finite area closed surface and therefore is a countable set.

We use the convention that the Euler characteristic $\chi(S)$ of a surface with boundary is the orbifold Euler characteristic of S with mirrored boundary (i.e. boundary edges and vertices are counted with multiplicity 1/2). Equivalently the Euler characteristic of S is half that of the double of S.

The main theorem of this paper is the following length spectrum identity

Theorem 1 *(Length Spectrum Identity) Let S be a finite area hyperbolic surface with non-empty totally geodesic boundary. Then the orthospectrum of S satisfies the equations*

$$\sum_i \mathcal{L}\left(\frac{1}{\cosh^2 \frac{l_i}{2}}\right) = \frac{\pi^2}{12}(6|\chi(S)| - N_S)$$

or equivalently

$$\sum_i \mathcal{L}\left(-\frac{1}{\sinh^2 \frac{l_i}{2}}\right) = \frac{\pi^2}{12}(6\chi(S) + N_S)$$

where $\mathcal{L}$ is the Rogers dilogarithm function (described below).

As the righthandside only depends on the underlying surface, the Length Spectrum Identity gives an identity on the moduli space $Mod(S)$ of the underlying surface.

2 Dilogarithms and Polylogarithms

The k^{th} polylogarithm function Li_k is defined by the Taylor series

$$\mathrm{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^k}$$

for $|z| < 1$ and by analytic continuation to C. In particular

$$\mathrm{Li}_0(z) = \frac{1}{1-z} \qquad\qquad \mathrm{Li}_1(z) = -\log(1-z).$$

Also

$$\mathrm{Li}_k'(z) = \frac{\mathrm{Li}_{k-1}(z)}{z} \qquad \text{giving} \qquad \mathrm{Li}_k(z) = \int_0^z \frac{\mathrm{Li}_{k-1}(z)}{z}\, dz.$$

Also the functions Li_k are related to the Riemann ζ function by $\mathrm{Li}_k(1) = \zeta(k)$.

The dilogarithm function is the function $\mathrm{Li}_2(z)$ and is given by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-z)}{z}\, dz.$$

Below is a brief description of some properties of the dilogarithm function. They can all be found in 1991 survey "Structural Properties of Polylogarithms" by L. Lewin (see [5]). From the power series representation, it is easy to see that the dilogarithm function satisfies the functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \frac{1}{2}\mathrm{Li}_2(z^2).$$

Other functional relations of the dilogarithm can be best described by normalizing the dilogarithm function. The (extended) Rogers $\mathcal{L}-$function (see [9]) is defined by

$$\mathcal{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log|x|\log(1-x) \qquad x \leq 1.$$

In terms of the Rogers $\mathcal{L}-$function, Euler's reflection relations for the dilogarithm are

$$\begin{aligned} \mathcal{L}(x) + \mathcal{L}(1-x) = \mathcal{L}(1) = \frac{\pi^2}{6} \qquad 0 \leq x \leq 1 \\ \mathcal{L}(x) + \mathcal{L}(x^{-1}) = 2\mathcal{L}(-1) = -\frac{\pi^2}{6} \qquad x < 0 \end{aligned} \tag{1}$$

Also in terms of $\mathcal{L}$, Landen's identity is

$$\mathcal{L}\left(\frac{-x}{1-x}\right) = -\mathcal{L}(x) \qquad 0 < x < 1 \tag{2}$$

and Abel's functional equation is

$$\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right). \tag{3}$$

A closed form for $\mathcal{L}(x)$ is known for certain values of x including

$$\mathcal{L}(1) = \frac{\pi^2}{6} \qquad \mathcal{L}\left(\frac{1}{2}\right) = \frac{\pi^2}{12} \qquad \text{(Euler)} \qquad \mathcal{L}(-\phi) = -\frac{\pi^2}{10} \qquad \mathcal{L}\left(\phi^{-2}\right) = \frac{\pi^2}{15} \qquad \text{(Landen)}$$

where ϕ is the golden ratio.

Finally we note that Ramanujan found the following linear identities (see [1], [3])

$$6\mathcal{L}\left(\frac{1}{3}\right) - \mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{3} \qquad 3\mathcal{L}\left(\frac{1}{4}\right) + \mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{6}. \tag{4}$$

3 Measure M_S

The Length Spectrum Identity Theorem follows immediately from the following theorem describing the measure M_S. This theorem is proved in a later section of the paper.

Theorem 2 *There exists a function $\rho : \mathbf{R}^2 \to \mathbf{R}$ such that infinitesimally*

$$dM_S = \left(\frac{4N_S x^2}{\sinh^2 x} + \sum_i \rho(l_i, x)\right) dx$$

where N_S is the number of cusps of S. Furthermore the total mass of the measure $\rho(l,x)dx$ on the real line is given by

$$F(l) = \int_0^\infty \rho(l,x)\ dx = 8\mathcal{L}\left(\frac{1}{\cosh^2 \frac{l}{2}}\right) = -8\mathcal{L}\left(-\frac{1}{\sinh^2 \frac{l}{2}}\right)$$

In particular the measure M_S depends only on the orthospectrum of S.

The Length Spectrum Identity now follows as a corollary of the above theorem.

Proof of Length Spectrum Identity: As $M_S = L_*\Omega$, then M_S has total mass equal to the volume of $T_1(S)$. Therefore $M_S(\mathbf{R}) = \Omega(T_1(S)) = 4\pi^2|\chi(S)|$. Summing up the masses of measures in theorem 2 above, we immediately obtain

$$4\pi^2|\chi(S)| = \int_0^\infty dM_S = 4N_S\int_0^\infty \frac{x^2}{\sinh^2 x}dx + \sum_i F(l_i).$$

An elementary calculation (see [2]) shows that

$$\int_0^\infty \frac{x^2}{\sinh^2 x}dx = \frac{\pi^2}{6}.$$

Therefore using $F(l) = 8\mathcal{L}\left(\frac{1}{\cosh^2\frac{l}{2}}\right)$ we get

$$4\pi^2|\chi(S)| = \frac{2N_S\pi^2}{3} + 8.\sum_i \mathcal{L}\left(\frac{1}{\cosh^2\frac{l_i}{2}}\right).$$

giving

$$\sum_i \mathcal{L}\left(\frac{1}{\cosh^2\frac{l_i}{2}}\right) = \frac{\pi^2(6|\chi(S)| - N_S)}{12}$$

Equivalently, using the form $F(l) = -8\mathcal{L}\left(-\frac{1}{\sinh^2\frac{l}{2}}\right)$ we get

$$\sum_i \mathcal{L}\left(-\frac{1}{\sinh^2\frac{l_i}{2}}\right) = \frac{\pi^2(6\chi(S) + N_S)}{12}$$

■

4 Identity on $Mod(S)$

As the Length Spectrum Identity has righthandside only depending on $\chi(S)$ and N_S, we obtain a summation identity on the Moduli space $Mod(S)$ of S. There is an obvious comparison to be made between the above Length Spectrum Identity and the (generalized) McShane identity of McShane and Mirzakhani (see [6] and [7]). Both identities are on the Moduli space of a surface and are given by the summation of a certain functions over a length spectrum of the surface. Alternately there does not seem to be a direct relation as both the spectra and functions used are completely different with the McShane identity being for embedded geodesics while the geodesics in the Length Spectrum Identity are not. Also the function summed in the McShane identity is an elementary hyperbolic trigonometric function unlike the Rogers L-function.

We note that the Length Spectrum Identity is an infinite relation except in the case when S is an ideal polygon. In the case of S being an ideal polygon, we will now show that the Length Spectrum Identities includes all the classical dilogarithm identities described above.

5 Classical Identities and the Moduli space of ideal polygons

For S an ideal n-gon, the Length Spectrum Identity is a finite summation relation. We will show that the associated relations give an infinite list of finite relations including the classical identities stated in the previous section.

