An Extension of the Weil-Petersson metric to Quasi-Fuchsian space

Martin Bridgeman [*]
Department of Mathematics, Boston College, Chestnut Hill, MA 02467
Edward C. Taylor [†]
Department of Mathematics, Wesleyan University, Middletown, CT 06257

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Abstract: We define a natural semi-definite metric on quasifuchsian space, derived from geodesic current length functions and hausdorf dimension, that extends the Weil-Petersson metric on the Teichmüller space. We use this to describe a new metric on Teichmüller space obtained by taking the second derivative of hausdorf dimension and show that this new metric is bounded below by the Weil-Petersson metric. We relate the change in hausdorf dimension under bending along a measured lamination to the length in the Weil-Petersson metric of the associated earthquake vector of the lamination.

1 Statement of Results

Let S be a closed hyperbolic surface and *Teichmüller space* $F(S)$, be the space of hyperbolic structures on S. We let w be the Weil-Petersson metric on $F(S)$. For simplicity we normalize the Weil-Petersson metric and define the *normalized Weil-Petersson metric* on $F(S)$ to be the metric

$$g = \left(\frac{2}{3\pi|\chi(S)|}\right) w \ .$$

Let $QF(S)$ be the quasifuchsian space of S with $F(S)$ naturally embedded in $QF(S)$. We call $F(S) \subseteq QF(S)$ the *fuchsian subspace*. We

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let $h : QF(S) \to \mathbb{R}$ be the hausdorf dimension function given by letting $h(X)$ be the hausdorf dimension of the limit set of X. In [19], Ruelle showed that h is real analytic on $QF(S)$.

Weil-Petersson Extension Theorem: *Let S be a closed hyperbolic surface. Then there exists a natural semi-definite metric G on $QF(S)$ that extends the normalized Weil-Petersson metric on the fuchsian subspace $F(S) \subseteq QF(S)$.*

Quasifuchsian space $QF(S)$ has a natural complex structure coming from the fact that $QF(S)$ is an open complex submanifold of the complex representation space $R(S) = Hom(\pi_1(S), PSL(2,\mathbb{C}))$(see [15]). Therefore if $T(QF(S))$ is the tangent bundle of $QF(S)$ and $\pi : T(QF(S)) \to QF(S)$ is the projection map, multiplication by i defines a bundle map $J : T(QF(S)) \to T(QF(S))$ with $\pi \circ J = \pi$ and $J^2 = -I$ where I the identity map on $T(QF(S))$.

We let $X \in F(S)$. Then $h(X) = 1$ and h is minimum at X. It follows that the Hessian of h evaluated at X is a well-defined symmetric bilinear form on $T_X(QF(S))$. We define $h''_X : T_X(QF(S)) \times T_X(QF(S)) \to \mathbb{R}$ to be the Hessian of h. Therefore if $u, v \in T_X(QF(S))$ we have

$$h''_X(u,v) = \frac{\partial^2 h}{\partial u \partial v}(X).$$

Using h''_X we describe a symmetric two-tensor H on $F(S)$. For each $X \in F(S)$, the associated symmetric bilinear map of H at X is denoted $H_X : T_X(F(S)) \times T_X(F(S)) \to \mathbb{R}$ and defined by letting

$$H_X(u,v) = h''_X(Ju, Jv).$$

H is a semi-definite metric on $F(S)$ and therefore has well-defined norm denoted $||.||_H$. The second result we obtain is;

H-metric Theorem: *The semi-definite metric H is a (positive-definite) metric on $F(S)$ satisfying*

$$||v||_H \geq ||v||_g$$

where $||.||_g$ is the norm of the (normalized) Weil-Petersson metric g.

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2 Kleinian groups and Geodesic Currents

Let $Isom_+(\mathbb{H}^n)$ $n \geq 2$ be the space of orientation preserving isometries of $\mathbb{H}^n$. As is well-known, this space of isometries can be given the topology induced by uniform convergence on compact sets. We define a *Kleinian group* Γ to be a discrete torsion-free subgroup of $Isom_+(\mathbb{H}^n)$. As such, Γ acts properly discontinuously on $\mathbb{H}^n$, and the quotient manifold $N = \mathbb{H}^n/\Gamma$ is a complete Riemannian manifold of constant curvature -1.

A Kleinian group Γ also acts as a discrete subgroup of conformal automorphisms of the sphere at infinity $\mathbb{S}^{n-1}_\infty$; this action partitions $\mathbb{S}^{n-1}_\infty$ into two disjoint sets. The *regular set* Ω_Γ is the largest open set in $\mathbb{S}^{n-1}_\infty$ on which Γ acts properly discontinuously, and the *limit set* Λ_Γ is its complement. In the case that Λ_Γ contains more than 2 points, it is characterized as being the smallest closed Γ-invariant subset of $\mathbb{S}^{n-1}_\infty$.

Define the *convex hull* $CH(\Lambda_\Gamma)$ of the limit set Λ_Γ to be the smallest convex subset of $\mathbb{H}^n$ so that all geodesics with both limit points in Λ_Γ are contained in $CH(\Lambda_\Gamma)$. We can take the quotient of $CH(\Lambda_\Gamma)$ by Γ (denoted by $C(\Gamma)$); this is the *convex core*. It is the smallest convex submanifold of $N = \mathbb{H}^n/\Gamma$ so that the inclusion map is a homotopy equivalence.

A Kleinian group is *convex co-compact* if its associated convex core is compact and it is *geometrically finite* if the volume of the unit neighborhood of the convex core is finite (see Bowditch [4]). This paper deals specifically with convex co-compact Kleinian groups. For the basics in the theory of Kleinian groups we refer the reader to Maskit [16].

If Γ_0 is a geometrically finite Kleinian group, we define the space $QC(\Gamma_0)$ of quasiconformal deformations of Γ_0 as follows; We consider pairs (f, Γ) such that $f : \mathbb{S}^{n-1}_\infty \to \mathbb{S}^{n-1}_\infty$ is a quasiconformal homeomorphism, conjugating Γ_0 to Kleinian group Γ, i.e. $\Gamma = f\ \Gamma_0\ f^{-1}$. The map f is called the *marking.* We define an equivalence relation by saying $(f_1, \Gamma_1) \equiv (f_2, \Gamma_2)$ if there exists a conformal map α conjugating Γ_1 to Γ_2, i.e.

$$f_2 \circ \gamma \circ f_2^{-1} = (\alpha \circ f_1) \circ \gamma \circ (\alpha \circ f_1)^{-1} \quad \forall \gamma \in \Gamma_0.$$

Then $QC(\Gamma_0)$ is the set of equivalence classes under this equivelence relation. For convenience, we will often supress the map f in describing a point of $QC(\Gamma_0)$ and just refer to it by the group.

We can identify a geodesic with its endpoints on $\mathbb{S}^{n-1}_\infty$ and therefore we identify the space of geodesics on $\mathbb{H}^n$ by $G(\mathbb{H}^n) \cong (\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{diagonal})/\mathbb{Z}_2$.

