

CENTRAL DERIVATIVES OF L-FUNCTIONS IN HIDA FAMILIES

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ABSTRACT. We prove a result of the following type: given a Hida family of modular forms, if there exists a weight two form in the family whose L -function vanishes to exact order one at $s = 1$, then all but finitely many weight two forms in the family enjoy this same property. The analogous result for order of vanishing zero is also true, and is an easy consequence of the existence of the Mazur-Kitagawa two-variable p -adic L -function.

1. INTRODUCTION

Fix embeddings $\mathbb{Q}^{\text{alg}} \hookrightarrow \mathbb{Q}_p^{\text{alg}}$ and $\mathbb{Q}^{\text{alg}} \hookrightarrow \mathbb{C}$, and let $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mu_{p-1}$ denote the Teichmüller lift. Suppose that M is a positive integer, $p \nmid M$ is prime, and

$$(1) \quad g = \sum_{n>0} a(n)q^n \in S_k(\Gamma_0(Mp), \omega^j, \mathbb{Q}^{\text{alg}})$$

is a normalized cusp form of weight $k \geq 2$, level $\Gamma_0(Mp)$, and nebentype ω^j . We assume that g is an eigenform for all Hecke operators T_ℓ with $\ell \nmid Mp$ and U_ℓ with $\ell \mid Mp$, that g is M -new, and that g is p -ordinary in the sense that $|a(p)|_p = 1$. Fixing a finite extension F/\mathbb{Q}_p inside of $\mathbb{Q}_p^{\text{alg}}$ large enough to contain all Fourier coefficients of g , let $\rho_g : G_{\mathbb{Q}} \rightarrow \text{GL}_2(F)$ be the Galois representation attached to g by Deligne. We assume throughout that p does not divide $6M\phi(M)$ (where ϕ is Euler's function) and that the residual Galois representation of ρ_g is absolutely irreducible.

To the form g Hida attaches a q -expansion $\mathbf{g} \in R[[q]]$ where R is an integral domain, finite and flat over the Iwasawa algebra $\Lambda = \mathcal{O}_F[[1 + p\mathbb{Z}_p]]$ (\mathcal{O}_F is the maximal order in F). If \mathfrak{p} is an arithmetic prime of R as defined in §2 and $F_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}$, then the image of \mathbf{g} in $F_{\mathfrak{p}}[[q]]$, denoted $g_{\mathfrak{p}}$, is the q -expansion of a p -adic modular form of some weight, level, and nebentype all of which depend on \mathfrak{p} . For a suitable choice of \mathfrak{p} we recover $g = g_{\mathfrak{p}}$. Let us denote by \mathfrak{P}_2 the set of all weight two arithmetic primes of R ; that is, the (infinite) set of arithmetic primes \mathfrak{p} for which the p -adic modular form $g_{\mathfrak{p}}$ has weight two. The family $\{g_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}_2\}$ contains only finitely many forms of any given nebentype, but there is a uniform way of twisting the entire family into a family of p -adic modular forms of trivial nebentype. Indeed, a choice of critical character $\theta : \mathbb{Z}_p^\times \rightarrow R^\times$ in the sense of §2 provides us with, for each $\mathfrak{p} \in \mathfrak{P}_2$, a character $\vartheta_{\mathfrak{p}} : \mathbb{Z}_p^\times \rightarrow F_{\mathfrak{p}}^\times$ whose square is the nebentype of $g_{\mathfrak{p}}$. Thus the form $f_{\mathfrak{p}} = g_{\mathfrak{p}} \otimes \vartheta_{\mathfrak{p}}^{-1}$ has trivial nebentype, and one may ask how the order of vanishing (i.e. the *analytic rank*) $\text{ord}_{s=1} L(s, f_{\mathfrak{p}})$ varies as \mathfrak{p} varies (strictly speaking one must choose an embedding $F_{\mathfrak{p}} \hookrightarrow \mathbb{Q}_p^{\text{alg}}$ in order to view

This research was supported in part by NSF grant DMS-0556174.
2000 Mathematics Subject Classification: 11G18, 11G40, 11R23.