If $\{l_i\}$ is an orthospectrum, we will define two parameterizations by letting

$$a_i = -\frac{1}{\sinh^2 \frac{l_i}{2}} \qquad\qquad b_i = \frac{1}{\cosh^2 \frac{l_i}{2}}.$$

We now consider the Poincaré disk model and let $x_i, i = 1, \ldots, n$ be the vertices in anticlockwise cyclic ordering around the circle. Let s_i be the side $x_i x_{i+i}$. Let l_{ij} be the length of the diagonal between s_i and s_j for $|i-j| \geq 2$. We define the cross-ratio by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_3)(z_4 - z_2)}.$$

As the cross ratio is invariant under Möbius transformations, we map the quadruple $(x_i, x_{i+1}, x_j, x_{j+1})$ to $(-1, 1, e^{l_{ij}}, -e^{l_{ij}})$.Then

$$[x_i, x_{i+1}, x_j, x_{j+1}] = [-1, 1, e^{l_{ij}}, -e^{l_{ij}}] = -\frac{(-1-1)(-e^{l_{ij}} - e^{l_{ij}})}{(-1-e^{l_{ij}})(-e^{l_{ij}}-1)} = \frac{4e^{l_{ij}}}{(e^{l_{ij}}+1)^2} = \frac{1}{\cosh^2(\frac{l_{ij}}{2})}$$

As S has area $(n-2)\pi$ and n cusps, $\chi(S) = (n-2)/2$ and $N_S = n$. Thus the Length Spectrum Identity becomes

$$\sum_{i,j} \mathcal{L}([x_i, x_{i+1}, x_j, x_{j+1}]) = \frac{(n-3)\pi^2}{6} \tag{5}$$

where the sum is over all ordered pairs i, j such that the sides s_i, s_j are disjoint (at infinity). In terms of dilogarithms we get

$$\sum_{i,j} \mathrm{Li}_2\left([x_i, x_{i+1}, x_j, x_{j+1}]\right) = \frac{(n-3)\pi^2}{6} - \frac{1}{2}\sum_{i,j} \log\left(1 - [x_i, x_{i+1}, x_j, x_{j+1}]\right) \log\left([x_i, x_{i+1}, x_j, x_{j+1}]\right) \tag{6}$$

5.1 Cases corresponding to classical relations

Quadrilateral and Euler reflection identity: The ideal quadrilateral has 4 cusps and two orthholengths l_1, l_2. By elementary hyperbolic geometry we have $\sinh(l_1/2).\sinh(l_2/2) = 1$. Therefore $a_1.a_2 = 1$ and letting $a = a_1$ the Length Spectrum identity is equivalent to the the classical reflection identity of Euler.

$$\mathcal{L}(a) + \mathcal{L}(a^{-1}) = -\frac{\pi^2}{6}. \tag{7}$$

Also we have

$$b_2 = \frac{1}{\cosh^2(l_2/2)} = \frac{1}{1+\sinh^2(l_2/2)} = \frac{1}{1+\frac{1}{\sinh^2(l_1/2)}} = \frac{\sinh^2(l_1/2)}{\cosh^2(l_1/2)} = 1 - \frac{1}{\cosh^2(l_1/2)} = 1 - b_1$$

Thus letting $b = b_1$, the Length Spectrum identity is equivalent to the Euler reflection identity

$$\mathcal{L}(b) + \mathcal{L}(1-b) = \frac{\pi^2}{6}. \tag{8}$$

Pentagon and Abel's Identity: If we choose a general ideal pentagon then there are 5 diagonals and therefore 5 parameters a_i. We send three of the vertices to $0, 1, \infty$ and the other two to u, v with $0 < u < v < 1$. Then the cross ratios in terms of u, v are

$$u, \qquad 1-v, \qquad \frac{v-u}{v}, \qquad \frac{v-u}{1-u}, \qquad \frac{u(1-v)}{v(1-u)}$$

Putting into the equation we obtain the following equation.

$$\mathcal{L}\left(u\right) + \mathcal{L}\left(1-v\right) + \mathcal{L}\left(\frac{v-u}{v}\right) + \mathcal{L}\left(\frac{v-u}{1-u}\right) + \mathcal{L}\left(\frac{u(1-v)}{v(1-u)}\right) = \frac{\pi^2}{3}. \tag{9}$$

Letting $x = u/v, y = v$, then we get

$$\mathcal{L}\left(xy\right) + \mathcal{L}\left(1-y\right) + \mathcal{L}\left(1-x\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) = \frac{\pi^2}{3}. \tag{10}$$

Now by applying Euler's reflection identities for x, y, we obtain Abel's identity for the Rogers $\mathcal{L}-$function.

$$\mathcal{L}\left(x\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(xy\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right). \tag{11}$$

General equation: We obtain similar finite identities in the general ideal $n-$gon case. In general we note that equation 5 will have $(n-3)$ independent variables and will be given by the summation of evaluating $\mathcal{L}$ on $\frac{n(n-3)}{2}$ rational functions in the $(n-3)$ variables.

5.2 Regular ideal n-gon relation

We now consider the dilogarithm equation for the specific case of a regular ideal n-gon. In this case the cross ratios can be calculated and the dilogarithm formulas for specific values of the dilogarithm function.

We consider a regular ideal $n-$gon in with center 0 in the Poincaré disk model and vertices at $v_k = u^k, k = 0, \ldots, n-1$ for $u = e^{\frac{2\pi i}{n}}$. Then equation 5 can be thought of as an equation on the roots of the polynomial $z^n = 1$. We have

$$[v_0, v_1, v_r, v_{r+1}] = -\frac{(1-u)(u^{r+1}-u^r)}{(1-u^r)(u^{r+1}-u)} = \frac{u^r(u-1)^2}{u.(u^r-1)^2} = \frac{\sin^2(\frac{\pi}{n})}{\sin^2(\frac{r\pi}{n})}$$

For $r < n/2$ there are exactly n distinct perpendiculars between sides separated by r sides and for $r = n/2$ there are $n/2$ such sides. To take care of the even and odd case simultaneously we let e_n be 1 if n is even and 0 if n is odd. Therefore we have

$$\sum_{r=2}^{\lceil n/2\rceil-1} n.\mathcal{L}\left(\frac{\sin^2(\frac{\pi}{n})}{\sin^2(\frac{r\pi}{n})}\right) + e_n\frac{n}{2}.\mathcal{L}\left(\sin^2\left(\frac{\pi}{n}\right)\right) = \frac{(n-3)\pi^2}{6} \tag{12}$$

where $\lceil x\rceil$ is the least integer greater than or equal to x. Dividing by n we get

$$\sum_{r=2}^{\lceil n/2\rceil-1} \mathcal{L}\left(\frac{\sin^2(\frac{\pi}{n})}{\sin^2(\frac{r\pi}{n})}\right) + \frac{e_n}{2}.\mathcal{L}\left(\sin^2\frac{\pi}{n}\right) = \frac{(n-3)\pi^2}{6n}$$

Limiting case: We let n go to infinity and obtain the equation

$$\lim_{n\to\infty}\left(\sum_{r=2}^{\lceil n/2\rceil-1} \mathcal{L}\left(\frac{\sin^2(\frac{\pi}{n})}{\sin^2(\frac{r\pi}{n})}\right) + \frac{e_n}{2}.\mathcal{L}\left(\sin^2\frac{\pi}{n}\right)\right) = \lim_{n\to\infty}\frac{(n-3)\pi^2}{6n} = \frac{\pi^2}{6}$$

This gives a Rogers $\mathcal{L}-$function series relation due to Lewin (see p. 298 of [5])

$$\sum_{r=2}^{\infty}\mathcal{L}\left(\frac{1}{r^2}\right) = \frac{\pi^2}{6} \tag{13}$$

Regular ideal quadrilateral: This case is trivial $a_1 = a_2 = -1$, $b_1 = b_2 = 1/2$ and equations 7,8 give the classical evaluations

$$\mathcal{L}(-1) = -\frac{\pi^2}{12} \qquad \text{and} \qquad \mathcal{L}(\frac{1}{2}) = \frac{\pi^2}{12}.$$

Regular ideal pentagon, Golden Mean: For the regular ideal pentagon, the orthospectrum consists of 5 geodesics each of the same length l. Using the formula above for $n = 5, r = 2$ we obtain that l satisfies

$$\cosh^2\left(\frac{l}{2}\right) = \frac{2}{\sqrt{5}+3} = \phi^2$$

where ϕ is the golden mean. Therefore as $\phi^2 = \phi + 1$

$$\sinh^2\left(\frac{l}{2}\right) = \phi^2 - 1 = \phi$$

and we have $a = -\phi^{-1}$. Thus the Length Spectrum Identity gives the classical relations of Landen

$$\mathcal{L}(-\phi^{-1}) = -\frac{\pi^2}{15} \qquad \mathcal{L}(\phi^{-2}) = \frac{\pi^2}{15}.$$