If N is a convex co-compact hyperbolic n-manifold, with $N = \mathbb{H}^n/\Gamma$, then each non-trivial homotopy class of closed curves in N corresponds to a unique multiple of a primitive closed geodesic. If α is a primitive closed geodesic in N, we lift α to get a discrete subset of $G(\mathbb{H}^n)$ which is Γ invariant. In this way we identify every non-trivial homotopy class of closed curves on $\mathbb{H}^n/\Gamma$ with a Γ invariant discrete subset of $G(\mathbb{H}^n)$ and a certain integral multiplicity. We then obtain a Γ invariant measure on $G(\mathbb{H}^n)$ by taking the Dirac measure on this discrete set times the multiplicity. This measure is the *geodesic current* associated with the closed curve. We have the following generalization;

Definition: A *geodesic current* for Kleinian group Γ is a positive measure on $G(\mathbb{H}^n)$ that is invariant under the action of Γ and supported on the set of geodesics with endpoints belonging to limit set Λ_Γ.

As geodesic currents are (Borel σ-finite) measures, we can add two geodesic currents and also multiply a geodesic current by a positive constant. A geodesic current which is a constant multiple of a closed geodesic is called a *discrete geodesic current.*

If Γ is a Kleinian group, we let $\mathcal{C}(\Gamma)$ be the space of geodesic currents defined for Γ. The natural topology on $\mathcal{C}(\Gamma)$, via the Radon-Riesz Representation Theorem, is the weak*-topology on the space of continuous functions with compact support in $G(\mathbb{H}^n)$.

Below is a basic fact we will need concerning the topology on $\mathcal{C}(\Gamma)$. The proof involves first showing that the geodesic flow on the unit tangent bundle has the specification property ([5] and [20]), and then applying Theorem 1 in [20].

Theorem 2.1 *Let Γ be a convex co-compact Kleinian group. Then the set of discrete geodesic currents is dense in $\mathcal{C}(\Gamma)$.*

3 The Length of Geodesic Currents

Let Γ be a Kleinian group and let $\mathcal{C}(\Gamma)$ be the space of geodesic currents defined above. We now describe the length of a geodesic current. Let

$PT(\mathbb{H}^n)$ be the projective tangent bundle of $\mathbb{H}^n$. The standard notation for a point $p \in PT(\mathbb{H}^n)$ is $p = (x, l)$ where $x \in \mathbb{H}^n$ is the base point and l is a line at x. The group Γ acts on $PT(\mathbb{H}^n)$ by defining $\gamma(x, v) = (\gamma(x), \gamma'(x).l)$. We define a fiber map $\pi : PT(\mathbb{H}^n) \rightarrow G(\mathbb{H}^n)$ by letting $\pi(p)$ be the geodesic with tangent line l. The fiber $\pi^{-1}(g)$ over geodesic g is equal to all tangent lines to geodesic g and therefore is naturally identified with g considered as a subset of $\mathbb{H}^n$. Therefore any geodesic current α can be lifted to a measure on α^* on $PT(\mathbb{H}^n)$ by taking the product of α with hyperbolic length measure λ on the fibers, i.e. $\alpha^* = \alpha \ \times \lambda$. The measure α^* is also invariant under the action of Γ on $PT(\mathbb{H}^n)$. Thus we define the *length of a current* α by $l_\Gamma(\alpha) = \alpha^*(PT(\mathbb{H}^n)/\Gamma)$. If Γ is convex co-compact then the length is finite as α^* has compact support on $PT(\mathbb{H}^n)/\Gamma$.

Lemma 3.1 *(Bridgeman,Taylor, [8]) If* Γ *is convex co-compact then the map* $l_\Gamma : \mathcal{C}(\Gamma) \rightarrow \mathbb{R}$ *is continuous.*

One particular geodesic current is the *Patterson-Sullivan geodesic current.* This is defined using the Poincaré series of Γ as follows.

Fix $s \in \mathbb{R}^+$. We define the *Poincaré series* of a Kleinian group Γ by

$$g_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}$$

where $x, y \in \mathbb{H}^n$ and d is the hyperbolic metric on $\mathbb{H}^n$. Let

$$\delta(\Gamma) = \inf\{s : \quad g_s < \infty\};$$

then $\delta(\Gamma)$ is called the *exponent of convergence* of the Poincaré series. We refer the reader to [18] for further details on the exponent of convergence.

Following the work of Patterson and Sullivan, a measure can be constructed on $\mathbb{S}^{n-1}_\infty$ which is supported on $\Lambda(\Gamma)$. For $x, y \in \mathbb{H}^n$ and $s > \delta(\Gamma)$, we consider the measure $\sigma_{x,s}$ to have Dirac point mass of weight $\frac{e^{-sd(x,\gamma y)}}{g_s(y,y)}$ at each point γy. The *Patterson-Sullivan* measure σ_x is constructed by taking a limit of these measures as $s \rightarrow \delta(\Gamma)^+$. The measure σ_x can be used to define a measure m on $G(\mathbb{H}^n) = (\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{diagonal})/\mathbb{Z}_2$ given differentially by

$$dm([a, b]) = \frac{d\sigma_x(a) d\sigma_x(b)}{|b - a|^{2\delta(\Gamma)}}.$$

This measure is Γ-invariant and supported on $G(\mathbb{H}^n) = (\Lambda(\Gamma) \times \Lambda(\Gamma) -$ diagonal$)/\mathbb{Z}_2$. Therefore it is a geodesic current and is called a *Patterson-Sullivan geodesic current.* By work of Sullivan ([21]), for Γ being geometrically finite, m is unique up to scalar multiple. Thus, in the case of Γ geometrically finite, we define $\mu \in \mathcal{C}(\Gamma)$ to be the unique *unit length Patterson-Sullivan geodesic current.*

4 Length function on $\mathcal{C}(\mathbf{\Gamma_0}) \times \mathbf{QC}(\mathbf{\Gamma_0})$

Let Γ_0 be a fixed convex co-compact Kleinian group and f a quasiconformal homeomorphism conjugating Γ_0 to Kleinian group Γ. Then f maps the limit set $\Lambda(\Gamma_0)$ to the limit set $\Lambda(\Gamma)$. Therefore we have the associated map (also called f) on the space of geodesics $f : G(\mathbb{H}^n) \to G(\mathbb{H}^n)$. As f maps $\Lambda(\Gamma_0)$ to $\Lambda(\Gamma)$, we therefore have the pushforward map $f_* : \mathcal{C}(\Gamma_0) \to \mathcal{C}(\Gamma)$ given by $(f_*\alpha)(\phi) = \alpha(\phi \circ f)$, for any $\alpha \in \mathcal{C}(\Gamma_0)$, and $\phi \in C_0(G(\mathbb{H}^n), \mathbb{R})$ (the space of continuous functions with compact support on $G(\mathbb{H}^n)$.) In the paper [8], we observed the following:

Lemma 4.1 *(Bridgeman,Taylor, [8]) The map $f_* : \mathcal{C}(\Gamma_0) \to \mathcal{C}(\Gamma)$ is a homeomorphism.*

Let $l_\Gamma : \mathcal{C}(\Gamma) \to \mathbb{R}$ be the length function on $\mathcal{C}(\Gamma)$ defined above. Then we can compose f_* with l_Γ to obtain the function $L_\Gamma : \mathcal{C}(\Gamma_0) \to \mathbb{R}$ given by

$$L_\Gamma = l_\Gamma \circ f_*. \tag{1}$$

As both l_Γ and f_* are continuous, we have the following;

Lemma 4.2 *If Γ_0 is convex co-compact, then the map $L_\Gamma : \mathcal{C}(\Gamma_0) \to \mathbb{R}$ is continuous.*

We first state an easy consequence of previous results:

Lemma 4.3 *Let $\Gamma_2 = \omega \circ \Gamma_1 \circ \omega^{-1}$, where Γ_1 is quasiconformally conjugate to Γ_0 and ω is a Möbius transformation. Then for each $\alpha \in \mathcal{C}(\Gamma_0)$ we have $L_{\Gamma_1}(\alpha) = L_{\Gamma_2}(\alpha)$.*

By the above, the length function $L : \mathcal{C}(\Gamma_0) \times QC(\Gamma_0) \to \mathbb{R}$ given by

$$L(\alpha, [(f, \Gamma)]) = L_\Gamma(f_*\alpha) \tag{2}$$

is well defined.

