Schur Polynomials and the Yang-Baxter Equation

Brubaker, Bump, Friedberg

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Tokuyama [33] proved a deformation of the Weyl character formula for $\mathrm{GL}_n(\mathbb{C})$. A substantial generalization of Tokuyama's deformation was found by Hamel and King [9]. The formula of Hamel and King expresses the Schur polynomial times a deformation of the Weyl denominator as a sum over states of the *two-dimensional ice* or *six-vertex model* in statistical mechanics. It turns out that there are two fundamentally distinct ways of doing this. We will call these *Gamma ice* and *Delta ice*. The Delta model is essentially that given by Hamel and King.

In statistical physics, the *partition function* is the sum of certain *Boltzmann weights* over all states of the system. The six-vertex model is an example that is much studied in the literature. If the Boltzmann weights are invariant under sign reversal the system is called *field-free*, corresponding to the physical assumption of the absence of an external field. For field-free weights, the six-vertex model was solved by Lieb [20] and Sutherland [32], in the sense that the partition function can be exactly computed. A very interesting treatment based on the "star-triangle relation" or Yang-Baxter equation ([14], [22]) was given by Baxter [1] and [2], Chapter 9. The papers of Lieb, Sutherland and Baxter assume periodic boundary conditions, but non-periodic boundary conditions were treated by Korepin [15] and Izergin [13]. Much of the literature assumes that the model is field free, but Baxter asserts that the six-vertex model can be solved even in the presence of fields. We do not know whether this has been carried out using the method of [1] and [2].

We will exhibit two particular choices of Boltzmann weights and boundary conditions in the six-vertex model giving systems $\mathfrak{S}_\lambda^\Gamma$ and $\mathfrak{S}_\lambda^\Delta$ for every partition λ of length $\leqslant n$. We will study the system by the method of [1] and [2]. The partition functions are

$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i<j}(t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n), \qquad Z(\mathfrak{S}_\lambda^\Delta) = \prod_{i<j}(t_j z_j + z_i) s_\lambda(z_1, \cdots, z_n), \quad (1)$$

where t_i are deformation parameters and s_λ is the Schur polynomial ([21]). The Boltzmann weights we use are not field-free.

To justify these evaluations of the partition function define

$$s_\lambda^\Gamma(z_1, \cdots, z_n; t_1, \cdots, t_n) = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)}. \tag{2}$$

Then one seeks to show that s_λ^Γ is symmetric in the sense that it is unchanged if the same permutation is applied to both z_i and t_i. Once this is known, it is possible to show that it is a polynomial in the z_i and t_i, then that it is independent of the t_i; finally, taking $t_i = -1$ one may invoke the Weyl character formula and conclude that it is equal to the Schur polynomial.

In order to prove the symmetry property of s_λ^Γ we will use an instance of the star-triangle relation in the form (9.6.8) of Baxter [2]. We thus obtain a new proof of Tokuyama's formula and of Corollary 5.1 in Hamel and King [9], which is our Theorem 7. A second instance of the star-triangle relation solves the same problem for the analogously defined s_λ^Δ, and a third instance shows directly, without using the above evaluations, that $s_\lambda^\Gamma = s_\lambda^\Delta$.

The star-triangle identity may be written

$$R_{12}M_{13}N_{23} = N_{23}M_{12}R_{12}, \tag{3}$$

where R, M, N are endomorphisms of $V \otimes V$ for some vector space V, and R_{ij} means apply R to the i-th and j-th components of $V \otimes V \otimes V$ and the identity map to the k-th component, where i, j, k are $1, 2, 3$ in some order. More precisely, for each pair z, t of spectral parameters, there is to be a vector space $V(z, t)$, and R, M, N are to depend on such parameters. But for this introduction we will suppress the spectral parameters; equation (3) is a shorthand for a parametrized equation such as (7) below (where $M = N = \Gamma$) or (12). The entries of the matrices of M and N with respect to suitable bases are Boltzmann weights associated with *commuting transfer matrices* (see Baxter [2]). The *R-matrix* R is to be supplied, and its existence implies the commutativity of these transfer matrices.

In Baxter's method M and N are implicit in the original problem of proving that the corresponding transfer matrices commute, and R is introduced as a method of proving this. But R itself turns out to be of fundamental importance. It satisfies $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, or a parametrized version (Theorem 6). This cubic version of the star-triangle relation is the (parametrized) *Yang-Baxter equation*.

We will prove the Yang-Baxter equation for several types of ice. More precisely, let X and Y be symbols which can be either Γ or Δ. We will construct an R-matrix

R_{XY} which has the effect of interchanging a strand of X ice with a strand of Y ice; thus in (3), M is of type X and N is of type Y. We will prove that $R_{\Gamma\Gamma}$ and $R_{\Delta\Delta}$ both satisfy the Yang-Baxter equation, and we will prove similar relations that involve all four types of ice R_{XY} in various combinations. These may be organized into a *Yang-Baxter system*, a structure whose significance we recall.

To explain this, recall that by Drinfeld [4], solutions to the Yang-Baxter equation may be found in $\mathrm{End}(V \otimes V)$ where V is any module over a *quasitriangular Hopf algebra*. (See also Majid [22], [23].) Conversely, given an R-matrix, that is, a solution to the Yang-Baxter equation, Drinfeld indicated that one expects to be able to reconstruct such a quasitriangular Hopf algebra. Faddeev, Reshetikhin and Takhtajan [5] gave details of such a construction which is further developed by Majid [22]. The version of Cotta-Ramusino, Lambe and Rinaldi [3] for R-matrices with spectral parameter is applicable in our case.

Another construction of Drinfeld [4] is that of the *quantum double*. This is an amalgamation of two Hopf algebras given a pairing between them. For example, this constructs the quantized enveloping algebra of a semisimple Lie algebra as the fusion of the enveloping algebras of its positive and negative nilpotent subalgebras. Vladimirov [33] revisited the Faddeev, Reshetikhin and Takhtajan construction, and clarified its relation to the quantum double. Snobl [28] is a development from Vladimirov's work. *Yang-Baxter systems* are described in Hlavaty [10] as a development from work of Vladimirov, and also work of Freidel and Maillet [7], where similar constructions appear in connection with integrable systems. We are able to realize our R-matrices as the ingredients of a Yang-Baxter system. That is, $R_{\Gamma\Gamma}$ and $R_{\Delta\Delta}$ are the R-matrices associated with a pair of triangular Hopf algebras, while $R_{\Gamma\Delta}$ and $R_{\Delta\Gamma}$ are connected with the pairing between them that produces the quantum double.

In the case where the t_i are equal, these Yang-Baxter systems are related to those previously found by Nichita and Parashar [26], [25].

Tokuyama's formula expresses what we have denoted $Z(\mathfrak{S}_\lambda^\Gamma)$ as a sum over strict Gelfand-Tsetlin patterns. These are triangular arrays of integers with descending rows that interleave (Section 2). The strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$ are in bijection with states of the model. This connection between states of the ice model and strict Gelfand-Tsetlin patterns has its historical origin in the literature for alternating sign matrices. In this case, the bijection with the set of alternating sign matrices and strict Gelfand-Tsetlin patterns (also called "monotone triangles") is in Mills, Robbins and Rumsey [24], while the connection with what are recognizably states of the six-vertex model is in Robbins and Rumsey [29]. This connection was used by Kuperberg [16] who gave a second proof (after the purely combinatorial one by Zeilberger [36]) of the conjecture of Mills, Robbins and Rumsey [24]. Kuperberg's paper follows Korepin [15] and Izergin [13] and makes use of the Yang-Baxter equation. It was observed by Okada [27] and Stroganov [31] that the number of $n \times n$ alternating sign matrices, that is, the value of Kuperberg's ice (with particular Boltzmann weights involving cube roots of unity) is a special value of the particular Schur function in $2n$ variables with $\lambda = (n, n, n-1, n-1, \cdots, 1, 1)$ divided by a power of 3. Moreover Stroganov gave a proof using the Yang-Baxter equation. See also Zinn-Justin [37] for a discussion. This occurrence of Schur polynomials in the six-vertex model is different from the one we discuss, and it would be quite interesting to find a relationship between the two.