the q -expansion coefficients of $f_{\mathfrak{p}}$ as elements of $\mathbb{Q}^{\text{alg}} \subset \mathbb{C}$, but we ignore this for the moment). A conjecture of Greenberg [4] predicts that as \mathfrak{p} varies over \mathfrak{P}_2 all but finitely many of the modular forms $f_{\mathfrak{p}}$ should have analytic rank zero or one, depending on the root number of the L -function of $f_{\mathfrak{p}}$. This root number is constant as \mathfrak{p} varies, except for a finite and explicit set of bad \mathfrak{p} 's which were determined by Mazur-Tate-Teitelbaum [10]. When the root number is $+1$ it follows easily from the existence of the Mazur-Kitagawa two-variable p -adic L -function that if one can find a single $\mathfrak{p} \in \mathfrak{P}_2$ for which $f_{\mathfrak{p}}$ has analytic rank zero, then $f_{\mathfrak{p}}$ has analytic rank zero for all but finitely many $\mathfrak{p} \in \mathfrak{P}_2$; see Theorem 7, which also includes information about $f_{\mathfrak{p}}$ of higher even weight. In the present article we prove a similar result in the case where the root number is -1 . If $\text{Gen}_2(\theta) \subset \mathfrak{P}_2$ denotes the set of arithmetic primes of weight two which are generic for θ in the sense of Definition 2, then we prove that if there is a single $\mathfrak{p} \in \text{Gen}_2(\theta)$ for which $f_{\mathfrak{p}}$ has analytic rank one then the analytic rank is one for all but finitely many $\mathfrak{p} \in \text{Gen}_2(\theta)$. The set $\text{Gen}_2(\theta)$ is very explicit and contains all but finitely many $\mathfrak{p} \in \mathfrak{P}_2$. The result is stated in the text as Theorem 8. This provides a method for verifying Greenberg's conjecture in any given case, as one need only exhibit a single form in the Hida family having the correct analytic rank.

The method of proof is based on the author's earlier papers [6] and [7]. The first of these papers contains the construction of a *big Heegner point* $\mathfrak{Z} \in H^1(K, \mathbf{T}^\dagger)$ where K is a quadratic imaginary field and \mathbf{T}^\dagger is a twist of the R -adic Galois representation attached to \mathfrak{g} . The second paper contains an extension of the Gross-Zagier theorem to modular forms of nontrivial nebentype which is general enough to include the forms $\{g_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Gen}_2(\theta)\}$. For any $\mathfrak{p} \in \text{Gen}_2(\theta)$ we make explicit the connection between the images $\mathfrak{Z}_{\mathfrak{p}} \in H^1(K, \mathbf{T}^\dagger \otimes_R F_{\mathfrak{p}})$ of \mathfrak{Z} for various \mathfrak{p} and the generalized Heegner points constructed in [7]. Assuming that there is a $\mathfrak{p} \in \text{Gen}_2(\theta)$ for which $f_{\mathfrak{p}}$ has analytic rank one, we use the main result of [7] to prove the nontriviality of $\mathfrak{Z}_{\mathfrak{p}}$ and then deduce that \mathfrak{Z} is not R -torsion (as predicted by [6, Conjecture 3.4.1]). From this it follows that $\mathfrak{Z}_{\mathfrak{p}}$ is non-trivial for all but finitely many $\mathfrak{p} \in \text{Gen}_2(\theta)$, and again applying the main result of [7] we deduce that the generic analytic rank of the $f_{\mathfrak{p}}$'s is one.

The sole reason for restricting attention to those forms $f_{\mathfrak{p}}$ having weight two is that the extension of the Gross-Zagier theorem proved in [7] applies only to modular forms of weight two. While higher even weight analogues of the Gross-Zagier theorem are known [12, 17], they are not strong enough to deduce higher weight analogues of Proposition 3 below.

If E is a number field we denote by \mathbb{A}_E its ring of adeles and by

$$\text{art}_E : \mathbb{A}_E^\times \rightarrow \text{Gal}(E^{\text{ab}}/E)$$

the Artin reciprocity map $\text{art}_E(x) = [x, E]$ normalized as in [16, §5.2].

2. THE HIDA FAMILY

In this section we quickly recall the basic facts of Hida theory which we will need, following the notation of [6, §2.1] (in which the reader will find references to facts stated here without proof). Set $\Gamma = 1 + p\mathbb{Z}_p$, define

$$\Lambda = \mathcal{O}_F[[\Gamma]] \quad \underline{\Lambda} = \mathcal{O}_F[[\mathbb{Z}_p^\times]],$$

and denote by $z \mapsto [z]$ the canonical inclusion of group-like elements $\mathbb{Z}_p^\times \hookrightarrow \underline{\Lambda}^\times$. For every integer $t > 0$ set

$$\Phi_t = \Gamma_0(M) \cap \Gamma_1(p^t) \subset \mathrm{SL}_2(\mathbb{Z})$$

and let $\mathfrak{h}_{r,t}$ denote the \mathcal{O}_F -algebra generated by the Hecke operators

$$\{T_\ell : (\ell, Mp) = 1\} \cup \{U_\ell : \ell \mid Mp\} \cup \{(d) : (d, p) = 1\}$$

acting on the space of p -adic cusp forms $S_r(\Phi_t, \mathbb{Q}_p^{\mathrm{alg}})$ of level Φ_t and weight $r \geq 2$. We view each $\mathfrak{h}_{r,t}$ as a $\underline{\Lambda}$ -algebra via $[z] \mapsto z^{r-2}\langle z \rangle$. Set $\mathfrak{h}_{r,t}^{\mathrm{ord}} = e^{\mathrm{ord}}\mathfrak{h}_{r,t}$ where $e^{\mathrm{ord}} \in \mathfrak{h}_{r,t}$ is Hida's ordinary projector and define

$$\mathfrak{h}^{\mathrm{ord}} = \varprojlim_t \mathfrak{h}_{r,t}^{\mathrm{ord}}.$$

By Hida's theory the $\underline{\Lambda}$ -algebra $\mathfrak{h}^{\mathrm{ord}}$ is independent of r and is finite and flat over $\underline{\Lambda}$.