Applying the quadrilateral relations 7, 8 we also get

$$\mathcal{L}(-\phi) = -\frac{\pi^2}{6} - \mathcal{L}(-\phi^{-1}) = -\frac{\pi^2}{10} \qquad .$$

Regular ideal Hexagon: For a regular ideal hexagon, there are 9 elements of the orthospectrum, with the 6 being perpendicular to sides one apart and three being perpendicular to opposite sides. Putting $n = 6$ into equation 12 above then gives the linear relation

$$6\mathcal{L}(\frac{1}{3}) + 3\mathcal{L}(\frac{1}{4}) = \frac{\pi^2}{2} \tag{14}$$

Ramanujan Identities: We can form an elementary ideal hexagon by gluing two regular ideal quadrilaterals along a common edge. Then calculating orthospectra, the length spectrum identity gives

$$2\mathcal{L}\left(\frac{1}{2}\right) + 2\mathcal{L}\left(\frac{1}{3}\right) + 4\mathcal{L}\left(\frac{1}{4}\right) + \mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{2}$$

Therefore as $\mathcal{L}\left(\frac{1}{2}\right) = \frac{\pi^2}{12}$ we obtain

$$2\mathcal{L}\left(\frac{1}{3}\right) + 4\mathcal{L}\left(\frac{1}{4}\right) + \mathcal{L}\left(\frac{1}{9}\right) = \frac{\pi^2}{3} \tag{15}$$

We see that taking linear combinations of equations 14, 15, we obtain the Ramanujan identities in equations 4.

6 Infinite Relations

The Length Spectrum Identity is an infinite relation on $Mod(S)$ when the surface S is not an ideal polygon. We now calculate the identity for some simple cases.

We consider the hyperbolic annulus with one boundary component a closed geodesic of length l and the other an infinite geodesic with one cusp. Then to find the orthospectrum, we lift the closed geodesic to the geodesic g joining $0, \infty$ in the upper half space model and such that a cusp is at the point 1. Then the covering group is $\mathbf{Z}$ and generated by the map $f(z) = \lambda z$ for $\lambda = e^l$. The lifts of the boundary components consists of the geodesic g joining $0, \infty$ for the closed boundary geodesic and the geodesics g_k joining λ^k to λ^{k+1} for k any integer, for the boundary component with cusp. As there are no orthgeodesics from the closed boundary component to itself, each orthogeodesic has a copy with endpoint on the geodesic g_0. Then the Length spectrum identity gives

$$\mathcal{L}([1, \lambda, 0, \infty]) + \sum_{k=2}^{\infty} \mathcal{L}([1, \lambda, \lambda^k, \lambda^{k+1}]) = \frac{\pi^2}{6}$$

Therefore calculating cross-ratios we get

$$\mathcal{L}\left(1 - e^{-l}\right) + \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{\sinh^2 \frac{l}{2}}{\sinh^2 \frac{kl}{2}}\right) = \frac{\pi^2}{6}$$

Using Euler's reflection identity we get

$$\mathcal{L}\left(e^{-l}\right) = \sum_{k=2}^{\infty} \mathcal{L}\left(\frac{\sinh^2 \frac{l}{2}}{\sinh^2 \frac{kl}{2}}\right) \tag{16}$$

If we let l tend to 0 we once more obtain Lewin's infinite identity (eqn. 13) so this can be thought of as a generalized version of Lewin's identity.

To obtain another relation, we glue two such annuli along their common closed geodesic boundary component by a dehn twist of size t. Then once again as above, we obtain a lift with the closed geodesic lifting to the geodesic g joining $0, \infty$ with one cusp lifting to 1 and another to $-x$ for some $x > 0$. Then one component of the boundary lifts to geodesics g_k and the other component to geodesics h_k with endpoints $-x\lambda^{k+1}, -x.\lambda^k$. Therefore $x = e^t$. The length spectrum identity is

$$\sum_{k=-\infty}^{\infty} \mathcal{L}([-x\lambda, -x, \lambda^k, \lambda^{k+1}]) + 2. \sum_{k=2}^{\infty} \mathcal{L}([1, \lambda, \lambda^k, \lambda^{k+1}]) = \frac{\pi^2}{3}$$

Subtracting the length spectrum identity for each side gives

$$\sum_{k=-\infty}^{\infty} \mathcal{L}([-x\lambda, -x, \lambda^k, \lambda^{k+1}]) = 2.\mathcal{L}\left(1 - e^{-l}\right).$$

Evaluating we get

$$\sum_{k=-\infty}^{\infty} \mathcal{L}\left(\frac{\cosh(l) - 1}{\cosh(kl - t) + \cosh(l)}\right) = 2.\mathcal{L}\left(1 - e^{-l}\right).$$

We now consider the geometry of ideal quadrilaterals in the hyperbolic plane.

7 Intersections with ideal quadrilaterals

Given two disjoint geodesics g_1, g_2 in $\mathbf{H}^2$ with perpendicular distance l between them, let Q be the ideal quadrilateral with opposite sides g_1, g_2. Then we can map Q by a Móbius transformation to the ideal quadrilateral Q_a in the upper half-plane with vertices $a, 0, 1, \infty \in \overline{\mathbf{R}}$ where $a < 0$. Similarly we can map Q to the ideal quadrilateral Q_b in the upper half-plane with vertices $0, b, 1, \infty \in \overline{\mathbf{R}}$ where $b > 0$. Using cross-ratios we have that

$$a = -\frac{1}{\sinh^2 \frac{l}{2}} \qquad b = \frac{1}{\cosh^2 \frac{l}{2}} \tag{17}$$

The choice of normalization Q_a, Q_b leads to the equivalent forms of the Length Spectrum Identity.

If $x, y \in \mathbf{R}, x \neq y$, we let $g(x, y)$ be the geodesic in the upper half plane with end points x, y. Then for $(x, y) \in (a, 0) \times (1, \infty)$, the geodesic $g(x, y)$ intersects Q_a in a definite length denoted $L_a(x, y)$. Similarly for $(x, y) \in (0, b) \times (1, \infty)$, the geodesic $g(x, y)$ intersects Q_b in a definite length denoted $L_b(x, y)$.

Lemma 3 *If $k = a$ or b, the map L_k is given*

$$L_k(x, y) = \frac{1}{2} \ln \left(\frac{y(y - k)(x - 1)}{x(x - k)(y - 1)}\right) = \frac{1}{2} \ln \frac{f_k(y)}{f_k(x)}.$$

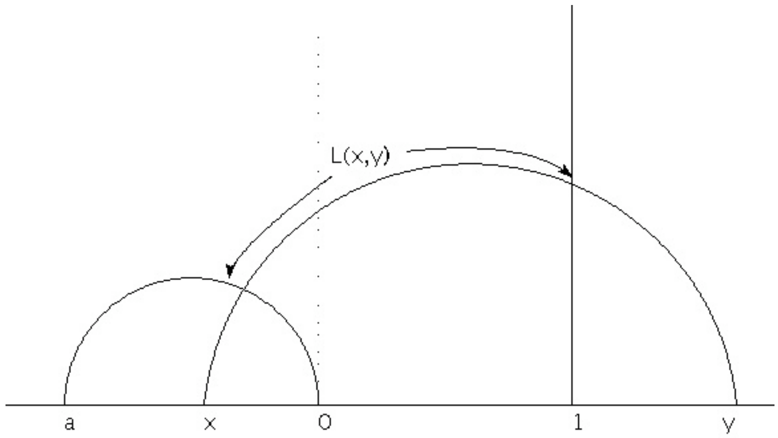

Figure 1: Length intersection function $L_a(x, y)$

where

$$f_k(x) = \frac{x(x-k)}{x-1}$$

Proof: Let T be the ideal triangle with vertices $0, 1, \infty$. Let $l_1 : (-\infty, 0) \times (0, 1) \to \mathbf{R}$ and $l_2 : (-\infty, 0) \times (1, \infty) \to \mathbf{R}$ be given by letting $l_1(x, y)$ be the length of the intersection of $g(x, y)$ with T and $l_2(x, y)$ be the length of the intersection of $g(x, y)$ with T. By a previous paper (see [2]) the functions l_i are given by

$$l_1(x, y) = \frac{1}{2} \ln \left(\frac{1-x}{1-y} \right) \qquad l_2(x, y) = \frac{1}{2} \ln \left(\frac{y(x-1)}{x(y-1)} \right)$$