Kuperberg [17] generalizes his work on alternating sign matrix enumeration to other symmetry types using the ice-type models and the Yang-Baxter equation, and Hamel and King [8] relate one of these classes to a symplectic analog of Tokuyama's theorem. Okada [28] enumerates many of the ice types of Kuperberg in terms of dimensions of highest weight representations of classical groups. In [27] he gives deformations of the Weyl character identity analogous to Tokuyama's result with $\lambda = 0$ for other classical groups. Fomin and Kirillov [6], [7] gave theories of the Schubert and Grothendieck polynomials of Lascoux and Schützenberger [19] based on the Yang-Baxter equation. Lascoux [18] gave six-vertex model representations of the Schubert polynomials with Yang-Baxter equations. Tsilevich [34] gives an interpretation of Schur polynomials and Hall-Littlewood polynomials in terms of a quantum mechanical system.

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1 Gamma Ice

If $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n$ then we may regard μ as an element of the $\mathrm{GL}_n(\mathbb{C})$ weight lattice and call it a *weight*. If $\mu_1 \geqslant \cdots \geqslant \mu_n$ we say it is *dominant*, and if $\mu_1 > \cdots > \mu_n$ we say it is *strictly dominant*. If μ is dominant and $\mu_n \geqslant 0$, it is a *partition*.

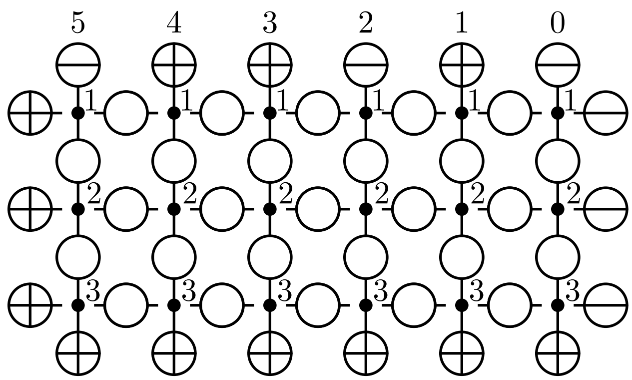
Note: The word "partition" occurs in two different senses in this paper. The partition function in statistical physics is different from partitions in the combinatorial sense. So for us a reference to a "partition" without "function" really means a partition. Another potentially ambiguous usage is that "weight" will sometimes refer to an element of the GL_n weight lattice, which we identify with $\mathbb{Z}^n$. Therefore if we mean Boltzmann weight, we will not omit Boltzmann.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a fixed partition. We will denote $\rho = (n-1, n-2, \cdots, 0)$. We will consider a rectangular grid with n rows and $\lambda_1 + n$ columns. We will number the columns of the lattice in descending order from $\lambda_1 + n - 1$ to 0.

A *state* of the model will consist of an assignment of "spins" $\pm$ to every edge. We will also assign labels to the vertices themselves, which will be integers between 1 and n. For Gamma ice the vertices in the i-th row will have the label i. The spins of the boundary edges are prescribed as follows.

Boundary Conditions determined by λ. *On the left and bottom boundary edges, we put +; on the right edges we put −. On the top, we put − at every column labeled $\lambda_i + n - i$ $(1 \leqslant i \leqslant n)$, that is, for the columns labeled with values in $\lambda + \rho$. Top edges not labeled by $\lambda_i + n - i$ for any i are given spin +.*

For example, suppose that $n = 3$ and $\lambda = (3, 1, 0)$, so that $\lambda + \rho = (5, 2, 0)$. Then the spins on the boundary are as follows.



(4)

The column labels are written at the top, and the vertex labels are written next to each vertex. The vertices are labeled • to indicate Gamma ice. The edge spins are marked inside circles. We have left the edge spins on the interior of the domain blank, since the boundary conditions only prescribe the spins we have written. The interior spins are not entirely arbitrary, since we require that at every vertex • the configuration of spins adjacent to the vertex be one of the six listed in Table 1 below under "Gamma ice."

Let $\mathfrak{S}^{\Gamma}_{\lambda}$ be the *Gamma ensemble determined by* λ, by which we mean the set of all such configurations, with the prescribed boundary conditions. If $x \in \mathfrak{S}^{\Gamma}_{\lambda}$, we assign a value $w(x)$ called the *Boltzmann weight*. Indeed, Table 1 assigns a Boltzmann weight to every vertex, and $w(x)$ is just the product over all the vertices of these Boltzmann weights. The *partition function* $Z(\mathfrak{S})$ of an ensemble $\mathfrak{S}$ is $\sum_{x \in \mathfrak{S}} w(x)$.

As an example, suppose that $n = 2$ and $l = (0, 0)$ so $\lambda + \rho = (1, 0)$. In this case

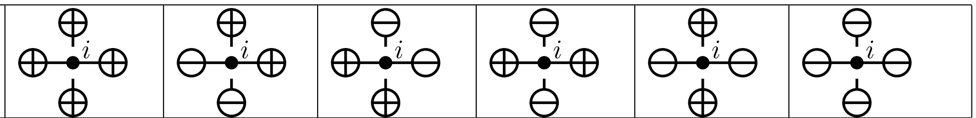
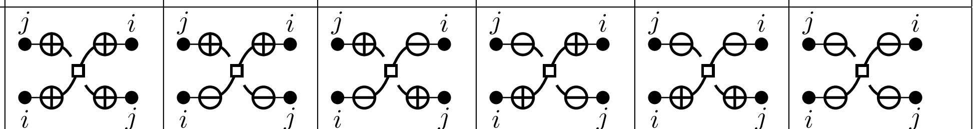
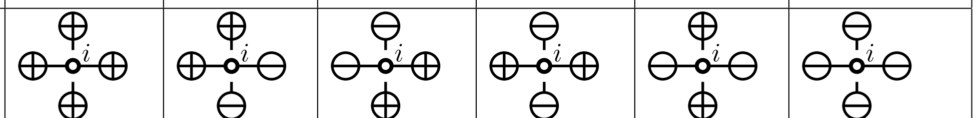
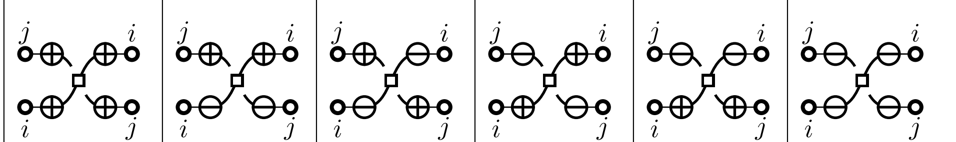
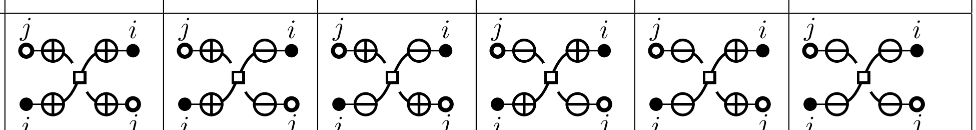
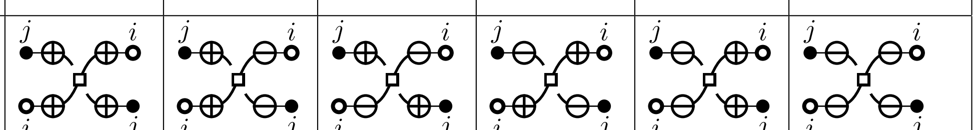
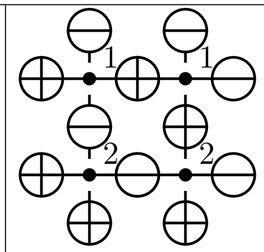
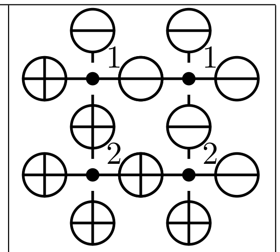
Gamma Ice						
Boltzmann weight	1	$z_i(t_i+1)$	1	t_i	z_i	z_i
Gamma-Gamma R-ice						
Boltzmann weight	$t_jz_i+z_j$	$(t_i+1)z_i$	z_i-z_j	$t_iz_j-t_jz_i$	$(t_j+1)z_j$	$t_iz_j+z_i$
Delta Ice						
Boltzmann weight	z_i	$z_i(t_i+1)$	1	z_it_i	1	1
Delta-Delta R-ice						
Boltzmann weight	$t_iz_i+z_j$	$z_j(t_j+1)$	$t_jz_j-t_iz_i$	z_i-z_j	$(t_i+1)z_i$	$z_i+t_jz_j$
Gamma-Delta R-ice						
Boltzmann weight	$t_it_jz_j-z_i$	$(t_j+1)z_j$	$t_iz_j+z_i$	$t_jz_j+z_i$	$(t_i+1)z_i$	z_i-z_j
Delta-Gamma R-ice						
Boltzmann weight	z_i-z_j	$(t_i+1)z_i$	$t_jz_i+z_j$	$t_iz_i+z_j$	$(t_j+1)z_j$	$-t_it_jz_i+z_j$