The cuspform (1) determines a homomorphism (again denoted g)

$$\mathfrak{h}^{\mathrm{ord}} \rightarrow \mathfrak{h}_{k,1}^{\mathrm{ord}} \rightarrow \mathcal{O}_F.$$

The ring $\mathfrak{h}^{\mathrm{ord}}$ is semi-local and there is a unique maximal ideal $\mathfrak{m} \subset \mathfrak{h}^{\mathrm{ord}}$ such that g factors through the local summand $\mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}$. The local ring $\mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}$ then has a unique minimal prime ideal \mathfrak{a} such that g factors through the integral domain $R = \mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a}$. We say that $\mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}$ is the *Hida family* of g , while R is the *branch* of $\mathfrak{h}_{\mathfrak{m}}^{\mathrm{ord}}$ on which g lives. A prime ideal $\mathfrak{p} \subset R$ is *arithmetic* if there is a homomorphism of \mathcal{O}_F -algebras $R \rightarrow \mathbb{Q}_p^{\mathrm{alg}}$ with kernel \mathfrak{p} such that the composition $\Gamma \rightarrow R^\times \rightarrow \mathbb{Q}_p^{\mathrm{alg},\times}$ has the form $\gamma \mapsto \psi(\gamma)\gamma^{r-2}$ with ψ of finite order and $r \geq 2$ an integer (called the *weight* of \mathfrak{p}). For such a \mathfrak{p} we define $F_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}$, a finite extension of F . The homomorphism $\Gamma \rightarrow R^\times \rightarrow F_{\mathfrak{p}}^\times$ then has the form $\gamma \mapsto \psi_{\mathfrak{p}}(\gamma)\gamma^{r-2}$ where $\psi_{\mathfrak{p}}$ is the *wild character* of \mathfrak{p} . By Hida's theory there is a formal q -expansion

$$\mathbf{g} = \sum_{n>0} \mathbf{a}(n)q^n \in R[[q]]$$

with the property that for any arithmetic prime \mathfrak{p} of weight r the image of \mathbf{g} in $F_{\mathfrak{p}}[[q]]$ is an M -new ordinary p -adic eigenform

$$g_{\mathfrak{p}} = \sum_{n>0} a_{\mathfrak{p}}(n)q^n \in S_r(\Gamma_0(Mp^t), \omega^{k+j-r}\psi_{\mathfrak{p}}, F_{\mathfrak{p}}).$$

Here $t = t(\mathfrak{p})$ is the smallest positive integer for which $1 + p^t\mathbb{Z}_p \subset \ker(\psi_{\mathfrak{p}})$ and we view both ω and $\psi_{\mathfrak{p}}$ as characters of

$$(\mathbb{Z}/p^t\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p)/(1 + p^t\mathbb{Z}_p).$$

A *critical character* is a homomorphism

$$\theta : \mathbb{Z}_p^\times \rightarrow R^\times$$

which satisfies $\theta^2(z) = [z]$ for all $z \in \mathbb{Z}_p^\times$. There are exactly two critical characters, and they differ by multiplication by $\omega^{(p-1)/2}$; see Proposition 2.1.3 and Remark 2.1.4 of [6]. We now fix, for the remainder of this article, a critical character θ and define

$$\Theta : G_{\mathbb{Q}} \rightarrow R^\times$$

by $\Theta(\sigma) = \theta(\epsilon_{\text{cyc}}(\sigma))$, where $\epsilon_{\text{cyc}} : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_p^{\times}$ is the cyclotomic character. The homomorphism $[\cdot] : \mathbb{Z}_p^{\times} \rightarrow R^{\times}$ has the property that for every arithmetic prime \mathfrak{p} of weight r the composition of $[\cdot]$ with $R^{\times} \rightarrow F_{\mathfrak{p}}^{\times}$ is given by

$$[z]_{\mathfrak{p}} = \omega^{k+j-r}(z)\psi_{\mathfrak{p}}(z)z^{r-2}$$

for every $z \in \mathbb{Z}_p^{\times}$. We denote by $\theta_{\mathfrak{p}}$ and $\Theta_{\mathfrak{p}}$ the compositions

$$\theta_{\mathfrak{p}} : \mathbb{Z}_p^{\times} \xrightarrow{\theta} R^{\times} \rightarrow F_{\mathfrak{p}}^{\times} \quad \Theta_{\mathfrak{p}} : G_{\mathbb{Q}} \xrightarrow{\Theta} R^{\times} \rightarrow F_{\mathfrak{p}}^{\times}.$$

and note that when \mathfrak{p} has weight two the nebentype of $g_{\mathfrak{p}}$ is $\theta_{\mathfrak{p}}^2 = [\cdot]_{\mathfrak{p}}$,