In the following sections, we consider the continuity and differentiability of the function L. In particular we note that to show that L is continuous on the product space, it is not sufficient to show that it is continuous on each factor.

The natural description of the analytic properties of L is via the associated length functions of geodesic currents.

Definition: If $\alpha \in \mathcal{C}(\Gamma_0)$ we define the *length function of* α to be the function $L_\alpha : QC(\Gamma_0) \to \mathbb{R}$ given by $L_\alpha([f,\Gamma]) = L(\alpha, [(f,\Gamma)])$.

5 Analyticity of Length Function

In this section we restrict Γ_0 to be a convex co-compact Kleinian group acting on $\mathbb{H}^3$. We consider the analyticity properties of the length function $L : \mathcal{C}(\Gamma_0) \times QC(\Gamma_0) \to \mathbb{R}$ in terms of the natural complex structure on $QC(\Gamma_0)$(see [15]).

By the invariance of the trace function under Möbius transformations, for each $\gamma \in \Gamma_0$,we have the well-defined trace function $t_\gamma : QC(\Gamma_0) \to \mathbb{C}$ given by

$$t_\gamma([f,\Gamma]) = trace(f \circ \gamma \circ f^{-1}).$$

With respect to the complex structure on $QC(\Gamma_0)$, the functions t_γ are holomorphic for all $\gamma \in \Gamma_0$ (see [15]). As this is the only property of the complex structure we use, we will not describe the complex structure in further detail. We will now use the theory of pluriharmonic functions to describe the analytic properties of the length function L.

A *complex line* l in $\mathbb{C}^n$ is the set $l = \{a + bz \mid z \in \mathbb{C}\}$ where $a, b \in \mathbb{C}^n$ and $b \neq 0$.

Definition: Let $\Omega \subseteq \mathbb{C}^n$ be open and $u : \Omega \to \mathbb{R}$ be a map. Then u is *pluriharmonic* if $u \in C^2(\Omega, R)$ (derivatives continuous up to order two), and for each complex line l the map $u : \Omega \cap l \to \mathbb{R}$ is harmonic.

Similarly if M is a complex manifold and $\Omega \subseteq M$ is an open subset, then $u : \Omega \to \mathbb{R}$ is pluriharmonic if for any complex chart (U, ϕ) the map $u \circ \phi^{-1} : \phi(U \cap \Omega) \to \mathbb{R}$ is pluriharmonic.

We have the following characterization of pluriharmonicity (see [14]).

Lemma 5.1 *Let M be a complex manifold. Then a function $f : M \to \mathbb{R}$ is pluriharmonic if and only if f is locally the real part of a holomorphic function.*

If α is a closed geodesic in $\mathbb{H}^n/\Gamma_0$ we have an associated holomorphic trace function $t_\alpha : QC(\Gamma_0) \to \mathbf{C}$. Then there exists a holomorphic function, $\mathcal{L}_\alpha : QC(\Gamma_0) \to \mathbb{C}$ defined by $t_\alpha = 2\cosh(\mathcal{L}_\alpha/2)$. The function $\mathcal{L}_\alpha$ is called the *complex length of* α and is well-defined up to addition of $2\pi i$. Also by definition, the real length function of α satisfies $L_\alpha = \mathit{Real}\ \mathcal{L}_\alpha$. Therefore L_α is pluriharmonic. An immediate consequence is the following lemma;

Lemma 5.2 *If μ is a discrete geodesic current then the length function L_μ is pluriharmonic.*

We now show that the length functions converges uniformly on compact sets.

Lemma 5.3 *Let $\mu_i \to \mu$ in $\mathcal{C}(\Gamma_0)$. Then the functions $L_{\mu_i} \to L_\mu$ uniformly on compacts sets.*

Proof: Let $K \subseteq QC(\Gamma_0)$ be a compact set. If L_{μ_i} do not converge uniformly to L_μ on K then there exists an $\epsilon > 0$ and a subsequence i_n such that for each $n \in N$

$$|L_{\mu_{i_n}}(X) - L_\mu(X)| \not< \epsilon \text{ for all } X \in K.$$

Therefore for each $n \in N$ there exists an $X_n \in K$ such that

$$|L_{\mu_{i_n}}(X_n) - L_\mu(X_n)| \geq \epsilon.$$

By compactness of K, by passing to a subsequence, we can assume that X_n is a convergent to $X \in K$.

By [9], given any $K > 1$ we can choose a neighborhood U about $X \in QC(\Gamma_0)$ such that if $Y \in U$ then X, Y are K quasi-isometric. Therefore if $Y \in U$ and $\alpha \in \mathcal{C}(\Gamma_0)$, then by definition of the length function we have

$$\frac{1}{K} L_\alpha(X) \leq L_\alpha(Y) \leq K.L_\alpha(X).$$

As $X_n \to X$, there exists an n_1 such that if $n > n_1$ then $X_n \in U$. Therefore

$$|L_{\mu_{i_n}}(X_n) - L_\mu(X_n)| \leq Max\left\{|KL_{\mu_{i_n}}(X) - \frac{1}{K}L_\mu(X)|, |\frac{1}{K}L_{\mu_{i_n}}(X) - KL_\mu(X)|\right\}$$

By lemma 4.2 the function l_X given by $l_X(\alpha) = L_\alpha(X)$ is continuous. Therefore as $\mu_i \to \mu$, given $\epsilon_1 > 0$ there exists an n_2 such that if $n > n_2$ then

$$|L_{\mu_{i_n}}(X) - L_\mu(X)| = |l_X(\mu_{i_n}) - l_X(\mu)| < \epsilon_1.$$

Therefore for $n > Max(n_1, n_2)$ we have

$$|L_{\mu_{i_n}}(X_n) - L_\mu(X_n)| \leq (K - \frac{1}{K})L_\mu(X) + K\epsilon_1$$

By suitable choice of K and ϵ_1 we can make the righthandside less than ϵ. This gives the necessary contradiction. $\square$

Definition: Let M be a smooth manifold and $C^\infty(M, R)$ be the set of smooth real valued functions M. Then the *C^∞-topology* on $C^\infty(M, R)$ is given by $f_i \to f$ if the derivatives of f_i converge uniformly on any compact subset of M to the derivatives of f.

We are now ready to prove the Analyticity Theorem.

Analyticity Theorem: *Let Γ_0 be an arbitrary co-infinite convex co-compact Kleinian group acting on $\mathbb{H}^3$. Let $\alpha \in \mathcal{C}(\Gamma_0)$ and $L_\alpha : QC(\Gamma_0) \to \mathbb{R}$ be the length function of α. Then L_α is real-analytic and the function*

$$L : \mathcal{C}(\Gamma_0) \to C^\infty(QC(\Gamma_0), R)$$

given by $L(\alpha) = L_\alpha$ is continuous with respect to the C^∞-topology on the space $C^\infty(QC(\Gamma_0), R)$ of smooth real-valued functions on $QC(\Gamma_0)$.