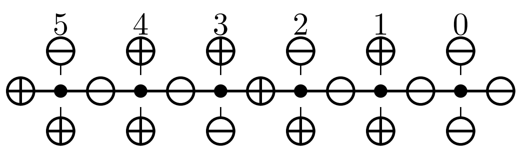
Table 1: Boltzmann weights for various types of ice with spectral parameters (z_i, t_i) and (z_j, t_j).

$\mathfrak{S}_\lambda^\Gamma$ has cardinality two, and $Z(\mathfrak{S}_\lambda^\Gamma) = t_1 z_2 + z_1$. The states are:

state		
Boltzmann weight	$t_1 z_2$	z_1

2 Gelfand-Tsetlin patterns

Let us momentarily consider a Gamma ice with just one layer of vertices, so there are two rows of edges, top and bottom. Let $\alpha_1, \cdots, \alpha_m$ be the locations (from left to right) of $-$ in the top row of edges with this labeling, and let $\beta_1, \cdots, \beta_{m'}$ be the locations of $-$ in the bottom row of edges. For example, in this ice:

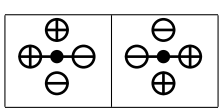


we have $m = 3$, $m' = 2$, $(\alpha_1, \alpha_2, \alpha_3) = (5, 2, 0)$ and $(\beta_1, \beta_2) = (3, 0)$. Since the columns are labeled in decreasing order, we have $\alpha_1 > \alpha_2 > \cdots$ and $\beta_1 > \beta_2 > \cdots$.

Lemma 1 *Suppose that the spin at the left edge is $+$. Then we have $m = m'$ or $m' + 1$ and $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \ldots$. If $m = m'$ then the spin at the right edge is $+$, while if $m = m' + 1$ it is $-$.*

We express the condition that $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \ldots$ by saying that the sequences $\alpha_1, \alpha_2, \cdots$ and $\beta_1, \beta_2, \cdots$ *interleave.*

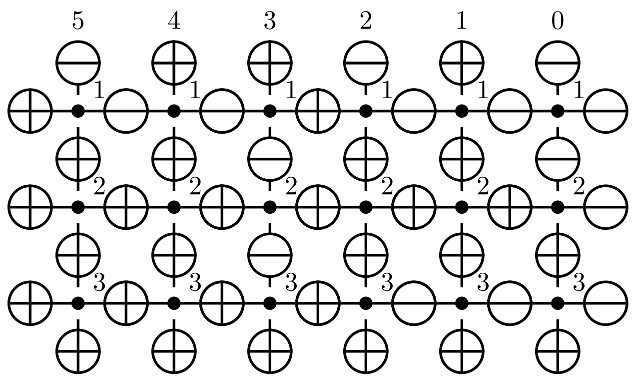
Proof The spins in the middle row are determined by those in the top and bottom rows and the left-most spin in the middle row, which is $+$, since the edges at each vertex have an even number of $+$ spins. If the rows do not interleave then one of the illegal configurations



will occur. Thus $\alpha_1 \geqslant \beta_1$ since if not, the vertex in the β_1 column would be surrounded by spins in the first illegal configuration. Now $\beta_1 \geqslant \alpha_2$ since otherwise the vertex in the α_2 column would be surrounded by spins in the first above illegal configuration, and so forth. The last statement is a consequence of the observation that the total number of spins must be even. □

We recall that a *Gelfand-Tsetlin pattern* is a triangular array of dominant weights, in which each row has length one less than the one above it, and the rows interleave. The pattern is called *strict* if the rows are strictly dominant.

It follows from Lemma 1 that taking the locations of $-$ in the rows of vertical edges gives a sequence of strictly dominant weights forming a strict Gelfand-Tsetlin pattern. For example, given the state



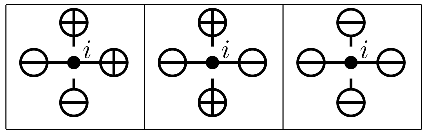
the pattern is

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} 5 & & 2 & & 0 \\ & 3 & & 0 & \\ & & 3 & & \end{array} \right\}. \tag{5}$$

It is not hard to see that this gives a bijection between strict Gelfand-Tsetlin patterns and states with boundary conditions determined by λ. Let us say that the *weight* of a state is $(\mu_1, \cdots, \mu_n)$ if the Boltzmann weight is the monomial $\boldsymbol{z}^\mu = \prod z_i^{\mu_i}$ times a polynomial in t_i. If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, let $d_k(\mathfrak{T})$ be the sum of the k-th row. We let $d_{n+1}(\mathfrak{T}) = 0$.

Lemma 2 *If $\mathfrak{T}$ is the Gelfand-Tsetlin pattern corresponding to a state of weight μ, then $\mu_k = d_k(\mathfrak{T}) - d_{k+1}(\mathfrak{T})$.*

Proof From Table 1, μ_k is the number of vertices in the k-th row that have an edge configuration of one of the three forms:



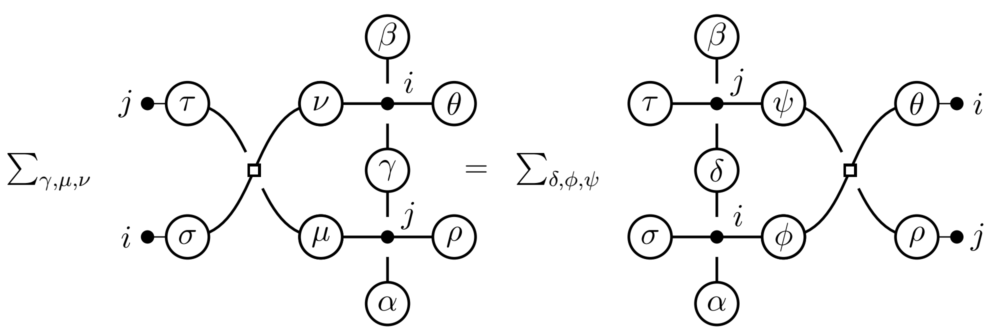
Let α_i and β_i be, respectively, the numbers of the columns where the top edge spin or the bottom edge spin of the vertex in the k-th row and i-column is $-$ (with columns numbered in descending order, as always). By Lemma 1 we have $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{n+1-k}$. It is easy to see that the vertex in the i-column has one of the above configurations if and only if its column number i satisfies $\alpha_j > i \geqslant \beta_j$ for some j. Therefore the number of such i is $\sum \alpha_j - \sum \beta_j = d_k(\mathfrak{T}) - d_{k+1}(\mathfrak{T})$. ☐

3 The Star-Triangle Relation

We fix spectral parameters (z_i, t_i) and (z_j, t_j). We will consider interactions between two strands of Gamma ice, one with vertices labeled i using the first pair of spectral parameters, and the second, labeled j using the second pair of spectral parameters. The star-triangle relation of Baxter gives a way of switching these strands.