For each $t > 0$ let $X_t = X(\Phi_t)$ be the modular curve over $\text{Spec}(\mathbb{Q})$ classifying elliptic curves with Φ_t level structure, set $J_t = \text{Jac}(X_t)$, and define an $\mathfrak{h}_{2,t}^{\text{ord}}[[G_{\mathbb{Q}}]]$ -module

$$\mathbf{Ta}_p^{\text{ord}}(J_t) = e^{\text{ord}}(\mathbf{Ta}_p(J_t) \otimes_{\mathbb{Z}_p} \mathcal{O}_F).$$

Passing to the limit over t we define an $\mathfrak{h}^{\text{ord}}$ -module $\mathbf{Ta}^{\text{ord}} = \varprojlim \mathbf{Ta}_p^{\text{ord}}(J_t)$. According to [11, Théorème 7] our hypothesis on the irreducibility of the residual Galois representation attached to g implies that

$$\mathbf{Ta}_m^{\text{ord}} = \mathbf{Ta}^{\text{ord}} \otimes_{\mathfrak{h}^{\text{ord}}} \mathfrak{h}_m^{\text{ord}} \quad \mathbf{T} = \mathbf{T}^{\text{ord}} \otimes_{\mathfrak{h}^{\text{ord}}} R$$

are free of rank two over $\mathfrak{h}_m^{\text{ord}}$ and R , respectively. The R -module \mathbf{T} carries a natural $G_{\mathbb{Q}}$ -action, and the twist $\mathbf{T}^{\dagger} = \mathbf{T} \otimes \Theta^{-1}$ admits a perfect alternating pairing $\mathbf{T}^{\dagger} \times \mathbf{T}^{\dagger} \rightarrow R(1)$ where $R(1)$ is the Tate twist of R . For E any number field denote by

$$\text{Sel}_{\text{Gr}}(E, \mathbf{T}^{\dagger}) \subset H^1(E, \mathbf{T}^{\dagger})$$

the *strict Greenberg Selmer group* defined in [6, Definition 2.4.2]. For any arithmetic prime \mathfrak{p} define $V_{\mathfrak{p}} = \mathbf{T} \otimes_R F_{\mathfrak{p}}$ and $V_{\mathfrak{p}}^{\dagger} = \mathbf{T}^{\dagger} \otimes_R F_{\mathfrak{p}}$.

Definition 1. Suppose $f \in S_2(\Gamma_0(M), \mathbb{Q}_p^{\text{alg}})$ is a p -ordinary newform with q -expansion $f = \sum_{n>0} b(n)q^n$ and let α denote the unit root of the polynomial $X^2 - b(p)X + p$. The p -stabilization of f is the p -adic modular form of level Mp with q -expansion

$$\sum_{n>0} b(n)q^n + \frac{p}{\alpha} \sum_{n>0} b(n)q^{pn}.$$

We note that the p -stabilization of f is an eigenform for U_p with eigenvalue α .

Definition 2. An arithmetic prime $\mathfrak{p} \subset R$ of weight two is said to be generic for θ if one of the following (mutually exclusive) hypotheses is satisfied:

- (a) $g_{\mathfrak{p}}$ is the p -stabilization of a newform $f \in S_2(\Gamma_0(M), F_{\mathfrak{p}})$ and $\theta_{\mathfrak{p}}$ is trivial;
- (b) $g_{\mathfrak{p}}$ is new of level Mp and $\theta_{\mathfrak{p}} = \omega^{\frac{p-1}{2}}$;
- (c) $g_{\mathfrak{p}}$ has nontrivial nebentype.

Let $\text{Gen}_2(\theta)$ denote the set of all arithmetic weight two primes of R which are generic for θ . Clearly all but finitely many weight two arithmetic primes are generic for θ as all but finitely many satisfy condition (c).

3. RESULTS OVER QUADRATIC IMAGINARY FIELDS

Let K be a quadratic imaginary field of discriminant $-D$ prime to Mp and let $\epsilon_K = \left(\frac{-D}{\cdot}\right)$ denote the quadratic Dirichlet character attached to K . Assume that every prime divisor of M splits in K , so that $\epsilon_K(M) = 1$, and fix an ideal $\mathfrak{M} \subset \mathcal{O}_K$ of the maximal order of K such that $\mathcal{O}_K/\mathfrak{M} \cong \mathbb{Z}/M\mathbb{Z}$. Let H denote the Hilbert class field of K . Taking $c = 1$ in the construction of [6, §2.2] gives a canonical cohomology class, the *big Heegner point* of conductor 1

$$\mathfrak{X} = \mathfrak{X}_1 \in H^1(H, \mathbf{T}^\dagger).$$

We define

$$\mathfrak{Z} = \text{Cores}_{H/K}(\mathfrak{X}) \in H^1(K, \mathbf{T}^\dagger)$$

and for any arithmetic prime $\mathfrak{p} \subset R$ let $\mathfrak{Z}_{\mathfrak{p}}$ denote the image of \mathfrak{Z} in $H^1(K, V_{\mathfrak{p}}^\dagger)$.