To calculate L_a, we split the quadrilateral Q_a by the vertical line at $x = 0$ into two ideal triangles T_1, T_2 where T_1 has vertices $0, 1, \infty$ and T_2 has vertices $a, 0, \infty$. Then $T_1 = T$ and $f_2(z) = z/a$ sends T_2 to T. Therefore

$$L_a(x, y) = l_2(x, y) + l_1(y/a, x/a)$$

Therefore

$$L_a(x, y) = \frac{1}{2} \ln \left(\frac{y(x-1)}{x(y-1)} \right) + \frac{1}{2} \ln \left(\frac{1-y/a}{1-x/a} \right) = \frac{1}{2} \ln \left(\frac{y(x-1)(a-y)}{x(y-1)(a-x)} \right) = \frac{1}{2} \ln \left(\frac{y(y-a)(x-1)}{x(x-a)(y-1)} \right)$$

To calculate L_b we note that the map

$$g(z) = \frac{z-b}{1-b}$$

sends Q_b to Q_a where $a = g(0) = \frac{-b}{1-b}$. Therefore

$$L_b(x, y) = L_a(g(x), g(y)) = \frac{1}{2} \ln \left(\frac{\frac{y-b}{1-b} \left(\frac{y-b}{1-b} - \frac{-b}{1-b} \right) \left(\frac{x-b}{1-b} - 1 \right)}{\frac{x-b}{1-b} \left(\frac{x-b}{1-b} - \frac{-b}{1-b} \right) \left(\frac{y-b}{1-b} - 1 \right)} \right).$$

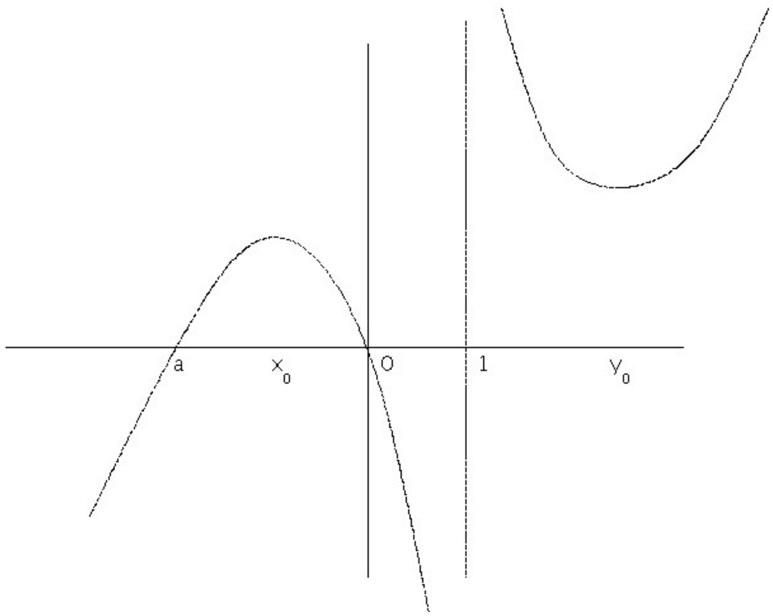

Figure 2: Graph of function $f_a(x)$

Simplifying we get

$$L_b(x,y) = \frac{1}{2}\ln\left(\frac{y(y-b)(x-1)}{x(x-b)(y-1)}\right)$$

■

We consider the rational function $f_k(x)$ defined above. Differentiating we have

$$f_k'(x) = \frac{(2x-k)(x-1) - 1.(x^2-kx)}{(x-1)^2} = \frac{x^2-2x+k}{(x-1)^2}$$

Therefore $f_k(x)$ has two critical points $1 \pm \sqrt{1-k}$. We label the critical points $x_0 = 1-\sqrt{1-k}$ and $y_0 = 1+\sqrt{1-k}$.

8 Summation Identity for F

In this section we prove that the first part of theorem 2 and a summation identity for F.

Lemma 4 *Let S be a finite area hyperbolic surface with non-empty totally geodesic boundary. Then the measure M_S has form*

$$dM_S = \left(\frac{4N_S x^2}{\sinh^2 x} + \sum_i \rho(l_i, x)\right) dx$$

where N_S is the number of cusps of S.

Proof: By definition

$$(L_*\Omega)(\phi) = \int_{T_1(S)} \phi(L(v))\ d\Omega.$$

Let α, β be two arcs in S with endpoints on ∂S. Then we say $\alpha \sim \beta$ if they are homotopic relative to the boundary ∂S.

We define the sets $A_i = \{v \in T_1(S) | \alpha(v) \sim \alpha_i\}$. Also for each cusp c we define $A_c = \{v \in T_1(S)|\alpha(v) \sim c\}$ where $\alpha(v) \sim c$ if $\alpha(v)$ can be homotoped (rel boundary) out the cusp c. Note that for $v \in A_i$ or $v \in A_c$, $L(v)$ is finite. Finally we define the set A_∞ to be all v not in any A_i or A_c. By definition, the sets A_i, A_c, A_∞ form a partition of $T_1(S)$. If we double S along its boundary, the geodesic arcs α_i correspond to a subset of the geodesics of the doubled surface. Therefore as the length spectrum of the doubled surface is countable, so is the collection of arcs α_i in S. Also, by ergodicity of geodesic flow on S (see [4]), the set A_∞ is a measure zero.

Therefore

$$(L_*\Omega)(\phi) = \sum_i \int_{A_i} \phi(L(v))\ d\Omega + \sum_c \int_{A_c} \phi(L(v))\ d\Omega.$$

We let

$$a_i = -\frac{1}{\sinh^2 \frac{l_i}{2}}$$

Then setting $Q_i = Q_{a_i}$, we have that Q_i is a quadrilateral with perpendicular of length l_i. We lift α_i to the upper half plane so that it is the perpendicular of length l_i in Q_i. We lift each cusp c to the ideal vertex at infinity between the vertical geodesics $x = 0, x = 1$. Let T be the ideal triangle with vertices $0, 1, \infty \in \overline{\mathbf{R}}$.

If $v \in T_1(\mathbf{H}^2)$ in the upper half plane, we define $g(v)$ to be the geodesic with tangent vector v. We also denote the endpoints of $g(v)$ by $(x(v), y(v))$.

We lift the set A_i to the set $A'_i \subseteq T_1(Q_i)$. Then for $v \in A'_i$ the geodesic arc $\alpha'(v) = Q_i \cap g(v)$ is a lift of $\alpha(v)$. Similarly we lift A_c to the set $A'_c \subseteq T_1(T)$.Then for $v \in A'_c$ the geodesic arc $\alpha'(v) = T \cap g(v)$ is a lift of $\alpha(v)$. By abuse of notation we also let Ω be the volume measure on $T_1(\mathbf{H}^2)$. We parameterize $T_1(\mathbf{H}^2)$ by $(x, y, l) \in \overline{\mathbf{R}} \times \overline{\mathbf{R}} \times \mathbf{R}$ where (x, y, l) corresponds to the vector v such that $g(v)$ has ordered endpoints (x, y) and v has basepoint on $g(v)$ a distance l from the highest point of $g(v)$ in the upper half-plane. Then the volume form Ω can be written as (see [8])

$$d\Omega = \frac{2dxdydl}{(x-y)^2}.$$

Therefore

$$\int_{A_c} \phi(L(v))\ d\Omega = \int_{A'_c} \frac{2.\phi(L(v))\ dxdydl}{(x-y)^2}$$

We note that $L(v)$ only depends on the endpoints and therefore we can write $L(v) = L(x, y)$. If $v \in A'_c$ then either (x, y) or $(y, x) \in (-\infty, 0) \times (1, \infty)$. Integrating over l we have

$$\int_{A'_c} \frac{2.\phi(L(v))\ dxdydl}{(x-y)^2} = \int_{-\infty}^{0} \int_{1}^{\infty} \frac{4.\phi(L(x,y))L(x,y)\ dxdy}{(x-y)^2}.$$