In addition to the "ordinary" vertices in the lattice, we will need another type of vertex which we will draw rotated 45°. This type of *R-vertex* will sit between two columns of ordinary vertices. These vertices are the ones labeled "Gamma-Gamma R-ice" in Table 1. Such an R-vertex also has six admissible configurations, and will be assigned Boltzmann weights by Table 1.

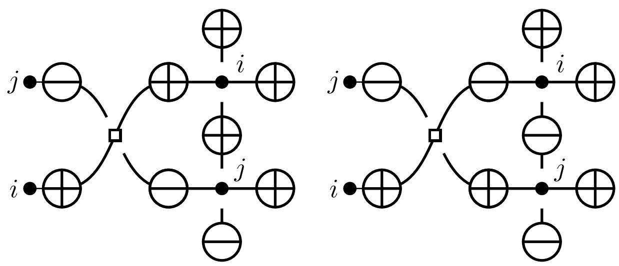
Theorem 1 *The star-triangle identity*



is true in the following sense. Fix edge spins $\alpha, \beta, \sigma, \tau, \rho, \theta$ and sum over edge spins γ, μ, ν on the left hand side of the following diagram. This equals the sum over edge spins δ, ϕ, ψ on the right hand side.

We may regard this as the equality of the partition functions of two ensembles, each consisting of three vertices. Two vertices (labeled $\bullet$) are Gamma ice, and one (labeled $\square$) is Gamma-Gamma R-ice, another type of vertex whose Boltzmann weights are listed in Table 1.

We will leave the proof to the reader. One may do this by checking all 32 cases, or by directly checking (7) below with a computer. But to clarify the statement we consider an example. Suppose that $\alpha = \tau = -$, while $\sigma = \beta = \theta = \rho = +$. Then we have the following two configuratons on the left-hand side of the equation.



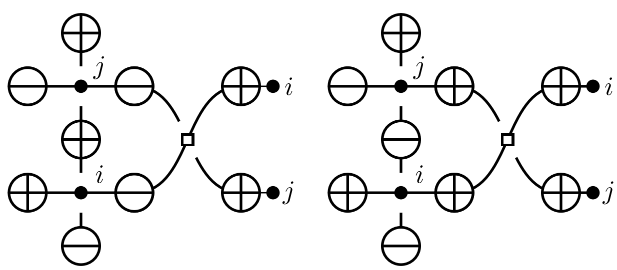
Consulting Table 1, the first configuration has Boltzmann weight

$$(t_i z_j - t_j z_i) \times 1 \times z_j(t_j + 1)$$

and the second has weight

$$(t_j + 1)z_j \times z_i(t_i + 1) \times t_j.$$

On the other side of the equation, there are two ways of assigning δ, ϕ, ψ such that the number of $-$ spins adjacent to each vertex is even. These are



But the first configuration is inadmissible since it has an inadmissible configuration at the i vertex. The second has weight

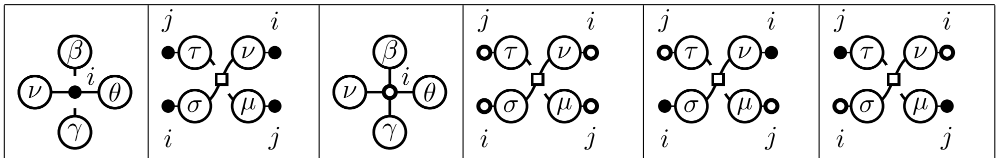
$$z_j(t_j + 1) \times t_i \times (t_j z_i + z_j)$$

Since

$$(t_i z_j - t_j z_i) z_j (t_j + 1) + (t_j + 1) z_j z_i (t_i + 1) t_j = z_j (t_j + 1) t_i (t_j z_i + z_j)$$

the star-triangle identity is satisfied.

If $1 \leqslant i, j \leqslant n$ and $i \neq j$, and if lower case Greek letters are assigned values $\pm$, make coefficients $R_{\Gamma\Gamma}(i,j)_{\sigma\tau}^{\nu\mu}$ and $\Gamma(i)_{\nu\beta}^{\theta\gamma}$ from the Boltzmann weights in Table 1 by means of the following labeling, so $R_{\Gamma\Gamma}(i,j)_{++}^{++} = t_j z_i + z_j$, etc.



$\Gamma(i)_{\nu\beta}^{\theta\gamma}$	$R_{\Gamma\Gamma}(i,j)_{\sigma\tau}^{\nu\mu}$	$\Delta(i)_{\nu\beta}^{\theta\gamma}$	$R_{\Delta\Delta}(i,j)_{\sigma\tau}^{\nu\mu}$	$R_{\Gamma\Delta}(i,j)_{\sigma\tau}^{\nu\mu}$	$R_{\Gamma\Delta}(i,j)_{\sigma\tau}^{\nu\mu}$

(6)

Only the first two types will be used in this section but we will return to the others in Section 7. Let V be a two-dimensional vector space with basis v_+ and v_-. Define endomorphisms of $V \otimes V$ by

$$R_{\Gamma\Gamma}(i,j) v_\sigma \otimes v_\tau = \sum_{\mu\nu} R_{\Gamma\Gamma}(i,j)_{\sigma\tau}^{\nu\mu} v_\nu \otimes v_\mu, \qquad \Gamma(i) v_\nu \otimes v_\beta = \sum_{\gamma,\theta} \Gamma(i)_{\nu\beta}^{\theta\gamma} v_\theta \otimes v_\gamma.$$

If $\phi \in \mathrm{End}(V \otimes V)$ then define endomorphisms of $V \otimes V \otimes V$ as follows. Let $\phi_{12} = \phi \otimes 1_V$, $\phi_{23} = 1_V \otimes \phi$ and let ϕ_{13} be ϕ acting on the first and third factors of $V \otimes V \otimes V$, and 1_V acting on the second. Then

$$\Gamma(j)_{23} \Gamma(i)_{13} R_{\Gamma\Gamma}(i,j)_{12} (v_\sigma \otimes v_\tau \otimes v_\beta) = \sum_{\gamma,\mu,\nu} \Gamma(j)_{\mu\gamma}^{\rho\alpha} \Gamma(i)_{\nu\beta}^{\theta\gamma} R_{\Gamma\Gamma}(i,j)_{\sigma\tau}^{\nu\mu} (v_\theta \otimes v_\rho \otimes v_\alpha),$$

$$R_{\Gamma\Gamma}(i,j)_{12} \Gamma(i)_{13} \Gamma(j)_{23} (v_\sigma \otimes v_\tau \otimes v_\beta) = \sum_{\delta,\phi,\psi} R_{\Gamma\Gamma}(i,j)_{\phi\psi}^{\theta\rho} \Gamma_{\sigma\delta}^{\phi\alpha}(i)_{13} \Gamma_{\tau\beta}^{\psi\delta}(j) (v_\theta \otimes v_\rho \otimes v_\alpha).$$

Thus Theorem 1 may be written in the form

$$R_{\Gamma\Gamma}(i,j)_{12} \Gamma(i)_{13} \Gamma(j)_{23} = \Gamma(j)_{23} \Gamma(i)_{13} R_{\Gamma\Gamma}(i,j)_{12}. \tag{7}$$

4 Evaluation of Gamma Ice

In this section we will prove

Theorem 2 *Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be a partition. Then*

$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i<j} (t_i z_j + z_i) s_\lambda(z_1, \cdots, z_n).$$