For $\mathfrak{p} \subset R$ arithmetic of weight two, define an $F_{\mathfrak{p}}^\times$ -valued character of \mathbb{A}_K^\times

$$\chi_{\mathfrak{p}}(x) = \Theta_{\mathfrak{p}}(\text{art}_{\mathbb{Q}}(N_{K/\mathbb{Q}}(x)))$$

and denote by $\chi_{0,\mathfrak{p}}$ the restriction of $\chi_{\mathfrak{p}}$ to $\mathbb{A}_{\mathbb{Q}}^\times$. As before let $t = t(\mathfrak{p})$ be the smallest positive integer for which $\psi_{\mathfrak{p}}$ is trivial on $1 + p^t\mathbb{Z}_p$. The conductor of $\chi_{0,\mathfrak{p}}$ divides p^t (more precisely, $\chi_{0,\mathfrak{p}}$ is either trivial or of conductor p^t), and we may view $\chi_{0,\mathfrak{p}}$ as a character of $(\mathbb{Z}/p^t\mathbb{Z})^\times$ in the usual way. The nebentype of $g_{\mathfrak{p}}$ is then equal to $\chi_{0,\mathfrak{p}}^{-1}$. Indeed, it follows from §2 that $g_{\mathfrak{p}}$ has nebentype $[\cdot]_{\mathfrak{p}} = \theta_{\mathfrak{p}}^2$, and the equality $\theta_{\mathfrak{p}}^2 = \chi_{0,\mathfrak{p}}^{-1}$ follows from the relations

$$\chi_{0,\mathfrak{p}}(x) = \chi_{\mathfrak{p}}(x_p) = \Theta_{\mathfrak{p}}^2(\text{art}_{\mathbb{Q}}(x_p)) = \theta_{\mathfrak{p}}^{-2}(x)$$

where $x \in \mathbb{Z}_p^\times$, $x_p \in \mathbb{A}_{\mathbb{Q}}^\times$ is the idele with x in the p -component and 1 in all other components, and we have used the fact that the composition

$$\mathbb{Z}_p^\times \xrightarrow{x \mapsto x_p} \mathbb{A}_{\mathbb{Q}}^\times \xrightarrow{\text{art}_{\mathbb{Q}}} \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\epsilon_{\text{cyc}}} \mathbb{Z}_p^\times$$

is given by $x \mapsto x^{-1}$.

Although we will not recall here the construction of the class \mathfrak{X} , we will need an explicit description of $\mathfrak{Z}_{\mathfrak{p}}$ when \mathfrak{p} has weight two. Fix such a \mathfrak{p} and let t be as above. Let J_t denote the Jacobian of the modular curve $X_t = X(\Phi_t)$. Let H_{p^t} denote the ring class field of the order $\mathcal{O}_{p^t} \subset \mathcal{O}_K$ of conductor p^t and set $L_t = H_{p^t}(\mu_{p^t})$. As in [6, §2.2] (but suppressing $c = 1$ from the notation) denote by $h_t \in X_t(L_t)$ the point corresponding to the triple $(E_t, \mathfrak{m}_t, \pi_t)$ where $E_t(\mathbb{C}) = \mathbb{C}/\mathcal{O}_{p^t}$, $\mathfrak{m}_t \subset E_t(\mathbb{C})$ is the kernel of the cyclic M -isogeny

$$\mathbb{C}/\mathcal{O}_{p^t} \rightarrow \mathbb{C}/(\mathcal{O}_{p^t} \cap \mathfrak{M})^{-1},$$

and $\pi_t \in E_t(\mathbb{C})$ is any generator of the kernel of the cyclic p^t -isogeny

$$\mathbb{C}/\mathcal{O}_{p^t} \rightarrow \mathbb{C}/\mathcal{O}_K.$$

Fix a rational cusp $\mathfrak{c} \in X_t(\mathbb{Q})$ and define

$$Q = \sum_{\sigma \in \text{Gal}(L_t/K)} \chi_{\mathfrak{p}}^{-1}(\sigma)(h_t - \mathfrak{c})^\sigma \in J_t(L_t) \otimes_{\mathbb{Z}} F_{\mathfrak{p}}.$$

By the Manin-Drinfeld theorem [2, §2.1.5] the point Q does not depend on the choice of \mathfrak{c} . The natural map $\mathbf{T} \rightarrow V_{\mathfrak{p}}$ can be factored as

$$\mathbf{T} \rightarrow \text{Ta}_p^{\text{ord}}(J_t) \rightarrow V_{\mathfrak{p}}$$

yielding a map

$$J_t(L_t) \rightarrow H^1(L_t, \mathrm{Ta}_p(J_t)) \xrightarrow{e^{\mathrm{ord}}} H^1(L_t, \mathrm{Ta}_p^{\mathrm{ord}}(J_t)) \rightarrow H^1(L_t, V_p).$$