By our previous paper [2]

$$\int_{-\infty}^{0}\int_{1}^{\infty}\frac{4.\phi(L(x,y))L(x,y)\ dxdy}{(x-y)^2}.=\int_{0}^{\infty}\frac{4.\phi(L).L^2dL}{\sinh^2 L}.$$

Thus as there are N_S cusps we have

$$\sum_c\int_{A_c}\phi(L(v))\ d\Omega = N_S.\int_0^\infty\frac{4.\phi(L).L^2dL}{\sinh^2 L}=M_\infty(\phi)$$

where M_∞ is the measure with infinitesimal

$$dM_\infty=\frac{4N_Sx^2dx}{\sinh^2 x}.$$

Similarly we have by lifting A_i to A'_i that

$$\int_{A_i}\phi(L(v))\ d\Omega=\int_{A'_i}\frac{2.\phi(L(v))\ dxdydl}{(x-y)^2}.$$

If $v\in A'_i$ then either (x,y) or $(y,x)\in(a_i,0)\times(1,\infty)$. Integrating over l we have

$$\int_{A'_i}\frac{2.\phi(L(v))\ dxdydl}{(x-y)^2}=\int_{a_i}^{0}\int_1^\infty\frac{4.\phi(L(x,y))L(x,y)\ dxdy}{(x-y)^2}.$$

For $a<0$ we define $M_a(\phi)$ to be the righthandside of the above equation. Then

$$M_a(\phi)=\int_a^0\int_1^\infty\frac{4.\phi(L(x,y))L(x,y)\ dxdy}{(x-y)^2}.$$

Then

$$M_S=M_\infty+\sum_i M_{a_i}$$

By lemma 3 we have

$$L(x,y)=\frac{1}{2}\log\left(\frac{y(y-a)(x-1)}{x(x-a)(y-1)}\right)=\frac{1}{2}\log\left(\frac{f(y)}{f(x)}\right)\qquad\text{for}\qquad f(x)=\frac{x(x-a)}{x-1}.$$

Taking derivatives of the length function $L(x,y)$ we have

$$\frac{\partial L}{\partial x}=-\frac{f'(x)}{2f(x)}\qquad\frac{\partial L}{\partial y}=\frac{f'(y)}{2f(y)}.$$

By the previous section, the function f has critical points x_0,y_0. Furthermore on $(a,0)$ the function $f(x)$ has global maximum at x_0 and on $(1,\infty)$, f has global minimum at y_0. Therefore fixing x, the function $u:(1,\infty)\to\mathbf{R}$ given by $u(y)=L(x,y)$ is decreasing on $(1,y_0)$ and increasing on (y_0,∞). Therefore we make the change of variable $t=L(x,y), x=x$. Finding inverses for f we define the two function g_+,g_- by

$$g_\pm(x)=\frac{(a+x)\pm\sqrt{(a+x)^2-4x}}{2}.$$

Then solving $t = L(x,y)$ gives $f(y) = f(x)e^{2t}$. Therefore on $(1, y_0)$ we have $y = g_-(f(x)e^{2t})$ and on (y_0, ∞) we have $y = g_+(f(x)e^{2t})$ Therefore we have

$$M_a(\phi) = \int_a^0 \left(\int_1^{y_0} + \int_{y_0}^\infty \frac{4.\phi(L(x,y))L(x,y)\ dy}{(x-y)^2} \right) dx.$$

and

$$\int_1^{y_0} \frac{4.\phi(L(x,y))L(x,y)\ dy}{(x-y)^2} = \int_\infty^{L(x,y_0)} \frac{4.\phi(t)t.g'_-(f(x)e^{2t})2f(x)e^{2t}dt}{(x-g_-(f(x)e^{2t}))^2}$$
$$\int_{y_0}^\infty \frac{4.\phi(L(x,y))L(x,y)\ dy}{(x-y)^2} = \int_{L(x,y_0)}^\infty \frac{4.\phi(t)t.g'_+(f(x)e^{2t})2f(x)e^{2t}dt}{(x-g_+(f(x)e^{2t})^2}$$

Therefore combining we have

$$M_a(\phi) = \int_a^0 \int_{L(x,y_0)}^\infty 8.\phi(t).t.e^{2t}.f(x) \left(\frac{g'_+(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} - \frac{g'_-(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2} \right) dtdx$$

We switch the order of integration. The function $L(x, y_0)$ is minimum at x_0 with minimum value $l = L(x_0, y_0)$ being the length of the perpendicular (see figure 2). Thus we integrate t from l to infinity. The integral in the x direction is between the two x solutions of $t = L(x, y_0)$ which are solutions to $f(x) = f(y_0)e^{-2t}$. Thus we integrate x from $g_-(f(y_0)e^{-2t})$ to $g_+(f(y_0)e^{-2t})$ giving

$$M_a(\phi) = \int_{l_a}^\infty 8.\phi(t).t.e^{2t}dt \left(\int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} \left(\frac{g'_+(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} - \frac{g'_-(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2} \right) f(x)dx \right)$$

Therefore

$$M_a(\phi) = \int_0^\infty \phi(t).\rho(l,t)dt$$

where

$$\rho(l,t) = 8te^{2t}\chi_{[l,\infty)}. \left(\int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} \left(\frac{g'_+(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} - \frac{g'_-(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2} \right) f(x)dx \right)$$

$$\text{and } f(x) = \frac{x(x-a)}{x-1} \text{ where } a = -\frac{1}{\sinh^2 l/2}$$

Therefore

$$M_S(\phi) = (L_*\Omega)(\phi) = \int_0^\infty \phi(x)\rho(x)dx$$

where

$$\rho(x) = \frac{4N_S x^2}{\sinh^2 x} + \sum_i \rho(l_i, x)$$

■

We note that the equivalence relation $\sim$ on $T_1(S)$ described in the above lemma gives a partition of $T_1(S)$ which is quite complicated with each tangent plane at a point $x \in S$ being decomposed in to an (countably) infinite collection of intervals. In particular it does not arise from a decomposition of the surface S into regions.

We now define

$$F(l) = \int_0^\infty \rho(l,x)dx.$$

Then we have the following corollary

Corollary 5 *Let S be a finite area hyperbolic surface with non-empty totally geodesic boundary. Then the function F satisfies*

$$\sum_i F(l_i) = \frac{2\pi^2}{3}(6|\chi(S)| - N_S) \tag{18}$$

Proof: As $M_S = L_*\Omega$ it has total mass equal to the volume of $T_1(S)$ which is $4\pi^2|\chi(S)|$. Therefore

$$\Omega(T_1(S)) = 4\pi^2|\chi(S)| = M_S(1) = M_\infty(1) + \sum_i M_{a_i}(1) \tag{19}$$

By an elementary calculation (see [2])

$$\int_0^\infty \frac{x^2dx}{\sinh^2 x} = \frac{\pi^2}{6}.$$

Therefore

$$M_\infty(1) = \int_0^\infty \frac{4N_S x^2 dx}{\sinh^2 x} = 4N_S \int_0^\infty \frac{x^2dx}{\sinh^2 x} = 4N_S.\frac{\pi^2}{6} = \frac{2N_S\pi^2}{3}.$$

Using lemma 3 we substitute the formula for $L_a(x,y)$ to obtain

$$M_a(1) = \int_a^0 \int_1^\infty \frac{2.\log\left(\frac{y(y-a)(x-1)}{x(x-a)(y-1)}\right) dxdy}{(x-y)^2}.$$

By the previous lemma

$$M_a(1) = \int_0^\infty \rho(l,x)dx = F(l).$$

Therefore by equation 19 above we obtain

$$4\pi^2|\chi(S)| = \frac{2N_S\pi^2}{3} + \sum_i F(l_i)$$

giving the summation identity

$$\sum_i F(l_i) = 4\pi^2|\chi(S)| - \frac{2N_S\pi^2}{3} = \frac{2\pi^2}{3}(6|\chi(S)| - N_S)$$

■

We note that it follows from the above that for

$$a = -\frac{1}{\sinh^2 \frac{l}{2}} \qquad\qquad b = \frac{1}{\cosh^2 \frac{l}{2}}$$

then by the formulae for $L_a(x,y)$ and $L_b(x,y)$ (see lemma 3) that

$$F(l) = -\int_0^a \int_1^\infty \frac{2.\log\left(\frac{y(y-a)(x-1)}{x(x-a)(y-1)}\right)dxdy}{(x-y)^2} = \int_0^b \int_1^\infty \frac{2.\log\left(\frac{y(y-b)(x-1)}{x(x-b)(y-1)}\right)dxdy}{(x-y)^2}. \tag{20}$$

9 Integral Calculation of F

In this section we find a formula for $F(l)$ by calculating an integral. We note that by the previous section, we already know that the function F satisfies the functional equation 18. We will make use of this to reduce F to the form we wish independent of using any classical dilogarithm relations.