Proof: Let $\mu \in \mathcal{C}(\Gamma_0)$. Then by theorem 2.1, there exist $\mu_i \to \mu$ such that μ_i are discrete. Then by theorem 5.3, $L_{\mu_i} \to L_\mu$ uniformly on compact sets. As a uniform limit of pluriharmonic functions is pluriharmonic (see Theorem 1.23 in [1]), then L_μ is pluriharmonic. In particular L_μ is real-analytic and smooth. Therefore we can define $L : \mathcal{C}(\Gamma_0) \to C^\infty(QC(\Gamma_0), \mathbb{R})$ from the space of geodesic currents to the set of smooth real valued functions on $QC(\Gamma_0)$ given by $L(\mu) = L_\mu$.

Now if $\mu_i \to \mu$ (μ_i not necessarily discrete), then by theorem 5.3 $L_{\mu_i} \to L_\mu$ uniformly on compact sets. As L_{μ_i} are pluriharmonic, uniform convergence on compact sets implies uniform convergence of derivatives (see Theorem 1.23 of [1]). Therefore the function $L : \mathcal{C}(\Gamma_0) \to C^\infty(QC(\Gamma_0), \mathbb{R})$ is continuous in the C^∞ topology on $C^\infty(QC(\Gamma_0), \mathbb{R})$. $\square$

We now introduce some notation to deal with the analyticity properties of maps $f : X \times M \to N$, where X is a topological space and M, N are smooth, or, analytic manifolds.

Definition: Let X be a topological space and M, N smooth manifolds. Then a map $f : X \times M \to N$ is $(0, \infty)$-*differentiable* if for each $x \in X$ the map $f_x : M \to N$ given by $f_x(p) = f(x, p)$ is smooth and $F : X \to C^\infty(M, N)$ given by $F(X) = f_x$ is continuous in the C^∞-topology. Furthermore if M, N are analytic, then f is $(0, \omega)$-*differentiable* if the maps f_x are also analytic.

With respect to this new notation, the Analyticity Theorem states that the map $L : \mathcal{C}(\Gamma_0) \times QC(\Gamma_0) \to \mathbb{R}$ is $(0, \omega)$-differentiable. We will use this notation as to emphasize the continuity in the first parameter and the real-analyticity in the second.

6 Complex Length on Quasi-fuchsian Space

In this section we will show that in the quasifuchsian case, there is a natural way of extending the real length function L to a complex length function $\mathcal{L}$.

Recall that a *Fuchsian group* Γ is a finitely generated Kleinian group, with limit set Λ_Γ equal to the extended real line $\overline{\mathbb{R}} \subseteq \widehat{\mathbb{C}}$ and such that Γ preserves each component of $\widehat{\mathbb{C}} - \overline{\mathbb{R}}$. Then the hyperbolic plane $\mathbb{H}^2$ with boundary $\overline{\mathbb{R}}$ is invariant under Γ and $S = \mathbb{H}^2/\Gamma$ is a hyperbolic surface.

Let Γ_0 be convex co-compact and Fuchsian; we call the space $QC(\Gamma_0)$ *quasifuchsian space*. The quotient manifold $\mathbb{H}^3/\Gamma_0$ is homeomorphic to $S \times \mathbb{R}$, where S is the closed hyperbolic surface given by $\mathbb{H}^2/\Gamma_0$.

To emphasize that we are dealing with a special case, $QC(\Gamma_0)$ is called the *quasifuchsian space* of S and denoted by $QF(S)$. Also we denote the space of currents $\mathcal{C}(\Gamma_0)$ by $\mathcal{C}(S)$. Furthermore we will denote the Fuchsian elements of $QF(S)$ by $F(S)$.

From the Analyticity Theorem we have that if $\mu \in \mathcal{C}(S)$ then the length function $L_\mu : QF(S) \to \mathbb{R}$ is pluriharmonic. By analytic continuation we obtain;

Complex Length Theorem: *For each $\mu \in \mathcal{C}(S)$ there exists a unique holomorphic function $\mathcal{L}_\mu : QF(S) \to \mathbb{C}$ with real part L_μ and imaginary part satisfying $Im\ \mathcal{L}_\mu = 0$ on $F(S)$.*

Furthermore the function

$$\mathcal{L} : \mathcal{C}(S) \to C^\omega(QF(S), \mathbb{C})$$

given by $\mathcal{L}(\mu) = \mathcal{L}_\mu$ is continuous with respect to the topology of uniform convergence (on compacta) on the space $C^\omega(QF(S), \mathbb{C})$ of holomorphic functions on $QF(S)$.

In order to prove this, we will need the following lemma from several complex variables (see [14]).

Lemma 6.1 *Let* $u : B(0,r) \to \mathbb{R}$ *be pluriharmonic on the disk* $B(0,r)$ *of radius* r *about* 0 *in* $\mathbb{C}^n$*. Then there exists a unique function* $v : B(0,r) \to \mathbb{R}$ *such that* $f = u + iv$ *is holomorphic and* $v(0) = 0$*.*

Furthermore if $z = (z_1, \ldots, z_n)$ *with* $z_k = x_k + iy_k$ *then*

$$v(z) = \int_0^z \sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k} dy_k - \frac{\partial u}{\partial y_k} dx_k \right)$$

Proof of Complex Length Theorem: As $S = \mathbb{H}^2/\Gamma_0$ we denote the Fuchsian element of $QF(S)$ corresponding to Γ_0 by S. Let $\mu \in \mathcal{C}(S)$ and L_μ be the associated length function. As L_μ is pluriharmonic, L_μ can be extended to a holomorphic function in a neighborhood of each point of $QF(S)$. As $QF(S)$ is simply connected, by analytic continuation, we obtain a holomorphic map $\mathcal{L}_\mu : QF(S) \to \mathbb{C}$, with real part L_μ. Furthermore, by addition of a purely imaginary constant, we can choose the imaginary part of $\mathcal{L}_\mu$ to satisfy $Im\ \mathcal{L}_\mu(S) = 0$. We now let $\mathcal{L}_\mu = L_\mu + iT_\mu$ and call T_μ the *torsion function* of μ. Then by the previous lemma 6.1, T_μ is given by the integral of partial derivatives of L_μ. Also, from the lemma, it follows that $\mathcal{L}_\mu$ is unique.

Let F_μ be another holomorphic function with real part L_μ. Then the holomorphic function $\mathcal{L}_\mu - F_\mu$ has real part zero and is therefore a purely imaginary constant.

Given $X \in QF(S)$ we define $F_\mu^X = \mathcal{L}_\mu - iT_\mu(X)$. Then F_μ^X is the unique holomorphic function with real part L_μ and $Im\ F_\mu^X(X) = 0$. Furthermore, as $Im\ F_\mu^X(X) = 0$, the imaginary part of F_μ^X is uniquely given by the integral formula in lemma 6.1 in a neighborhood of X.

Now let $\mu_i \to \mu$. We define the set $U \subseteq QF(S)$ by the condition that $X \in U$ if and only if X has a neighborhood on which $\mathcal{L}_{\mu_i}$ converges uniformly to $\mathcal{L}_\mu$. We will show that $U = QF(S)$.

By definition U is open. Also by lemma 6.1, there is an open neighborhood of S where the torsion function T_μ is given by the integral of partial derivatives of L_μ. By the Analyticity Theorem, the partial derivatives of L_{μ_i} converge uniformly to L_μ on compact sets. Therefore it follows that T_{μ_i} converges uniformly to T_μ on a compact neighborhood of S. Thus $\mathcal{L}_{\mu_i}$ uniformly converges to $\mathcal{L}_\mu$ in a neighborhood of S. Therefore $S \in U$ and $U \neq \emptyset$.