If we define Q_p to be the image of Q under the induced map

$$(2) \quad J_t(L_t) \otimes_{\mathbb{Z}} F_p \rightarrow H^1(L_t, V_p)$$

then, tracing through the construction of \mathfrak{X} found in [6, §2.2], Q_p is equal (up to multiplication by an element of F_p^\times) to the image of \mathfrak{Z}_p under

$$H^1(K, V_p^\dagger) \xrightarrow{\mathrm{res}} H^1(L_t, V_p^\dagger)$$

after identifying the G_{L_t} -representations $V_p^\dagger \cong V_p \otimes \Theta_p^{-1} \cong V_p$. As the kernel of the above restriction map is both an F_p -vector space and is killed by $[L_t : K]$, it must be trivial, and we find that

$$(3) \quad Q_p = 0 \iff \mathfrak{Z}_p = 0.$$

Let $\mathrm{Emb}(\mathfrak{p})$ denote the set of all F -algebra embeddings $F_p \hookrightarrow \mathbb{Q}_p^{\mathrm{alg}}$. If for each $\iota \in \mathrm{Emb}(\mathfrak{p})$ we set $a_p^\iota(n) = \iota(a_p(n))$ then $\{a_p^\iota(n)\}$ generates a finite extension of \mathbb{Q} inside $\mathbb{Q}_p^{\mathrm{alg}}$, and so we may view

$$g_p^\iota = \sum_{n>0} a_p^\iota(n) q^n$$

as a classical modular form with Fourier coefficients in $\mathbb{Q}^{\mathrm{alg}}$. Denote by χ_p^ι the unique $\mathbb{Q}^{\mathrm{alg}, \times}$ -valued character of \mathbb{A}_K^\times satisfying $\iota(\chi_p(x)) = \chi_p^\iota(x)$ for all $x \in \mathbb{A}_K^\times$, and define $Q^\iota \in J_t(L_t) \otimes_{\mathbb{Z}} \mathbb{Q}^{\mathrm{alg}}$ in the same way as above, replacing χ_p by χ_p^ι . Using the chosen embedding $\mathbb{Q}^{\mathrm{alg}} \hookrightarrow \mathbb{C}$, we may form the Rankin-Selberg convolution L -function $L(s, \chi_p^\iota, g_p^\iota)$ as in [7, §1]. One could also accept (11) below as the definition of $L(s, \chi_p^\iota, g_p^\iota)$.

Proposition 3. *Suppose $\mathfrak{p} \in \mathrm{Gen}_2(\theta)$ satisfies either (b) or (c) in Definition 2. For any $\iota \in \mathrm{Emb}(\mathfrak{p})$ the L -function $L(s, \chi_p^\iota, g_p^\iota)$ vanishes to odd order at $s = 1$ and*

$$\mathfrak{Z}_p = 0 \iff L'(1, \chi_p^\iota, g_p^\iota) = 0.$$

Proof. Let \mathbb{T} denote the \mathbb{Z} -algebra generated by the Hecke operators T_n with $(n, Mp) = 1$ and the diamond operators $\langle d \rangle$ with $(d, Mp) = 1$ acting on the space of cusp forms $S_2(\Phi_t, \mathbb{C})$. Let $e_p \in \mathbb{T} \otimes_{\mathbb{Z}} F_p$ be the idempotent which projects onto the maximal summand on which $T_n = a_p(n)$ for all $(n, Mp) = 1$ and $\langle d \rangle^{-1} = \chi_{0, \mathfrak{p}}(d)$ for all $(d, Mp) = 1$. Similarly, define $e_p^\iota \in \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}^{\mathrm{alg}}$ to be the projector to the maximal summand on which $T_n = a_p^\iota(n)$ and $\langle d \rangle^{-1} = \chi_{0, \mathfrak{p}}^\iota(d)$ for all n and d as above. The algebra \mathbb{T} acts on $J_t(L_t)$ and we have

$$(4) \quad e_p Q = 0 \iff e_p^\iota Q^\iota = 0.$$

Our hypotheses imply that g_p is new of level Mp^t (where $t = t(\mathfrak{p})$ as above), so by Eichler-Shimura theory the summand

$$e_p(\mathrm{Ta}_p^{\mathrm{ord}}(J_t) \otimes_{\mathcal{O}_F} F_p) = e_p(\mathrm{Ta}_p(J_t) \otimes_{\mathbb{Z}_p} F_p)$$

is taken isomorphically to V_p under $\mathrm{Ta}_p^{\mathrm{ord}}(J_t) \otimes_{\mathcal{O}_F} F_p \rightarrow V_p$. It follows that the map

$$e_p(J_t(L_t) \otimes_{\mathbb{Z}} F_p) \rightarrow H^1(L_t, V_p)$$

induced by (2) is injective. As this map takes $e_{\mathfrak{p}}Q$ to $Q_{\mathfrak{p}}$, (3) and (4) imply that

$$e_{\mathfrak{p}}^t Q^t = 0 \iff \mathfrak{Z}_{\mathfrak{p}} = 0.$$

We may now apply the results of [7, §1] with $f = g_{\mathfrak{p}}^t$, $\chi = \chi_{\mathfrak{p}}^t$, $N = Mp^t$, and $C = S = p^t$. As we have assumed that $\epsilon_K(M) = 1$ the functional equation [7, (1)] forces $L(s, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t)$ to vanish to odd order at $s = 1$, and [7, Theorem A] asserts that $e_{\mathfrak{p}}^t Q_{\mathfrak{p}} = 0$ if and only if $L'(1, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t) = 0$. \square