Lemma 6 *For $t < 1$*

$$\mathcal{L}(t) = \frac{1}{4}\int_0^t \int_1^\infty \frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right| dxdy}{(x-y)^2}$$

Proof: We fix $t < 1$ and let

$$G(t) = \int_0^t \int_1^\infty \frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right| dxdy}{(x-y)^2}$$

Integrating by parts we get

$$\int \frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right| dx}{(x-y)^2} = -\frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right|}{x-y} + \int \frac{1}{x-y}\left(\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x-t}\right)dx.$$

Using

$$\int \frac{1}{(x-a)(x-b)}dx = \frac{1}{a-b}\left(\log|x-a| - \log|x-b|\right)$$

we get

$$\int \frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right| dx}{(x-y)^2} = -\frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right|}{x-y} + \frac{1}{y-1}\left(\log|x-y| - \log|x-1|\right) +$$
$$-\frac{1}{y}\left(\log|x-y| - \log|x|\right) - \frac{1}{y-t}\left(\log|x-y| - \log|x-t|\right)$$
$$= \frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right|}{y-x} - \frac{\log|x-1|}{y-1} + \frac{\log|x|}{y} + \frac{\log|x-t|}{y-t} + \log|x-y|)\left(\frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-t}\right)$$

We define

$$I(y) = \int_0^t \frac{\log\left|\frac{y(y-t)(x-1)}{x(x-t)(y-1)}\right| dx}{(x-y)^2}$$

To evaluate the improper integral $I(y)$ we gather the divergent terms to find their limits. Therefore

$$\begin{aligned} I(y) = \lim_{x\to 0} \log|x| \left(\frac{1}{y-x} - \frac{1}{y}\right) + \lim_{x\to t} \log|x-t| \left(\frac{1}{y-t} - \frac{1}{y-x}\right) \\ -\frac{\log\left|\frac{y(y-t)}{t(y-1)}\right|}{y} - \frac{\log|t|}{y-t} - \log|y| \left(\frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-t}\right) \\ +\frac{\log\left|\frac{y(y-t)(t-1)}{t(y-1)}\right|}{y-t} - \frac{\log|t-1|}{y-1} + \frac{\log|t|}{y} + \log|t-y| \left(\frac{1}{y-1} - \frac{1}{y} - \frac{1}{y-t}\right) \end{aligned}$$

By elementary calculus, both limits are zero. As $y > 1$ and $t < 1$, when we gather the remaining terms by common denominators and get

$$\begin{aligned} I(y) = \frac{2\log|t| - 2\log(y-t) + \log(y-1)}{y} + \frac{\log(y-t) - \log(1-t) - \log(y)}{y-1} + \\ + \frac{\log(1-t) + 2\log(y) - 2\log|t| - \log(y-1)}{y-t} \end{aligned}$$

We now rewrite in the following form

$$I(y) = \left(\frac{\log(y-1)}{y} - \frac{\log(y)}{y-1}\right) + 2\left(\frac{\log\left|\frac{y}{t}\right|}{y-t} - \frac{\log\left|\frac{y-t}{t}\right|}{y}\right) + \left(\frac{\log\left(\frac{y-t}{1-t}\right)}{y-1} - \frac{\log\left(\frac{y-1}{1-t}\right)}{y-t}\right) = I_1(y) + I_2(y) + I_3(y). \tag{21}$$

Before we calculate the integral of $I(y)$ we note some properties of the dilogarithm. As the dilogarithm function Li_2 satisfies

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$$

Then for $x < 1$ we have that $\mathcal{L}$ has derivative

$$\mathcal{L}'(x) = \frac{d}{dx}\left(\mathrm{Li}_2(x) + \frac{1}{2}\log|x|\log(1-x)\right) = -\frac{\log(1-x)}{x} + \frac{1}{2}\left(\frac{\log(1-x)}{x} - \frac{\log|x|}{1-x}\right) = -\frac{1}{2}\left(\frac{\log(1-x)}{x} + \frac{\log|x|}{1-x}\right)$$

We let

$$J(y) = 2\mathcal{L}(1-y) - 4\mathcal{L}\left(\frac{t}{y}\right) - 2\mathcal{L}\left(\frac{1-y}{1-t}\right) = J_1(y) + J_2(y) + J_3(y)$$

Then differentiating we get

$$J_1'(y) = 2.\left(-\frac{1}{2}\right).(-1).\left(\frac{\log(1-(1-y)}{1-y} + \frac{\log|1-y|}{1-(1-y)}\right) = \left(\frac{\log(y-1)}{y} - \frac{\log(y)}{y-1}\right) = I_1(y).$$

$$J_3'(y) = -2.\left(-\frac{1}{2}\right).\left(-\frac{1}{1-t}\right).\left(\frac{\log\left(1-\frac{1-y}{1-t}\right)}{\frac{1-y}{1-t}} + \frac{\log\left|\frac{1-y}{1-t}\right|}{1-\frac{1-y}{1-t}}\right) = \left(\frac{\log\left(\frac{y-t}{1-t}\right)}{y-1} - \frac{\log\left(\frac{y-1}{1-t}\right)}{y-t}\right) = I_3(y).$$

$$J_2'(y) = -4.\left(-\frac{1}{2}\right).\left(-\frac{t}{y^2}\right).\left(\frac{\log\left(1-\frac{t}{y}\right)}{\frac{t}{y}} + \frac{\log\left|\frac{t}{y}\right|}{1-\frac{t}{y}}\right) = -2\left(\frac{\log\left(\frac{y-t}{y}\right)}{y} + \frac{t.\log\left(\frac{|t|}{y}\right)}{y(y-t)}\right).$$

As

$$\frac{t}{y(y-t)} = \frac{1}{y-t} - \frac{1}{y}$$

we have

$$J_2'(y) = -2\left(\frac{\log\left(\frac{y-t}{y}\right) - \log\left(\frac{|t|}{y}\right)}{y} + \frac{\log\left(\frac{|t|}{y}\right)}{y-t}\right) = -2\left(\frac{\log\left(\frac{y-t}{|t|}\right)}{y} - \frac{\log\left(\frac{y}{|t|}\right)}{y-t}\right) = I_2(y).$$

Then we have $J'(y) = I(y)$ and therefore we have an antiderivative for I. Integrating we get

$$G(t) = \int_1^\infty I(y)dy = J(y)|_1^\infty = \lim_{y\to\infty} J(y) - \lim_{y\to 1^+} J(y).$$

We let $\mathcal{L}_\infty$ be the limit if $\mathcal{L}(x)$ as x tends to $-\infty$. Then

$$\lim_{y\to 1^+} J(y) = 2\mathcal{L}(0) - 4\mathcal{L}(t) - 2\mathcal{L}(0) = 4\mathcal{L}(t) \qquad\qquad \lim_{y\to\infty} J(y) = -4\mathcal{L}(0).$$

As $\mathcal{L}(0) = 0$ we have

$$G(t) = 4\mathcal{L}(t) - 4\mathcal{L}(0) = 4\mathcal{L}(t)$$

■

Therefore using equation 20 we have that

$$F(l) = -8\mathcal{L}(a) = 8\mathcal{L}(b)$$

10 Volume interpretation of $\mathcal{L}$

Let g_1, g_2 be disjoint geodesics in $\mathbf{H}^2$ with perpendicular distance l and endpoints x_1, y_1 and x_2, y_2 respectively on $\mathbf{S}^1$. Given $v \in T_1(S)$ let g_v be the associated oriented geodesic with tangent v. Then we define the set

$$C(g_1, g_2) = \{v \in T_1(S) \mid g_v \cap g_1 \neq \emptyset,\ g_v \cap g_2 \neq \emptyset\}$$

Let $t = [x_1, y_1, x_2, y_2]$, then depending on the ordering of the points on the circle we have

$$t = [-1, 1, e^l, -e^l] = \frac{1}{\cosh^2(l/2)} \qquad \text{or} \qquad t = [-1, 1, -e^l, e^l] = -\frac{1}{\sinh^2(l/2)}.$$

It follows from the invariance of volume on $T_1(S)$, that the volume of $C(g_1, g_2)$ in $T_1(S)$ only depends on t. We therefore define $V(t) = \text{Volume}(S(g_1, g_2))$.