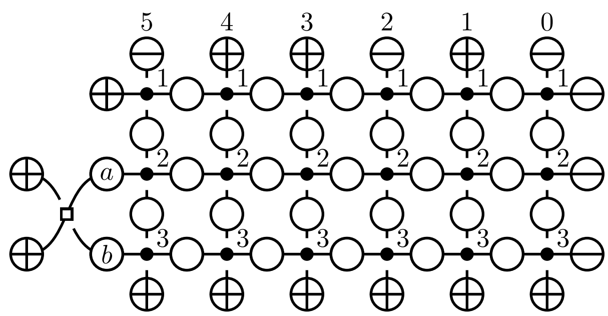
To begin with, define

$$s_\lambda^\Gamma(z_1, \cdots, z_n; t_1, \cdots, t_n) = \frac{Z(\mathfrak{S}_\lambda^\Gamma)}{\prod_{i<j}(t_i z_j + z_i)}. \tag{8}$$

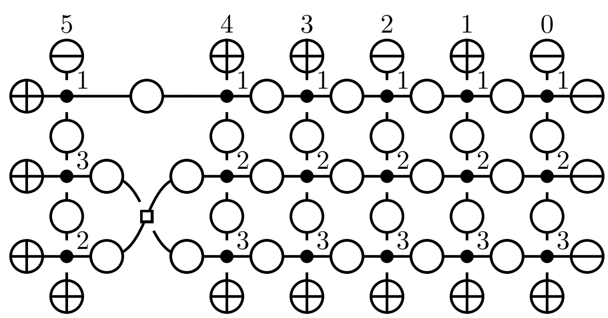
We will eventually show that s_λ^Γ is the Schur polynomial s_λ. But *a priori* it is not obvious from this definition that s_λ^Γ is symmetric, nor that it is a polynomial, nor that it is independent of t.

Lemma 3 *The expression $(t_{k+1} z_k + z_{k+1}) Z(\mathfrak{S}_\lambda^\Gamma)$ is invariant under the interchange of the spectral parameters: $(z_k, t_k) \longleftrightarrow (z_{k+1}, t_{k+1})$.*

Proof We modify the ice by adding a Gamma-Gamma R-vertex (that is, one of the vertices from the bottom row in Table 1) to the left of the k and $k+1$ rows. Thus (4) becomes (with $k = 2$ for illustrative purposes)

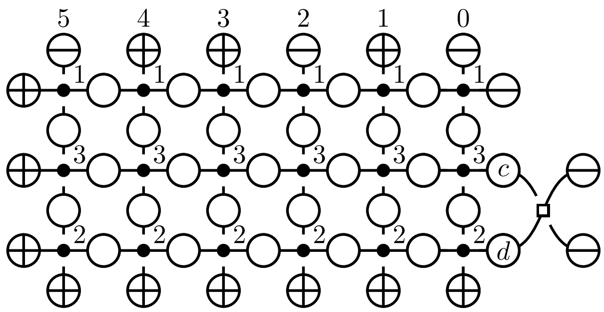


which is a new boundary value problem. The only legal values for a and b are $+$, so every state of this problem determines a unique state of the original problem, and the partition function for this state is the original partition function multiplied by the Boltzmann weight of the R-vertex, which is $t_{k+1} z_k + z_{k+1}$. Now we apply the star-triangle identity, and obtain equality with the the following configuration.



Thus if $\mathfrak{S}'$ denotes this ensemble the partition function $Z(\mathfrak{S}') = (t_{k+1}z_k + z_{k+1})Z(\mathfrak{S}^\Gamma_\lambda)$.

Repeatedly applying the star-triangle identity, we eventually obtain the configuration in which the R-vertex is moved entirely to the right.



Now there is only one legal configuration for the R-vertex, so $c = d = -$. The Boltzmann weight at the R-vertex is therefore $t_k z_{k+1} + z_k$. Note that (z_k, t_k) and (z_{k+1}, t_{k+1}) have been interchanged. This proves that $(t_{k+1}z_k + z_{k+1})Z(\mathfrak{S}^\Gamma_\lambda)$ is unchanged by switching (z_k, t_k) and (z_{k+1}, t_{k+1}). □

Proposition 1 *s^Γ_λ is a symmetric polynomial in $z_1, \cdots, z_n$, and is independent of the t_i.*

Proof Consider

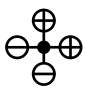
$$\prod_{i<j}(t_j z_i + z_j)Z(\mathfrak{S}^\Gamma_\lambda). \tag{9}$$

We will show that this is invariant under the interchange $k \leftrightarrow k+1$. This means that we interchange both z_k with z_{k+1} and t_k with t_{k+1}. Indeed, we may write (9) as $(t_{k+1}z_k + z_{k+1})Z(\mathfrak{S}^\Gamma_\lambda)$ times the product of all factors $t_j z_i + z_j$ with $i < j$ *except* $(i, j) = (k, k+1)$. These factors are permuted by $k \leftrightarrow k+1$, so the statement follows from Lemma 3. Thus (9) is invariant under permutations of the indices, where it is understood that the same permutation is applied to the t_i as to the z_i. Now (9) equals $\prod_{i\neq j}(t_j z_i + z_j)\, s^\Gamma_\lambda(z_1, \cdots, z_n)$, so it follows that s^Γ_λ is also invariant under such permutations. Moreover, (9) is divisible by each $t_j z_i + z_j$ with $i < j$ in the unique factorization ring $\mathbb{C}[z_1, \cdots, z_n, t_1, \cdots, t_n]$. The symmetry property implies that it is also divisible by $t_i z_j + z_i$ with $i < j$, and since these are coprime to $\prod_{i<j}(t_j z_i + z_j)$ it follows that $Z(\mathfrak{S}^\Gamma_\lambda)$ is divisible by these. Therefore s^Γ_λ is a polynomial in $\mathbb{C}[z_1, \cdots, z_n, t_1, \cdots, t_n]$.

It remains to be seen that it is independent of t. In

$$s^\Gamma_\lambda = \frac{Z(\mathfrak{S}^\Gamma_\lambda)}{\prod_{i<j}(t_i z_j + z_i)},$$

we regard the numerator and the denominator as both being elements of $R[t_i]$ where $R = \mathbb{C}[z_1, \cdots, z_n, t_j(j \neq i)]$. From what we have shown, s_λ^Γ is a polynomial. We claim that both the numerator and denominator have the same degree $i-1$ in t_i. For the denominator, this is clear. For the numerator, the number of $-$ in the top row of vertical edge spins is n by the boundary conditions, and it follows from Lemma 1 that each successive row has one fewer $-$. This means that there are $i-1$ vertices labeled i such that the spin on the edge below it is $-$, and from Table 1, it follows that the number of Boltzmann weights equal to $z_i(t_i+1)$ or t_i in any particular state is $\leqslant i-1$. The degree of the numerator is thus $\leqslant i-1$ and since the degree of the denominator is $i-1$, and the quotient is a polynomial, both numerator and denominator must have degree $i-1$ in t_i. Thus the quotient has degree zero, and does not involve t_i. $\square$

We may now conclude the proof of Theorem 2 by showing that $s_\lambda^\Gamma = s_\lambda$. Since s_λ^Γ is independent of t_i, we may take all $t_i = -1$. Now in (8) the denominator becomes $\prod_{i<j}(z_i - z_j)$. Since this is skew-symmetric under permutations, the numerator $Z(\mathfrak{S}_\lambda^\Gamma)$ is also skew-symmetric. With $t_i = -1$ any state containing a vertex in configuration  has Boltzmann weight 0, so we are limited to states omitting this configuration. In view of the bijection between states and strict Gelfand-Tsetlin patterns, this means that the corresponding Gelfand-Tsetlin pattern $\mathfrak{T}$ has the property that every entry from any row but the first is equal to one of the two entries directly above it. It is easy to see that the weight μ of such a coefficient, described by Lemma 2, is a permutation σ of the top row of $\mathfrak{T}$, that is, of $\lambda+\rho$. These weights are all distinct since $\lambda+\rho$ is strongly dominant, i.e. without repeated entries. Since it is skew-symmetric, its value is $\text{sgn}(\sigma)$ times a constant times $\prod z_j^{\mu_j} = z_j^{\rho_{\sigma(j)}+\lambda_{\sigma(j)}}$. To determine the constant, we may take the state whose Gelfand-Tsetlin pattern is

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} \lambda_1+\rho_1 & & \lambda_2+\rho_2 & & \cdots & & \lambda_n \\ & \lambda_2+\rho_2 & & & & \lambda_n & \\ & & \ddots & & \iddots & & \\ & & & \lambda_n & & & \end{array} \right\}.$$

This has weight $\prod z_j^{\lambda_j+\rho_j}$ and so

$$s_\lambda^\Gamma(z_1, \cdots, z_n) = \frac{\sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod z_j^{\rho_{\sigma(j)}+\lambda_{\sigma(j)}}}{\prod_{i<j}(z_j - z_i)}$$

which equals $s_\lambda(z_1, \cdots, z_n)$ by the Weyl character formula.