Now let $X \in \partial U$. As $Im\, F^X_\nu(X) = 0$ for all $\nu \in \mathcal{C}(S)$, it follows again from lemma 6.1 that $F^X_{\mu_i}$ converges uniformly to F^X_μ in a neighborhood of X. We denote this neighborhood by V. As $X \in \partial U$, there exists a $Y \in U \cap V$. Therefore both $L_{\mu_i}(Y) \rightarrow L_\mu(Y)$ and $F^X_{\mu_i}(Y) \rightarrow F^X_\mu(Y)$. As $\mathcal{L}_\nu - F^X_\nu = iT_\nu(X)$, it follows that the sequence $T_{\mu_i}(X) \rightarrow T_\mu(X)$. From before we have $\mathcal{L}_\nu = F^X_\nu + iT_\nu(X)$. Therefore as $F^X_{\mu_i} \rightarrow F^X_\mu$ uniformly in V and the sequence $T_{\mu_i}(X) \rightarrow T_\mu(X)$, the function $\mathcal{L}_{\mu_i} \rightarrow \mathcal{L}_\mu$ uniformly in V. Therefore $X \in U$ and U is closed.

As $QF(S)$ is connected, this implies that $U = QF(S)$. Therefore $\mathcal{L}_{\mu_i}$ must converge uniformly on a neighborhood of each point of $QF(S)$. Thus it follows that $\mathcal{L}_{\mu_i}$ must converge uniformly on compact sets to $\mathcal{L}_\mu$. Therefore the map $\mathcal{L} : \mathcal{C}(S) \rightarrow C^\omega(QF(S), \mathbb{C})$ given by $\mathcal{L}(\mu) = \mathcal{L}_\mu$ is continuous with respect to the topology of uniform convergence (on compacta) on the space $C^\omega(QF(S), \mathbb{C})$ of holomorphic functions on $QF(S)$. Also we note that as the maps are holomorphic, by the Cauchy integral formula for several complex variables (see [14]), the partial derivatives of $\mathcal{L}_{\mu_i}$ converge uniformly on compact sets to the partial derivatives of $\mathcal{L}_\mu$.

To show that the torsion function $T_\mu = 0$ on $F(S)$ we first consider $\mu = \alpha$ where α is a closed geodesic. Then for $X \in F(S)$, $T_\alpha(X)$ must be a multiple of 2π. As T_α is continuous on the connected subset $F(S) \subseteq QF(S)$, and $T_\alpha(S) = 0$, we have that $T_\alpha = 0$ on $F(S)$. Similarly if $\mu = k\alpha$ where $k \in \mathbb{R}_+$ then for $X \in F(S)$, $T_\mu(X)$ is a multiple of $2\pi k$. Therefore it follows once again that $T_\mu = 0$ on $F(S)$. Finally if μ is not discrete, then by lemma 2.1 there exists $\mu_i \rightarrow \mu$ with μ_i discrete. Therefore for $X \in F(S)$ we have $T_{\mu_i}(X) \rightarrow T_\mu$. As $T_{\mu_i}(X) = 0$, we have that $T_\mu = 0$ on $F(S)$. $\square$

Let $X \in QF(S)$ and $f_X : \mathcal{C}(S) \rightarrow \mathcal{C}(X)$ be the natural homeomorphism between geodesic current spaces coming from the marking. We let $\mu_X \in \mathcal{C}(X)$ be the unit Patterson-Sullivan geodesic current of X. Then we define the *complex Patterson-Sullivan length function* of X to be the function $\mathcal{L}_{\mathcal{X}} : QF(S) \rightarrow \mathbb{C}$ given by

$$\mathcal{L}_X = \mathcal{L}_{f_X^{-1}(\mu_X)}.$$

Similarly we define the *Patterson-Sullivan length function* $L_X : QF(S) \rightarrow \mathbb{R}$ by $L_X = Re(\mathcal{L}_X)$.

In [8], we defined the length distortion function $R : QF(S) \times QF(S) \rightarrow \mathbb{R}$ by $R(X, Y) = L_X(Y)$. We have the following immediate application of Analyticity Theorem.

Corollary 6.2 *Let S be a closed hyperbolic surface. Then the length distortion function $R : QF(S) \times QF(S) \to \mathbb{R}$ is $(0, \omega)$-differentiable.*

Proof: By Hamenstädt (see [11]), the map $I : QF(S) \to \mathcal{C}(S)$ given by $I(X) = f_X^{-1}(\mu_X)$ is a continuous and injective map onto its image. By the Analyticity Theorem, the length function $L : \mathcal{C}(S) \to C^\infty(QF(S), \mathbb{R})$ is continuous. We consider $R : QF(S) \to C^\infty(QF(S), \mathbb{R})$ by letting $R(X) = L_X$. Then $R = L \circ I$, and is a composition of two continuous functions. Thus we have that R is continuous. Thus R is $(0, \omega)$-differentiable. $\square$

7 Second derivatives and metric structures

We now consider metric structures on Teichmüller space and quasifuchsian space. To do so we first define some basic facts about smooth manifolds.

Let M be a smooth $n-$manifold and $F : T^2(M) \to \mathbb{R}$ be a symmetric two-tensor defined on the product bundle $T^2(M) = T(M) \times T(M)$ of M. We also use the notation $< .,. >_F$ for F. Then for $x \in M$ we denote the associated symmetric bilinear map by $F_x : T_x(M) \times T_x(M) \to \mathbb{R}$. If F_x is positive-definite for all $x \in M$ we call F a *metric* on M. Similarly if F_x is positive semi-definite for all $x \in M$ we call F a *semi-definite metric* M. If F is a definite or semi-definite metric, we denote the associated norm by $||.||_F$.

To define symmetric two-tensors we will use the second derivative of smooth real-valued functions. Let $f : M \to \mathbb{R}$ be a smooth real-valued function on M. Then the differential df is a well-defined 1-form with associated linear maps $f'(x) : T_x(M) \to \mathbb{R}$. In general derivatives of order two or higher will not be well-defined. We define the Hessian of f with respect to a chart $u : U \to \mathbb{R}^n$ to be the map $f''_u(x) : T_x(M) \times T_x(M) \to \mathbb{R}$, given by

$$f''_u(x) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial u^i \partial u^j}(x)\ du^i \otimes du^j.$$

If $v : V \to \mathbb{R}^n$ is another coordinate chart, then

$$\frac{\partial^2 f}{\partial v^k \partial v^l}(x) = \sum_{i,j=1}^{n} \left(\frac{\partial^2 f}{\partial u^i \partial u^j}(x)\ \frac{\partial u^i}{\partial v^k} \frac{\partial u^j}{\partial v^l} \right) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)\ \frac{\partial^2 u^i}{\partial v^k \partial v^l}.$$

Because of the second term on the righthandside, we do not have a well-defined second derivative. In order to obtain a well-defined second derivative, we must restrict to $x \in M$ such that $f'(x) = 0$. Then the obstructing term disappears and we have that $f''_u(x), f''_v(x)$ are equal and we have a well-defined second derivative denoted $f''(x)$. The map $f''(x) : T_x(M) \times T_x(M) \rightarrow \mathbb{R}$ is then a symmetric bilinear form.