Then it follows from the main theorem that

$$\mathcal{L}(t) = \pm\frac{1}{8}V(t)$$

where the sign is given by the sign of t. Therefore we can interpret the Rogers $\mathcal{L}-$function as a signed volume function on $T_1(S)$ for the sets $G(g_1, g_2)$.

11 Asymptotic behavior

In this section we study the asymptotic behavior of the function $\rho(l,t)$ for large t.

For functions of a single variable, we write $f(x) \simeq g(x)$ *as* x *tends to* x_0 if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1.$$

Furthermore for functions of more than one variable, we write $f(x,y) \simeq_x g(x,y)$ *as* x *tends to* x_0 if

$$\lim_{x\to x_0} \frac{f(x,y)}{g(x,y)} = 1.$$

Theorem 7 *The measure $\rho(l,t)dx$ on the real line satisfies*

$$\lim_{t\to\infty} \frac{\rho(l,t)}{16t^2e^{-2t}} = r(l)$$

uniformly on compact subsets of $(0,\infty)$ where

$$r(l) = \frac{-2a^2+5a-2}{a(1-a)} \qquad \textit{for} \qquad a = -\frac{1}{\sinh^2\left(\frac{l}{2}\right)}$$

Proof: We now show $\lim_{t\to\infty}\rho(l,t) = r(l)$ converges uniformly on compact subsets of $(0,\infty)$. Let $I \subseteq (0,\infty)$ be a compact interval. Now let $l \in I$. As before we let $a = -1/\sinh^2(l/2)$ and define $f(x) = x(x-a)/(x-1)$ with inverses $g_\pm$ and critical values x_0, y_0. Let

$$G(t,x) = 8te^{2t}\left(\frac{g_+'(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} - \frac{g_-'(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2}\right)f(x)$$

Then for $t > l$ we have

$$\rho(l,t) = \int_{g_-(f(y_0)e^{-2t})}^{g_+(f(y_0)e^{-2t})} G(t,x)dx$$

For $C > 0$, we further define

$$\rho(C,l,t) = \int_{g_-(f(y_0)Ce^{-2t})}^{g_+(f(y_0)Ce^{-2t})} G(t,x)dx \tag{22}$$

On the interval $[a,0]$ f has maximum at x_0. Therefore $\rho(C,l,t)$ is defined for all t such that $f(y_0)Ce^{-2t} < f(x_0)$ or

$$t > K_0(C) = \frac{1}{2}\ln C + \frac{1}{2}\ln\left(\frac{f(y_0)}{f(x_0)}\right) = l + \frac{1}{2}\ln C$$

Considering $g_\pm(x)$ for large x we have

$$g_\pm(x) = \frac{(a+x) \pm \sqrt{(a+x)^2 - 4x}}{2} \simeq \frac{(a+x)}{2}\left(1 \pm \left(1 - \frac{2x}{(a+x)^2}\right)\right)$$

Therefore
$$g_-(x) \simeq \frac{(a+x)}{2}\left(1-1+\frac{2x}{(a+x)^2}\right) = \frac{x}{a+x} \simeq 1-\frac{a}{x}$$
and
$$g_+(x) \simeq \frac{(a+x)}{2}\left(1+1-\frac{2x}{(a+x)^2}\right) = (a+x)-\frac{x}{a+x} \simeq (a-1)+x+\frac{a}{x}$$
Taking leading terms we have
$$g_-(x) \simeq 1 \qquad g_-'(x) \simeq \frac{a}{x^2} \qquad g_+(x) \simeq x \qquad g_+'(x) \simeq 1 \tag{23}$$

We let $I_C = [g_-(f(y_0)Ce^{-2t}), g_+(f(y_0)Ce^{-2t})]$. Then for $x \in I_C$ we have $f(x)e^{2t} \geq C.f(y_0)$. Therefore for C sufficiently large we use the above approximations to approximate $G(t,x)$ on I_C. We substitute the approximations 23 into the formula for $G(t,x)$ to define
$$G_1(t,x) = 8te^{2t}.\left(\frac{1}{(x-f(x)e^{2t})^2} - \frac{\frac{a}{(f(x)e^{2t})^2}}{(x-1)^2}\right) f(x)$$
Simplifying we have
$$G_1(t,x) = 8te^{-2t}.\left(\frac{1}{(1-\frac{x}{f(x)e^{2t}})^2} - \frac{a}{(x-1)^2}\right)\frac{1}{f(x)}.$$
Noting that $f(x)e^{2t} > Cf(y_0)$ on I_C, then for large C the quantity $\frac{x}{f(x)e^{2t}}$ is small and we obtain the approximation
$$G_2(t,x) = 8te^{-2t}.\left(1-\frac{a}{(x-1)^2}\right)\frac{1}{f(x)}.$$

Therefore given an $\epsilon > 0$ we can find a $K_1(\epsilon)$ such that
$$\frac{G(t,x)}{G_2(t,x)} \in [1-\epsilon, 1+\epsilon] \qquad \text{for all } C > K_1(\epsilon), t > K_0(C), x \in I_C.$$
Therefore integrating
$$\frac{1}{\rho(C,l,t)}\left(8te^{-2t}.\int_{g_-(f(y_0)Ce^{-2t})}^{g_+(f(y_0)Ce^{-2t})}\left(1-\frac{a}{(x-1)^2}\right)\frac{1}{f(x)}dx\right) \in [1-\epsilon, 1+\epsilon]$$
for $C > K_1(\epsilon)$ and $t > K_0(C)$. We fix a $K > K_1(\epsilon)$ and define
$$\rho_K(l,t) = 8te^{-2t}.\left(\int_{g_-(f(y_0)Ke^{-2t})}^{g_+(f(y_0)Ke^{-2t})}\left(1-\frac{a}{(x-1)^2}\right)\frac{1}{f(x)}dx\right)$$
$$= 8te^{-2t}.\left(\int_{g_-(f(y_0)Ke^{-2t})}^{g_+(f(y_0)Ke^{-2t})}\left(\frac{x-1}{x.(x-a)} - \frac{a}{x(x-a)(x-1)}\right)dx\right)$$
Integrating we have
$$\int\left(\frac{x-1}{x.(x-a)} - \frac{a}{x(x-a)(x-1)}\right)dx = \left(\frac{1-a}{a}\ln|x| - \frac{a}{1-a}\ln|x-1| - \frac{a^2-3a+1}{a(1-a)}\ln|x-a|\right)$$

Therefore

$$\rho_K(l,t) = 8te^{-2t}.\left(\frac{1-a}{a}\ln|x| - \frac{a}{1-a}\ln|x-1| - \frac{a^2-3a+1}{a(1-a)}\ln|x-a|\right)\Big|^{g_+(f(y_0)Ke^{-2t})}_{g_-(f(y_0)Ke^{-2t})}$$

For x small we have

$$g_\pm(x) = \frac{(a+x)\pm\sqrt{(a+x)^2-4x}}{2} \simeq \frac{(a+x)}{2}\left(1\mp\left(1-\frac{2x}{(a+x)^2}\right)\right)$$

Therefore

$$g_-(x) \simeq (a+x) - \frac{x}{a+x} \simeq a - \frac{(1-a)x}{a} \qquad g_+(x) \simeq \frac{x}{a+x} \simeq \frac{x}{a}$$

Therefore

$$\rho_K(l,t) \simeq_t 8te^{-2t}.\left(\frac{1-a}{a}\ln\left|\frac{Kf(y_0)e^{-2t}}{a^2}\right| - \frac{a}{1-a}\ln\left|\frac{1}{a-1}\right| - \frac{a^2-3a+1}{a(1-a)}\ln\left|\frac{a^2}{(1-a)f(y_0)Ke^{-2t}}\right|\right)$$