Proposition 4. *Suppose $\mathfrak{p} \in \text{Gen}_2(\theta)$ satisfies condition (a) in Definition 2. For any $\iota \in \text{Emb}(\mathfrak{p})$ the L -function $L(s, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t)$ vanishes to odd order at $s = 1$ and*

$$\mathfrak{Z}_{\mathfrak{p}} = 0 \implies L'(1, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t) = 0.$$

Proof. Our hypotheses imply that $t = 1$, $\chi_{\mathfrak{p}}$ is trivial, and there is a normalized newform

$$f = \sum_{n>0} b(n)q^n \in S_2(\Gamma_0(M), F_{\mathfrak{p}})$$

such that $g_{\mathfrak{p}}$ is the p -stabilization of f . Define $f^t \in S_2(\Gamma_0(M), \mathbb{Q}^{\text{alg}})$ to be the modular form with Fourier coefficients $b^t(n) = \iota(b(n))$. The L -functions of $g_{\mathfrak{p}}^t$ and f^t are related by

$$\left(1 - \frac{p^{1-s}}{a_{\mathfrak{p}}^t(p)}\right) L(s, f^t) = L(s, g_{\mathfrak{p}}^t)$$

while the Rankin-Selberg L -functions are related by

$$\left(1 - \frac{p^{1-s}}{a_{\mathfrak{p}}^t(p)}\right) \left(1 - \epsilon_K(p) \frac{p^{1-s}}{a_{\mathfrak{p}}^t(p)}\right) L(s, \chi_{\mathfrak{p}}^t, f^t) = L(s, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t).$$

As $a_{\mathfrak{p}}^t(p)$ is a root of $X^2 - b^t(p)X + p$, and hence has complex absolute value $p^{1/2}$, we have $a_{\mathfrak{p}}^t(p) \neq \pm 1$. Therefore the Euler factors do not vanish at $s = 1$ and the two Rankin-Selberg L -functions have the same order of vanishing, which must be odd as the sign of the functional equation of the L -function on the left is $-\epsilon_K(M) = -1$ by [5, IV (0.2)] or [7, (1)]. The central derivative of the L -function on the left is determined by the original Gross-Zagier theorem, and we will relate our Heegner class $\mathfrak{Z}_{\mathfrak{p}}$ to the classical Heegner point considered by Gross-Zagier.

Consider the degeneracy maps

$$\alpha, \beta : X_0(Mp) \rightarrow X_0(M)$$

defined on moduli by

$$\alpha(E, C \times D) = (E, C) \quad \beta(E, C \times D) = (E/D, (C \times D)/D)$$

where E is an elliptic curve (over some \mathbb{Q} -scheme) and C and D are cyclic subgroup schemes of E of orders M and p , respectively. The Hecke algebra \mathbb{T} defined in the proof of Proposition 3 acts on J_t , $J_0(Mp)$, and $J_0(M)$, and α and β induce \mathbb{T} -equivariant maps

$$\alpha^*, \beta^* : \text{Ta}_{\mathfrak{p}}(J_0(M)) \rightarrow \text{Ta}_{\mathfrak{p}}(J_0(Mp))$$

which satisfy the relations

$$(5) \quad U_p \circ \alpha^* = p\beta^* \quad U_p \circ \beta^* = \beta^* \circ T_p - \alpha^*.$$

Gently abusing notation, we abbreviate

$$e_{\mathfrak{p}} \text{Ta}_{\mathfrak{p}}(J) = e_{\mathfrak{p}}(\text{Ta}_{\mathfrak{p}}(J) \otimes_{\mathbb{Z}_p} F_{\mathfrak{p}})$$

for J any one of J_t , $J_0(M)$, or $J_0(Mp)$. Whenever $(n, Mp) = 1$ we have $b(n) = a_p(n)$, thus T_n acts as $b(n)$ on $e_p \text{Ta}_p(J_0(M))$. It follows from strong multiplicity one that also T_p acts as $b(p)$ on $e_p \text{Ta}_p(J_0(M))$. From this and (5) it follows that U_p acts semi-simply on

$$(6) \quad \begin{aligned} e_p \text{Ta}_p(J_t) &\cong e_p \text{Ta}_p(J_0(Mp)) \\ &\cong \alpha^* e_p \text{Ta}_p(J_0(M)) \oplus \beta^* e_p \text{Ta}_p(J_0(M)) \end{aligned}$$