5 Tokuyama's theorem

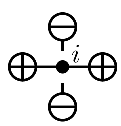
We recall some definitions from Tokuyama [33]. Each entry (not in the top row) is classified as *left-leaning* if it equals the entry above it and to the left. It is *right-leaning* if it equals the entry above it and to the right. It is *special* if it is neither left- nor right-leaning. Thus in (5), the 3 in the bottom row is left-leaning, the 0 in the second row is right-leaning and the 3 in the middle row is special. If $\mathfrak{T}$ is a Gelfand-Tsetlin pattern, let $l(\mathfrak{T})$ be the number of left-leaning entries. Let $d_k(\mathfrak{T})$ be the sum of the k-th row of $\mathfrak{T}$, and $d_{n+1}(\mathfrak{T}) = 0$.

Theorem 3 (Tokuyama) *We have*

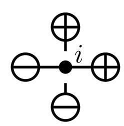
$$\sum_{\mathfrak{T}} \left(\prod_{k=1}^{n} z_k^{d_k(\mathfrak{T}) - d_{k+1}(\mathfrak{T})} \right) t^{l(\mathfrak{T})} (t+1)^{s(\mathfrak{T})} = \prod_{i<j} (z_i + t z_j) s_\lambda(z_1, \cdots, z_n),$$

where the sum is over all strict Gelfand-Tsetlin patterns with top row $\lambda + \rho$.

Proof If $\mathfrak{T}$ corresponds to a state of the Gamma ice with boundary conditions determined by λ, then we will show that the Boltzmann weight of the state is the term on the left-hand side. From Lemma 2 the powers of z are correct. It is easy to see that if the k-th row of $\mathfrak{T}$ is left leaning (respectively special), and that value is j, then the entry in the j-column and the k-th row of the ice is



respectively

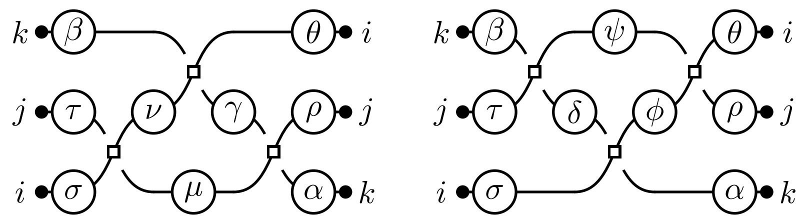


so from Table 1, it follows that the powers of t_i are also correct. The statement now follows from Theorem 2. □

6 The Yang-Baxter equation for Gamma-Gamma ice

If $\phi, \psi \in \mathrm{End}(V \otimes V)$ we have followed Baxter in using the term "star-triangle identity" to mean a relation of the form $R_{12} \circ \phi_{13} \circ \psi_{23} = \psi_{23} \circ \phi_{13} \circ R_{12}$ with R another endomorphism of $V \otimes V$. The term *Yang-Baxter equation* is often reserved for the special case where $\phi = \psi = R$, or where ϕ, ψ, R are all drawn from the same family of endomorphisms. The result of this section is of this sort.

We will prove a star-triangle relation that only involves Gamma-Gamma ice. Let us think of Gamma ice as being organized into strands of horizontal edges, with every Gamma vertex of the strand having the same label i. We may think of Gamma-Gamma ice as a tool that switches two strands. The following result states that this tool respects the braid relation. We have drawn this picture differently from that in Theorem 1 since this Yang-Baxter equation involves only horizontal edges, while that in Theorem 1 involves both horizontal and vertical edges.



With $\sigma, \tau, \beta, \alpha, \rho, \theta$ fixed, we may regard these two configurations as ensembles each involving three Gamma-Gamma vertices. The Yang-Baxter equation says that they have the same partition function.

Theorem 4 *The Yang-Baxter equation is true in the form*

$$\sum_{\mu,\nu,\gamma} R(j,k)^{\rho\alpha}_{\mu\gamma} R(i,k)^{\theta\gamma}_{\nu\beta} R(i,j)^{\nu\mu}_{\sigma\tau} = \sum_{\delta,\phi,\psi} R(j,k)^{\psi\delta}_{\tau\beta} R(i,k)^{\phi\alpha}_{\nu\delta} R(i,j)^{\theta\rho}_{\phi\psi},$$

with $R = R_{\Gamma\Gamma}$.

Proof This equation may be written

$$R(i,j)_{12} R(i,k)_{13} R(j,k)_{23} = R(j,k)_{23} R(i,k)_{13} R(i,j)_{12}. \tag{10}$$

With respect to the basis $v_+ \otimes v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+$,v $v_- \otimes v_-$, $R(i,j)$ is the matrix

$$R_{\Gamma\Gamma}(i,j) = R_{\Gamma\Gamma}(z_i, t_i, z_j, t_j) = \left(\begin{array}{cccc} t_j z_i + z_j & 0 & 0 & 0 \\ 0 & t_i z_j - t_j z_i & (t_j+1) z_j & 0 \\ 0 & (t_i+1) z_i & z_i - z_j & 0 \\ 0 & 0 & 0 & t_i z_j + z_i \end{array} \right). \tag{11}$$

In this form the identity may easily be checked using a computer, and we leave it to the reader. □

7 Delta Ice

As we mentioned in the introduction, there is a second, fundamentally distinct six-vertex model representing Schur polynomials, that we call *Delta ice*. The Boltzmann weights are given in Table 1.

This also leads to star-triangle relations. These require several types of new ice, which we will call Gamma-Delta ice, Delta-Gamma ice and Delta-Delta ice, listed in Table 1. In addition to (11) we have:

$$R_{\Delta\Delta}(z_i, t_i, z_j, t_j) = \begin{pmatrix} z_i t_i + z_j & & & \\ & z_i - z_j & z_i t_i + z_i & \\ & z_j t_j + z_j & z_j t_j - z_i t_i & \\ & & & z_j t_j + z_i \end{pmatrix},$$

$$R_{\Gamma\Delta}(z_i, t_i, z_j, t_j) = \begin{pmatrix} -z_i + t_i t_j z_j & & & z_j t_j + z_j \\ & z_j t_j + z_i & & \\ & & z_j t_i + z_i & \\ z_i t_i + z_i & & & z_i - z_j \end{pmatrix},$$

$$R_{\Delta\Delta}(z_i, t_i, z_j, t_j) = \begin{pmatrix} z_i - z_j & & & z_i t_i + z_i \\ & z_i t_i + z_j & & \\ & & z_i t_j + z_j & \\ z_j t_j + z_j & & & z_j - t_i t_j z_i \end{pmatrix}.$$

Also we recall from Section 3 that

$$\Gamma(z_i, t_i) = \begin{pmatrix} 1 & & & \\ & t_i & 1 & \\ & (t_i + 1) z_i & z_i & \\ & & & z_i \end{pmatrix}, \qquad \Delta(z_i, t_i) = \begin{pmatrix} z_i & & & z_i(t_i + 1) \\ & z_i t_i & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}.$$

Theorem 5 *If* $X, Y \in \{\Gamma, \Delta\}$ *then*

$$R_{XY}(z_i, t_i, z_j, t_j)\, X(z_i, t_i) Y(z_j, t_j) = Y(z_j, t_j) X(z_i, t_i) R_{XY}(z_i, t_i, z_j, t_j)\,. \tag{12}$$

This is a generalization of the star-triangle identity. For example if $X = Y = \Gamma$ this (7), which we have already noted is equivalent to Theorem 1. We leave the formulation of these statements in the form similar to Theorem 1 to the reader, but we mention the following interpretation: R_{XY} is a "braid" that can be used to interchange a strand of X ice with a strand of Y ice. We will give two applications below.