We now apply this construction to maps on quasifuchsian space $QF(S)$.

Let $h : QF(S) \rightarrow \mathbb{R}$ be the hausdorf dimension function given by letting $h(X)$ be the hausdorf dimension of the limit set of X. In [19], Ruelle showed that h is real analytic on $QF(S)$. Also $h(X) = 1$ for $X \in F(S)$ and Bowen proved that $h(Y) > 1$ for $Y \notin F(S)$ (see [6]). Therefore if $X \in F(S)$ then h is minimum at X. Therefore $h'(X) = 0$ and $h''(X)$ is a well-defined symmetric bilinear form $h''(X) : T_X(QF(S)) \times T(QF(S)) \rightarrow \mathbb{R}$. Also, as h is minimum at X, the bilinear form $h''(X)$ is positive semi-definite.

Given a point $X \in QF(S)$ we consider the Patterson-Sullivan length function $L_X : QF(S) \rightarrow \mathbb{R}$. We multiply the functions h and L_X to obtain the function $(h.L_X) : QF(S) \rightarrow \mathbb{R}$ given by $(h.L_X)(Y) = h(Y).L_X(Y)$. In [8] we prove

Theorem 7.1 *(Bridgeman, Taylor, [8]) The map* $(h.L_X) : QF(S) \rightarrow \mathbb{R}$ *is minimum at* X*.*

It follows that $(h.L_X)'(X) = 0$ and therefore second derivative $(h.L_X)''(X)$ is a semi-definite symmetric bilinear form. We define the semi-definite metric G on $QF(S)$ by letting $G_X : T_X(QF(S)) \times T_X(QF(S)) \rightarrow \mathbb{R}$ satisfy

$$G_X = (h.L_X)''(X).$$

We will see in the next section that G is a natural extension of the Weil-Petersson metric on Teichmüller space.

Lemma 7.2 *Let* $X \in F(S)$*, then* $L'_X(X) = 0$ *and*

$$G_X = h''(X) + L''_X(X).$$

Proof: As $(h.L_X)'(X) = 0$, then by the product rule we have

$$h'(X).L_X(X) + h(X).L'_X(X) = 0.$$

Let $X \in F(S)$. Then $h'(X) = 0$, and we have $L'_X(X) = 0$. It follows that $L''_X(X)$ is a well-defined symmetric bilinear form $L''_X(X) : T_X(QF(S)) \times T_X(QF(S)) \to \mathbb{R}$. Then the desired equation follows by applying the product rule again. □

8 An extension of the Weil-Petersson metric

In [2], Bonahon introduced geodesic currents on surfaces to study the Teichmüller space. We outline this work.

Let $S = \mathbb{H}^2/\Gamma$ be a closed hyperbolic surface, and $\mathcal{C}(S) = \mathcal{C}(\Gamma)$ be the space of geodesic currents for S. If α, β are closed geodesics in S, let $i_S(\alpha, \beta)$ be the geometric intersection number of α, β and let $l_S(\alpha), l_S(\beta)$ be the lengths of α, β respectively. By linearity, the maps i_S and l_S naturally extend to geodesic currents which are multiples of closed geodesics. Bonahon proved that

Theorem 8.1 *(Bonahon [2]) There exists continuous maps $i_S : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}$ and $l_S : \mathcal{C}(S) \to \mathbb{R}$, which extend the intersection pairing and length function on multiples of closed geodesics.*

Considering $G(\mathbb{H}^2) = (\mathbb{S}^1_\infty \times \mathbb{S}^1_\infty - \text{diagonal})/\mathbb{Z}_2$ we associate with the hyperbolic structure S, a *Liouville geodesic current* m_S of S by letting

$$m_S([[a, b] \times [c, d]]) = |\ln |(a, b, c, d)|| .$$

where (a, b, c, d) is the cross-ratio of the points $a, b, c, d \in \mathbb{S}^1_\infty$. By the invariance of the cross ratio under Möbius transformations, m_S is invariant under the action of Γ on $G(\mathbb{H}^2)$ and is therefore a geodesic current for S.

Theorem 8.2 *(Bonahon [2]) The geodesic current m_S is uniquely determined by the property that*

$$i_S(\alpha, m_S) = l_S(\alpha) \text{ for all } \alpha \in \mathcal{C}(S).$$

Furthermore it follows that $l_S(m_S) = \pi^2|\chi(S)|$.

We normalize m to obtain the *unit Liouville geodesic current*

$$\mu_S = \frac{m_S}{l_S(m_S)}.$$

Let $X \in F(S)$ and $f_X : \mathcal{C}(S) \to \mathcal{C}(X)$ be the homeomorphism obtained from the marking. Then we have the length function $\widehat{L} : \mathcal{C}(S) \times F(S) \to \mathbb{R}$ by $\widehat{L}(\alpha, X) = i_X(f_X(\alpha), m_X)) = l_X(f_X(\alpha))$ where $l_X : \mathcal{C}(X)$ is the length function on $\mathcal{C}(X)$. Then $\widehat{L}(\alpha, X)$ is the length of α in X. Furthermore if $X \in F(S)$ we let $\widehat{L}_X : F(S) \to \mathbb{R}$ be the length function of the unit Liouville geodesic current given by $\widehat{L}_X(Y) = \widehat{L}(f_X^{-1}(\mu_X), Y)$.

As the definition of the length function $L : \mathcal{C}(S) \times QF(S) \to \mathbb{R}$ is a direct generalization of Bonahon's length function, we have that $\widehat{L}$ is the restriction of L to the domain $\mathcal{C}(S) \times F(S)$. Also by definition of the Liouville geodesic current m_X, it is also a Patterson-Sulliivan measure for $X \in F(S) \subseteq QF(S)$. Then by uniqueness, the unit Liouville geodesic current μ_X is also the unit Patterson-Sullivan geodesic current of X. Then by the above, $\widehat{L}_X : F(S) \to \mathbb{R}$ is just the restriction of $L_X : QF(S) \to \mathbb{R}$ to $F(S) \subseteq QF(S)$.

In [23], Wolpert gave a description of the Weil-Petersson metric on Techmüller space $F(S)$ in terms of the limit of second derivatives of certain geodesic length functions. In a subsequent paper, Bonahon described the same construction using the theory of geodesic currents, showing that the Weil-Petersson metric at $X \in F(S)$ is just the second derivative of the Liouville length function $\widehat{L}_X$ of X evaluated at X (see [2]).

Theorem 8.3 *(Wolpert [23], Bonahon [2]) Let S be a closed surface and g the normalized Weil-Petersson metric on $F(S)$. If $X \in F(S)$ then*

$$g_X = \widehat{L}''_X(X).$$

The proof of the Weil-Petersson Extension Theorem follows from this theorem and the fact that the hausdorf dimension term in the definition of G vanishes in $F(S)$.