Taking limits as we have

$$\rho_K(l,t) \simeq_t (16t^2e^{-2t}).\left(-\frac{1-a}{a} - \frac{a^2-3a+1}{a(1-a)}\right) = (16t^2e^{-2t}).\frac{-2a^2+5a-2}{a(1-a)}$$

Therefore given $\epsilon > 0$ there exists $K_1(\epsilon) > 0$ such that for any $C > K_1(\epsilon)$ both

$$\liminf_{t\to\infty}\frac{\rho(C,l,t)}{16t^2e^{-2t}r(a)} \quad\text{and}\quad \limsup_{t\to\infty}\frac{\rho(C,l,t)}{16t^2e^{-2t}r(a)} \text{ are in } [1-\epsilon,1+\epsilon]. \tag{24}$$

where

$$r(a) = \frac{-2a^2+5a-2}{a(1-a)}$$

We now define

$$\rho_-(C,l,t) = \int_{g_-(f(y_0)e^{-2t})}^{g_-(f(y_0)Ce^{-2t})} G(t,x)dx \qquad\text{and}\qquad \rho_+(C,l,t) = \int_{g_+(f(y_0)Ce^{-2t})}^{g_+(f(y_0)e^{-2t})} G(t,x)dt.$$

Then by definition

$$\rho(l,t) = \rho(C,l,t) + \rho_-(C,l,t) + \rho_+(C,l,t).$$

We now bound the functions $\rho_\pm(C,l,t)$. Let I_C^-, I_C^+ be the given intervals.

On the interval I, $g_\pm(f(x)e^{2t}) > 1$ and $x < 0$ so $(x - g_\pm(f(x)e^{2t}))^2 > 1$. Also as $g_-'(f(x)e^{2t}) < 0$ we have

$$\begin{aligned}|G(t,x)| &= (8t.e^{2t}).\left(\frac{g_+'(f(x)e^{2t})}{(x-g_+(f(x)e^{2t}))^2} - \frac{g_-'(f(x)e^{2t})}{(x-g_-(f(x)e^{2t}))^2}\right)f(x).\\ &\le 8t.e^{2t}.\left(g_+'(f(x)e^{2t}) - g_-'(f(x)e^{2t})\right)f(x).\end{aligned}$$

The derivative of $g_\pm(x)$ is given by

$$g_\pm'(x) = \frac{1}{2} \pm \frac{1}{2}\frac{x+a-2}{\sqrt{(a+x)^2-4x}}.$$

Therefore

$$g_+'(x) - g_-'(x) = \frac{x+a-2}{\sqrt{(a+x)^2-4x}}.$$

As f has critical values $f(x_0)$ and $f(y_0)$ we have that

$$g'_+(x) - g'_-(x) = \frac{x + a - 2}{\sqrt{(x - f(x_0))(x - f(y_0))}}.$$

We note that on $I_C^{\pm}$ we have $f(y_0) < f(x)e^{2t} < Cf(y_0)$ then

$$g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t}) \leq \frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0))(f(x)e^{2t} - f(y_0))}}.$$
$$\leq \left(\frac{Cf(y_0) + a - 2}{\sqrt{f(y_0) - f(x_0)}} \right) \frac{e^{-t}}{\sqrt{f(x) - f(y_0)e^{-2t}}}$$

The function $f(x) = x(x - a)/(x - 1)$ has maximum at x_0 on $(a, 0)$. Therefore for $b < f(x_0)$

$$f(x) - b = \frac{(x - g_-(b))(x - g_+(b))}{(x - 1)}$$

As $x \in (a, 0)$ we have

$$f(x) - b \geq (x - g_-(b))(g_+(b) - x)$$

Therefore

$$g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t}) \leq \left(\frac{Cf(y_0) + a - 2}{\sqrt{f(y_0) - f(x_0)}} \right) \frac{e^{-t}}{\sqrt{(x - g_-(f(y_0)e^{-2t}))(g_+(f(y_0)e^{-2t}) - x)}}$$

Now restricting to I_C^+ we have $x > g_+(f(y_0)Ce^{-2t})$. Therefore for $x \in I_C^+$,

$$g'_+(f(x)e^{2t}) - g'_-(f(x)e^{2t}) \leq \left(\frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0))(g_+(f(y_0)Ce^{-2t}) - g_-(f(y_0)e^{-2t}))}} \right) \frac{e^{-t}}{\sqrt{g_+(f(y_0)e^{-2t}) - x}}$$

Therefore we have

$$\rho_+(C, l, t) \leq \int_{I_C^+} |G(t, x)| dx \leq D(t) 8te^t. \int_{I_C^+} \frac{f(x)}{\sqrt{g_+(f(y_0)e^{-2t}) - x}} dt$$

where $D(t)$ is the constant

$$D(t) = \left(\frac{Cf(y_0) + a - 2}{\sqrt{(f(y_0) - f(x_0))(g_+(f(y_0)Ce^{-2t}) - g_-(f(y_0)e^{-2t}))}} \right)$$

As $f(x) = x(x - a)/(x - 1)$ then, $0 < f(x) \leq ax$ on $(a, 0)$ we have

$$\rho_+(C, l, t) \leq \int_{I_C^+} |G(t, x)| dx \leq D(t).8.a.t.e^t. \int_{I_C^+} \frac{x}{\sqrt{g_+(f(y_0)e^{-2t}) - x}} dx$$

By integration we have

$$\int_a^b \frac{x}{\sqrt{b - a}} dx = \frac{2}{3}(2b + a)\sqrt{b - a}$$

Therefore

$$\rho_+(C, l, t) \leq 16.D(t).a.t.e^t. \left(2g_+(f(y_0)e^{-2t}) + g_+(f(y_0)Ce^{-2t})\right) \sqrt{g_+(f(y_0)e^{-2t}) - g_+(f(y_0)Ce^{-2t})}.$$

Now for t large we have

$$\lim_{t\to\infty} D(t) = \left(\frac{Cf(y_0)+a-2}{\sqrt{(f(y_0)-f(x_0)).|a|}}\right) = D.$$

We note for x small $g_+(x) \simeq x/a$. Therefore

$$\limsup_{t\to\infty}\left|\frac{\rho_+(C,l,t)}{t^2e^{-2t}}\right| \leq \limsup_{t\to\infty}\frac{16.D.a.t.e^t.\left(\frac{2f(y_0)e^{-2t}+f(y_0)Ce^{-2t}}{a}\right)\sqrt{\frac{f(y_0)e^{-2t}-f(y_0)Ce^{-2t}}{a}}}{t^2e^{-2t}}.$$

$$\limsup_{t\to\infty}\left|\frac{\rho_+(C,l,t)}{t^2e^{-2t}}\right| \leq \limsup_{t\to\infty}\frac{16.D.f(y_0)^{3/2}(C+2)\sqrt{C-1}}{t.\sqrt{-a}} = 0.$$

Thus

$$\lim_{t\to\infty}\frac{\rho_+(C,l,t)}{t^2e^{-2t}} = 0.$$

Similarly for $\rho_-(C,l,t)$ we once again have that

$$\lim_{t\to\infty}\frac{\rho_-(C,l,t)}{t^2e^{-2t}} = 0.$$

Therefore given $\epsilon > 0$ we can find $K(\epsilon)$ such that for $C > K(\epsilon)$ by equations 24

$$\limsup_{t\to\infty}\frac{\rho(l,t)}{16t^2e^{-2t}r(a)} = \limsup_{t\to\infty}\left(\frac{\rho_-(C,l,t)}{16t^2e^{-2t}r(a)} + \frac{\rho(C,l,t)}{16t^2e^{-2t}r(a)} + \frac{\rho_+(C,l,t)}{16t^2e^{-2t}r(a)}\right) = \limsup_{t\to\infty}\frac{\rho(C,l,t)}{16t^2e^{-2t}r(a)} \in [1-\epsilon, 1+\epsilon]$$

As ϵ is arbitrary we have

$$\limsup_{t\to\infty}\frac{\rho(l,t)}{16t^2e^{-2t}r(a)} = 1$$

Similarly

$$\liminf_{t\to\infty}\frac{\rho(l,t)}{16t^2e^{-2t}r(a)} = 1$$

■

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