Proof This is easily checked with a computer. □

For every choice of z and t and $X \in \{\Gamma, \Delta\}$, let $V^X(z,t)$ be a two-dimensional vector space with basis $v_+^X(z,t)$ and $v_-^Y(z,t)$. Then $R^{XY}(z_1,t_1,z_2,t_2)$ defines an endomorphism of $V^X(z_1,t_1) \otimes V^Y(z_2,t_2)$ by

$$R(v_\sigma \otimes v_\tau) = \sum_{\mu\nu} R_{\sigma\tau}^{\nu\mu} v_\nu \otimes v_\mu, \qquad R = R^{XY}(z_1,t_1,z_2,t_2).$$

Now we turn to generalizations of the Yang-Baxter equation. If U, V, W are three vector spaces and $R \in \mathrm{End}(U \otimes V)$, $S \in \mathrm{End}(U \otimes W)$ and $W \in \mathrm{End}(V \otimes W)$ then R_{12}, S_{13}, T_{23}, defined as in Section 3, are endomorphisms of $U \otimes V \otimes W$. We define the Yang-Baxter commutator

$$[R, S, T] = R_{12} S_{13} T_{23} - T_{23} S_{13} R_{12}.$$

Theorem 6 *If* $X, Y, Z \in \{\Gamma, \Delta\}$ *then we have*

$$[R_{XY}(z_1,t_1,z_2,t_2), R_{XZ}(z_1,t_1,z_3,t_3), R_{YZ}(z_2,t_2,z_3,t_3)] = 0.$$

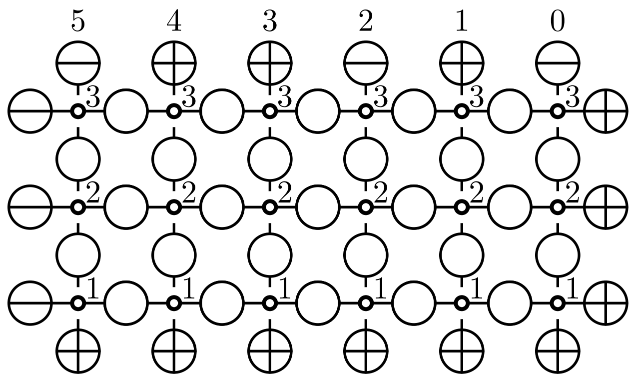
Moreover

$$[R_{XY}(z_2,t_1,z_1,t_2), R_{XZ}(z_3,t_1,z_1,t_3), R_{YZ}(z_3,t_2,z_2,t_3)] = 0.$$

Proof We leave this verification to the reader. □

We now describe the boundary conditions for Delta ice in the ensemble $\mathfrak{S}_\lambda^\Delta$ that appears in the second identity in (1). The columns are labeled, as with the Gamma ice, in decreasing order. However we label the vertices in decreasing row order, so the labels of the vertices of the top row are n, and so forth.

The Delta ice boundary conditions are as follows. We again fix a partition λ. On the left boundary edges, we put $-$; on the right and bottom edges we put $+$. On the top, we put $-$ at every column labeled $\lambda_i + n - i$ $(1 \leqslant i \leqslant n)$, that is, for the columns labeled with values in $\lambda + \rho$. Top edges not labeled by $\lambda_i + n - i$ for any i are given spin $+$. Thus if $\lambda = (3, 1, 0)$, here is the Delta ice. (To indicate that this is Delta ice, the vertices are marked $\circ$.)



Theorem 7 *The partition function is*

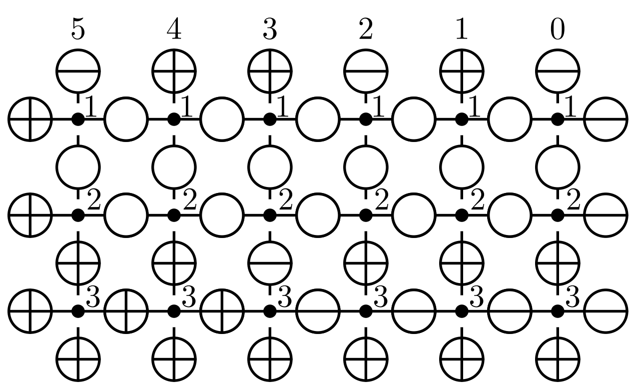
$$Z(\mathfrak{S}_\lambda^\Delta)(z_1, \cdots, z_n; t_1, \cdots, t_n) = \prod_{i<j} (t_j z_j + z_i) s_\lambda(z_1, \cdots, z_n).$$

Proof This is proved analogously to Theorem 2, using the case $X = Y = \Delta$ of Theorem 5. We leave the details of the proof to the reader. $\square$

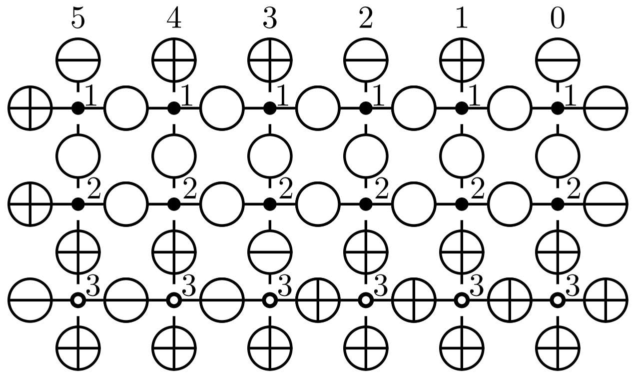
Theorem 5 may be used to show that

$$\prod_{i<j} (t_j z_j + z_i) Z(\mathfrak{S}_\lambda^\Gamma) = Z(\mathfrak{S}_\lambda^\Delta) \prod_{i<j} (t_i z_j + z_i) \tag{13}$$

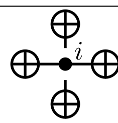
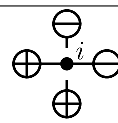
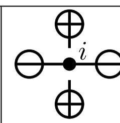
directly without invoking Theorems 2 and 7. Begin with an element x of $\mathfrak{S}_\lambda^\Gamma$, say (for example with $\lambda = (3, 1, 0)$):



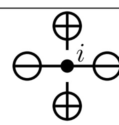
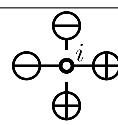
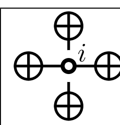
(The unlabeled edges can be filled in arbitrarily.) We wish to transform this into an element of an ensemble that has a row of Delta ice so that we may use the mixed star-triangle relation. We simply change the signs of all the entries on the edges in the 3 row:



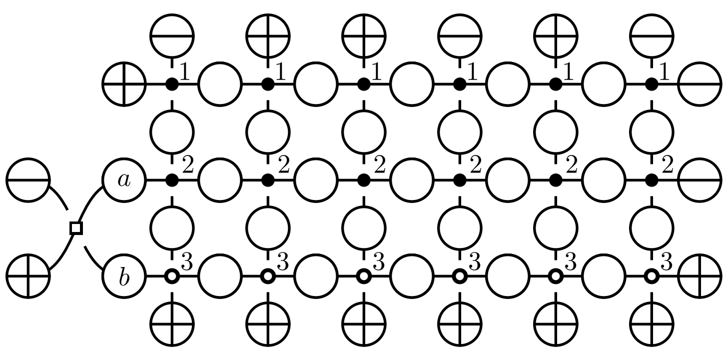
Let x' be this element of the mixed ensemble $\mathfrak{S}'$. We observe that the Boltzmann weights satisfy $w(x) = w(x')$. Indeed, in the bottom row only the following types of Gamma ice can appear:

Gamma Ice			
	1	1	z_i

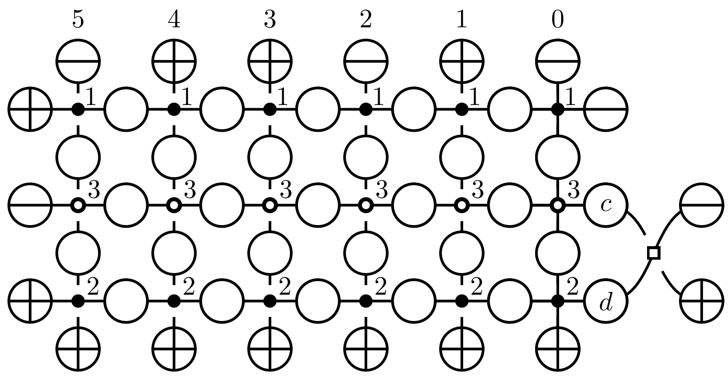
These change to:

Delta Ice			
	1	1	z_i

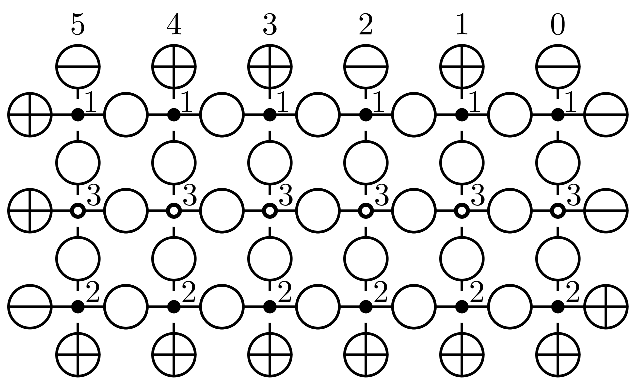
Observe that the weights are unchanged. *Note that this would not work in any row but the last* because it is essential that there be no $-$ on the bottom edge spins. Now we add a Gamma-Delta R-vertex.



If $\mathfrak{S}''$ is this ensemble, we claim that $Z(\mathfrak{S}'') = (t_3z_3 + z_2)Z(\mathfrak{S}') = (t_3z_3 + z_2)Z(\mathfrak{S}^\Gamma_\lambda)$. Indeed, from Table 1, the values of a and b must be $+, -$ respectively and so the value of the R-vertex is $t_3z_3 + z_2$ for every element of the ensemble. Now using the star-triangle, we $Z(\mathfrak{S}'') = Z(\mathfrak{S}''')$ where $\mathfrak{S}'''$ is the ensemble:



Here we must have $c, d = +, -$and so $(t_3z_3+z_2)Z(\mathfrak{S}^\Gamma_\lambda) = Z(\mathfrak{S}''') = (t_2z_3+z_2)Z(\mathfrak{S}^{(\mathrm{iv})})$ where $\mathfrak{S}^{(\mathrm{iv})}$ is the ensemble:



We repeat the process, first moving the Delta layer up to the top, then introducing another Delta layer at the bottom, etc., until we have the ensemble $\mathfrak{S}^\Delta_\lambda$, obtaining (13).

8 Yang-Baxter Systems

An important property of the R-matrices $R_{XY}(z_i, t_i, z_j, t_j)$ is that they are *projectively triangular*. That is,

$$R_{XY}(z_i, t_i, z_j, t_j)^{-1} = c_{XY}(z_i, t_i, z_j, t_j)P\, R_{YX}(z_j, t_j, z_i, t_i)\, P \tag{14}$$

where $c_{XY}(z_i, t_i, z_j, t_j)$ is a scalar and

$$P = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}.$$

The constant c_{XY} may be eliminated by multiplying R_{XY} by suitable scalar - for example in the case $X = Y = \Gamma$ if $R'_{\Gamma\Gamma}(z_i, t_i, z_j, t_j) = (z_j t_i + z_i)^{-1} R_{\Gamma\Gamma}(z_i, t_i, z_j, t_j)$ then R' satisfies (14) without the c_{XY}, at the cost of introducing denominators.

Yang-Baxter systems occur with varying degrees of generality in connection with different problems. One type occurs in the work of Vladimirov [35] on quantum doubles; another type occurs in Hlavaty [11] on quantized braided groups. The most general formulation [12], [10] involves four types of matrices which correspond to our R_{XY}, $X, Y \in \{\Gamma, \Delta\}$.

The axioms for a parametrized (or "colored") Yang-Baxter system in the most general definition require four types of matrices, A, B, C, D, depending on the parameter z and subject to the properties

$$\begin{aligned} [[A,A,A]] &= 0, & [[D,D,D]] &= 0, \\ [[A,C,C]] &= 0, & [[D,B,B]] &= 0, \\ [[A,B^\ddagger,B^\ddagger]] &= 0, & [[D,C^\ddagger,C^\ddagger]] &= 0, \\ [[A,C,B^\ddagger]] &= 0, & [[D,B,C^\ddagger]] &= 0, \end{aligned} \tag{15}$$

where the *Yang-Baxter commutator* is

$$[[X,Y,Z]](z_1,z_2,z_3) = X_{12}(z_1,z_2)Y_{13}(z_1,z_3)Z_{23}(z_2,z_3) - Z_{23}(z_2,z_3)Y_{13}(z_1,z_3)X_{12}(z_1,z_2)$$

and $X^\ddagger(z_1, z_2) = PX(z_2, z_1)P$. We have two spectral parameters z and t, so we interpret

$$X^\ddagger(z_1, t_1, z_2, t_2) = PX(z_2, t_2, z_1, t_1)P.$$

Theorem 8 *Let $X, Y \in \{\Gamma, \Delta\}$. Then*

$$A = R_{XX}, \qquad C = B^\ddagger = R_{XY}, \qquad D = R_{YY}{}^\ddagger$$

is a Yang-Baxter system satisfying (15).

Proof We leave the verification to the reader. □

Note that by projective triangularity we may replace B by $R_{YX}{}^{-1}$, which is a scalar multiple of $R_{XY}{}^\ddagger$. Thus if $X = \Gamma, Y = \Delta$ we have the Yang-Baxter system

$$A = R_{\Gamma\Gamma}, \qquad B = R_{\Delta\Gamma}{}^{-1}, \qquad C = R_{\Gamma\Delta}, \qquad D = R_{\Delta\Delta}{}^\ddagger,$$

which uses each of the four braided ice in Table 1 exactly once. It is probably most interesting to take $X \neq Y$, but worth noting that we can also make a Yang-Baxter system involving only $R_{\Gamma\Gamma}$ (or $R_{\Delta\Delta}$) playing all four roles. And we also obtain a Yang-Baxter system as follows by interchanging the z_i (but not the t_i) in the spectral parameters.

Theorem 9 *Another set of four Yang-Baxter systems may be obtained by taking*

$$A = \hat{R}_{XX}, \qquad C = B^{\ddagger} = \hat{R}_{XY}, \qquad D = \hat{R}^{\ddagger}_{YY},$$

where

$$\hat{R}_{XY}(z_1, t_1, z_2, t_2) = R_{XY}(z_2, t_1, z_1, t_2).$$

Proof We leave this to the reader. □

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