Proof of the Weil-Petersson Extension Theorem: Let $X \in F(S) \subseteq QF(S)$ and $v_1, v_2 \in T_X(F(S)) \subseteq T_X(QF(S))$. By definition of G we have

$$G_X(v_1, v_2) = (h.L_X)''(X)(v_1, v_2).$$

As $h(Y) = 1$ on $F(S)$ and $v_1, v_2 \in T_X(F(S))$, we have

$$G_X(v_1, v_2) = L''_X(X)(v_1, v_2).$$

By theorem 8.3 above, and the fact that L_X is an extension of $\widehat{L}_X$, we have

$$G_X(v_1, v_2) = L''_X(X)(v_1, v_2) = \widehat{L}''_X(v_1, v_2) = g_X(v_1, v_2).$$

Therefore restricting G_X to $T_X(F(S)) \subseteq T_X(QF(S))$ we obtain

$$G_X \big|_{T_X(F(S))} = g_X \ .$$

□

9 Calculating $L''_X(X)$

As $X \in F(S) \subseteq QF(S)$ we have the vector subspace $T_X(F(S)) \subseteq T_X(QF(S))$. By Bonahon, we have

Theorem 9.1 *(Bonahon [3]) If $X \in F(S)$ then*

$$T_X(QF(S)) = T_X(F(S)) \oplus J.T_X(F(S)).$$

We will see in the following section that the vector subspace $T_X(F(S))$ corresponds to the pure shearing directions and the vector subspace $J.T_X(F(S))$ corresponds to the pure bending directions.

We denote the formal complexification of a real vector space V by $\mathbb{C}.V$. Then by the above lemma we have $T_X(QF(S)) = \mathbb{C}.T_X(F(S))$. The Weil-Petersson metric defines a real symmetric bilinear form g_X on $T_X(F(S))$. Therefore there is a unique symmetric complex bilinear form $g_X^{\mathbb{C}}$ on $T_X(QF(S)) = \mathbb{C}.T_X(F(S))$ with $g_X^{\mathbb{C}} = g_X$ on $T_X(F(S))$. We have the following theorem.

Theorem 9.2 *Let $X \in F(S)$ and $\mathcal{L}_X : QF(S) \to \mathbb{C}$ be the complex Patterson-Sullivan length function of X. Then $\mathcal{L}''_X(X)$ is well-defined and satisfies*

$$\mathcal{L}''_X(X) = g_X^{\mathbb{C}}.$$

Proof: To show that $\mathcal{L}''_X(X)$ is well-defined, we must first show that $\mathcal{L}'_X(X) = 0$. As $L'_X(X) = 0$ and $\mathcal{L}_X = L_X$ on $F(S)$, if $u \in T_X(F(S))$, then $\mathcal{L}'_X(X)(u) = L'_X(X)(u) = 0$. Therefore $\mathcal{L}'_X$ is zero on $T_X(F(S))$. As $\mathcal{L}_X(X)$ is holomorphic, $\mathcal{L}'_X(X)$ is complex linear on $T_X(QF(S)) =$

$\mathbb{C}.T_X(F(S))$. Therefore $\mathcal{L}'_X(X)$ is the complexification of it's restriction to $T_X(F(S))$ and is thus zero. It follows $\mathcal{L}''_X(X)$ is a well-defined complex bilinear map. As $\mathcal{L}_X = L_X$ on $F(S)$ and theorem 8.3, we have that for $u, v \in T_X(F(S))$,

$$\mathcal{L}''_X(X)(u,v) = L''_X(X)(u,v) = \widehat{L}''_X(X)(u,v) = g_X(u,v).$$

As $\mathcal{L}''_X(X)$ is complex bilinear, we have $\mathcal{L}''_X(X) = g_X^{\mathbb{C}}$ $\square$

An immediate corollary is

Corollary 9.3 *Let $X \in F(S)$ and $L_X : QF(S) \to \mathbb{R}$ be the Patterson-Sullivan length function of X*

$$L''_X(X) = Re(g_X^{\mathbb{C}}).$$

In terms of the decomposition $T_X(QF(S)) = T_X(F(S)) \oplus J.T_X(F(S))$, we have

$$L''_X(X) == \left(\begin{array}{cc} g & 0 \\ 0 & (-g) \end{array} \right)$$

where $(-g)(J.u, J.v) = -g(u,v)$.

10 Proof of the H-metric Theorem

We now define a semi-definite metric H on $F(S)$. Let $X \in F(S)$ and $u, v \in T_X(F(S))$, then we let

$$H(u,v) = h''(X)(J.u, J.v).$$

Before we prove the H-metric Theorem, we restate it.

H-metric Theorem *The semi-definite metric H is a (positive-definite) metric on $F(S)$ and satisfies*

$$||u||_H \geq ||u||_g.$$

Proof: Let $X \in F(S)$ and $u \in T_X(F(S))$. As G is a semi-definite metric on $QF(S)$ we have $G_X(J.u, J.u) \geq 0$. Therefore by lemma 7.2

$$h''(X)(J.u, J.u) + L''_X(X)(J.u, J.u) = G_X(J.u, J.u) \geq 0.$$

By corollary 9.3 we have $L''_X(X) = Re(g^{\mathbb{C}})$ and therefore

$$L''_X(X)(J.u, J.u) = (-g)(J.u, J.u) = -g(u,u) = -||u||_g^2.$$

As $||u||_H^2 = H(u,u) = h''(X)(J.u, J.u)$ then

$$||u||_H^2 - ||u||_g^2 \geq 0$$

giving the desired inequality. Positive-definiteness of H now follows from positive-definiteness of g. $\Box$

As $h = 1$ on $F(S)$ we have $h''(X) = 0$ on $T_X(F(S))$. Therefore we have the following description of $h''(X)$.

Corollary 10.1 *Let $X \in F(S)$, then in terms of the decomposition $T_X(QF(S)) = T_X(F(S)) \oplus J.T_X(F(S))$, we have*

$$h''(X) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

where A is positive definite and given by $A(J.u, J.v) = H(u,v)$.

11 Complex earthquakes

We now describe the relation between hausdorf dimension and the Weil-Petersson metric given in the H-metric Theorem in terms of complex earthquakes. Let S be a closed hyperbolic surface. Then Thurston introduced measured laminations as a natural extension of simple closed geodesics (see [22]). Specifically a *measured lamination* on S is a geodesic current α with zero self-intersection, that is, $i_S(\alpha,\alpha) = 0$. A measured lamination is supported on a set of geodesics that are pairwise disjoint. The space of measured laminations on S is denoted $ML(S)$ and given the weak* topology. If α is a geodesic current corresponding to a multiple of a simple closed geodesic then α is a measured lamination and is called *discrete*. Also, Thurston proved that the set of discrete measured laminations is dense in $ML(S)$ (see [22]). Thus measured laminations can be considered as completion of the set of simple closed geodesics, in the same way that geodesic currents are considered a completion of the set of closed geodesics.

Let $X \in F(S) \subseteq QF(S)$ and $\beta \in ML(S)$. We now describe a complex earthquake deformation of X in $QF(S)$ along $\beta \in ML(S)$ (see Epstein and Marden's paper [10] for details).

We first consider the case when β is a simple closed geodesic. We let $X = \mathbb{H}^2/\Gamma$ where Γ is a Fuchsian subgroup of $PSL(2,\mathbb{C})$ acting on upper half space with limit set $\overline{\mathbb{R}} = \mathbb{R} \cup \infty$. Then Γ fixes the hyperbolic plane $H = \{(x,0,z) | z > 0\}$. We deform Γ as follows; Let $\gamma \in \Gamma$ be a hyperbolic element of Γ corresponding to the closed geodesic β. By composition, we can assume that γ has axis equal to the vertical line L with endpoints $0, \infty$.

To see how γ is deformed, we let $\overline{\beta} = f_X(\beta)$ be the closed geodesic in X corresponding to β. We then consider the lifts of $\overline{\beta}$ which intersect the axis L of γ. We choose one such lift of $\overline{\beta}$ and label it $\overline{\beta}_1$. We enumerate all other lifts by the order of the height of their intersection point with L starting with the intersection point of $\overline{\beta}_1$. Let n be such that $\gamma.\overline{\beta}_1 = \overline{\beta}_{n+1}$. Let $z = x + iy \in \mathbb{C}$. Then we define $R_i(z)$ to be the Möbius transformation corresponding to the composition of translating along $\overline{\beta}_i$ an amount x and rotating about $\overline{\beta}_i$ an amount y . Then under the complex earthquake $z.\beta$, γ is deformed to the element $\gamma(z)$ given by

$$\gamma(z) = R_1(z)R_2(z)\dots R_n(z)\gamma.$$

We let Γ_z be the subgroup of $PSL(2,\mathbb{C})$ obtained by performing the complex earthquake along β by $z \in \mathbb{C}$. The case when β is a general measured lamination is obtained by taking a limit of this process and using the fact that discrete measured laminations are dense (see [10]).

For small z, Γ_z will be a quasifuchsian group (see [15]). In fact, given a $\beta \in ML(S)$, there is an open set $V \subseteq \mathbb{C}$ with $\mathbb{R} \subseteq V$ such that $\mathbb{H}^3/\Gamma_z \in QF(S)$ for all $z \in V, X \in F(S)$. Therefore the *complex earthquake map* $E_\beta : V \times F(S) \to QF(S)$ is then defined by letting $E_\beta(z, X) = \mathbb{H}^3/\Gamma_z \in QF(S)$ (see [10], [17]).

Restricting E_β to z real, we obtain the map $S_\beta : \mathbb{R} \times F(S) \to F(S)$. The map S_β is called the *shearing map* along β. In the case when $\beta = t.\alpha$ where α is a simple closed geodesic, S_β corresponds to cutting X along α, twisting an amount t and then regluing. We define the shearing vector field s_β on $F(S)$ by letting $s_\beta(X) \in T_X(F(S))$ satisfy

$$s_\beta(X) = \frac{dS_b}{dt}(0, X).$$

Kerckhoff showed that the map S_β is real-analytic in t and that $T_X(F(S)) = \{s_\beta(X) \mid \beta \in ML(S)\}$ (see [12]).

Now restricting E_β to z purely imaginary, for $\epsilon > 0$ sufficiently small, we obtain the map $B_\beta : (-\epsilon, \epsilon) \times F(S) \to QF(S)$ given by $B_\beta(t, X) = E_\beta(it, X)$. The map B_β is called the *bending map* along β. In the case when $\beta = t.\alpha$ where α is a simple closed geodesic, B_β corresponds to bending X along α, by angle t. In general, the quasifuchsian space $B_\beta(t, X)$ will have convex core with one boundary component having intrinsic hyperbolic structure X and bending lamination $|t|.\beta$. The sign of t determines which side has bending lamination $|t|.\beta$. We define the bending vector field b_β on $F(S)$ by letting $b_\beta(X) \in T_X(QF(S))$ satisfy

$$b_\beta(X) = \frac{dB_b}{dt}(0, X).$$

In [3], Bonahon showed that shearing and bending behave nicely with respect to the complex structure on $QF(S)$.

Lemma 11.1 *(Bonahon, [3]) Let* $X \in F(S) \subseteq QF(S)$*, then* $T_X(QF(S)) = T_X(F(S)) \oplus J.T_X(F(S))$ *and* $b_\beta(X) = J.s_\beta(X)$*.*

It follows from this lemma and the H-metric Theorem that if we deform a hyperbolic surface $X \in F(S)$ by bending along β then

$$h''(X)(b_\beta, b_\beta) = H(s_\beta, s_\beta) \geq g(s_\beta, s_\beta) > 0.$$

Therefore we obtain the following relation between hausdorf dimension under bending and the Weil-Petersson metric,

$$h(B_\beta(t, X)) \geq 1 + \frac{t^2}{2}.g(s_\beta, s_b) + o(t^2).$$

In [6], Bowen proved that for $h(X) > 1$ for $X \notin F(S)$. The above equation can be considered an infinitesimal version of this theorem stating that the rate of increase of h is exactly second order on $F(S)$.

References

[1] S. Axler, P. Bourdon, W. Ramey, “Harmonic Function Theory ” *Graduate Texts in Mathematics* Springer-Verlag, **137**, (1992)

[2] Francis Bonahon “The geometry of Teichmüller space via geodesic currents,” *Invent. Math.* **92** (1988), pp. 139–162.

[3] F. Bonahon, “Shearing hyperbolic surfaces, bending pleated surfaces and Thurston’s symplectic form,”*Ann. Fac. Sci. Toulouse Math.* (6) **5** (1996), pp. 233- 297.

[4] B.H. Bowditch, “Geometrical finiteness for hyperbolic groups,” *J. Funct. Anal.* **113** (1993), pp. 245–317.

[5] R. Bowen, “Periodic orbits for hyperbolic flows,” *Amer. J. Math.* **94** (1972), pp. 1–30.

[6] R. Bowen, "Hausdorff dimension of quasi-circles," *Publ. Math. IHES,* **50** (1979), pp. 11–25.

[7] M. Bridgeman and E.C. Taylor, "Length distortion and the Hausdorff dimension of Kleinian groups," *American Journal of Mathematics,* **122** (2000), pp 465-482.

[8] M. Bridgeman and E. Taylor, "Patterson-Sullivan measures and quasiconformal deformations," *Communications in Analysis and Geometry,* **13(3)** (2005), 561-589

[9] A. Douady and C. Earle, "Conformally natural extensions of homeomorphisms of the circle," *Acta Math.,* **157** (1986), pp. 23–48.

[10] D. Epstein and A. Marden, "Convex hulls in hyperbolic space, a Theorem of Sullivan, and measured pleated surfaces," in *Analytical and Geometrical Aspects of Hyperbolic Space*, Cambridge University Press, 1987.

[11] U. Hamenstädt, *Ergodic properties of function groups*, preprint (2001).

[12] S. Kerckhoff, "Earthquakes are analytic," *Comment. Math. Helvetici,* **60** (1985),pp. 17–30.

[13] I. Kra and B. Maskit, "Deformation Space of a Kleinian Group" Amer. J. Math. **103**(5), (1980), pp. 1065–1102.

[14] S. Krantz, "Function Theory of Several Complex Variables," *AMS Chelsea Publishing*, (1992).

[15] A. Marden, "The geometry of finitely generated Kleinian groups," *Ann. of Math.* **99** (1974), pp. 383–462.

[16] B. Maskit, *Kleinian Groups*, Springer-Verlag, 1987.

[17] C. McMullen, "Complex earthquakes and Teichmüller theory," *Journal of the A.M.S.*, **11**, 1998.

[18] P. Nicholls, *The Ergodic Theory of Discrete Groups*, Cambridge University Press, 1989.

[19] D. Ruelle "Repellers for real analytic maps," *Ergodic Theory Dynamical Systems,* **2** (1982), pp. 99–107.

[20] K. Sigmund, "On dynamical systems with the specification property," *Trans. Amer. Math. Soc.,* **190** (1974), pp. 285–299.

[21] D. Sullivan, "Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups," *Acta Math.,* **153** (1984), pp. 259–277.

[22] W.P. Thurston, *The Geometry and Topology of* 3-*Manifolds*, Lecture Notes, Princeton University, 1979.

[23] S. Wolpert, "Thurston's Riemannian metric for Teichmüller space," *J. Diff. Geom.,* **23** (1986), pp. 173–174.