with characteristic polynomial

$$(X^2 - b(p)X + p)^2 = (X - a_p(p))^2 (X - p/a_p(p))^2,$$

and that e^{ord} and $1 - e^{\text{ord}}$ act as projections onto the $a_p(a)$ and $p/a_p(p)$ eigenspaces, respectively. Thus the map $\text{Ta}_p^{\text{ord}}(J_t) \otimes_{\mathcal{O}_F} F_p \rightarrow V_p$ takes the summand

$$e_p(\text{Ta}_p^{\text{ord}}(J_t) \otimes_{\mathcal{O}_F} F_p) = e^{\text{ord}} e_p \text{Ta}_p(J_t)$$

isomorphically to V_p , and it follows that the map

$$e^{\text{ord}} e_p(J_t(L_t) \otimes_{\mathbb{Z}} F_p) \rightarrow H^1(L_t, V_p)$$

induced by (2) is injective. As this map takes $e^{\text{ord}} e_p Q$ to Q_p , we deduce from (3)

$$\mathfrak{Z}_p = 0 \iff e^{\text{ord}} e_p Q = 0 \iff \left(U_p - \frac{p}{a_p(p)} \right) e_p Q = 0.$$

Let $x \in X_0(Mp)(H_p)$ be the image of h_t (still $t = 1$) under the degeneracy map $X_t \rightarrow X_0(Mp)$. Thus x corresponds to the cyclic Mp -isogeny given by the composition

$$\mathbb{C}/\mathcal{O}_p \rightarrow \mathbb{C}/\mathcal{O}_K \rightarrow \mathbb{C}/\mathfrak{M}^{-1}$$

Fix a rational cusp $\mathfrak{c} \in X_0(Mp)(\mathbb{Q})$ and define

$$P = \sum_{\sigma \in \text{Gal}(H_p/K)} (x - \mathfrak{c})^\sigma \in J_0(Mp) \otimes_{\mathbb{Z}} F_p.$$

As the first isomorphism of (6) identifies the Kummer images of $[L_t : H_p] e_p P$ and $e_p Q$, we have

$$\mathfrak{Z}_p = 0 \iff \left(U_p - \frac{p}{a_p(p)} \right) e_p P = 0.$$

The Albanese maps $\alpha_*, \beta_* : J_0(Mp) \rightarrow J_0(M)$ are related by $\beta_* U_p = p \alpha_*$, and applying β_* to the right hand side of the last \iff we find

$$(7) \quad \mathfrak{Z}_p = 0 \implies \left(\alpha_* - \frac{\beta_*}{a_p(p)} \right) e_p P = 0.$$

The point $\beta(x) \in X_0(M)(H)$ is none other than the classical Heegner point considered by Gross-Zagier corresponding to the cyclic M -isogeny $\mathbb{C}/\mathcal{O}_K \rightarrow \mathbb{C}/\mathfrak{M}^{-1}$, while $\alpha(x) \in X_0(M)(H_p)$ is the point corresponding to $\mathbb{C}/\mathcal{O}_p \rightarrow (\mathcal{O}_p \cap \mathfrak{M})^{-1}$. The Euler system relations of [15, §3.1] imply that

$$u \cdot \text{Tr}_{H_p/H} \alpha(x) = \begin{cases} T_p \cdot \beta(x) & \text{if } \epsilon_K(p) = -1 \\ (T_p - \sigma_p - \sigma_p^*) \cdot \beta(x) & \text{if } \epsilon_K(p) = 1 \end{cases}$$

as divisors on $X_0(M)/H$, where $2u = |\mathcal{O}_K^\times|$ and $\sigma_p, \sigma_p^* \in \text{Gal}(H/K)$ are the Frobenius automorphisms of the two primes above p when $\epsilon_K(p) = 1$. From this and $u[H_p : H] = p - \epsilon_K(p)$ we deduce

$$\alpha_*(P) = \frac{T_p - 1 - \epsilon_K(p)}{p - \epsilon_K(p)} \cdot \beta_*(P).$$

We may use this, $a_{\mathfrak{p}}(p)^2 - b(p)a_{\mathfrak{p}}(p) + p = 0$, and $a_{\mathfrak{p}}(p) \neq \pm 1$ to deduce from (7) the implications

$$\begin{aligned} \mathfrak{Z}_{\mathfrak{p}} = 0 &\implies (a_{\mathfrak{p}}(p) - 1)(a_{\mathfrak{p}}(p) - \epsilon_K(p)) \cdot \beta_*(e_{\mathfrak{p}}P) = 0 \\ &\implies \beta_*(e_{\mathfrak{p}}P) = 0. \end{aligned}$$

Finally, according to the Gross-Zagier theorem [5] the element

$$e_{\mathfrak{p}}\beta_*(P) \in J_0(M) \otimes_{\mathbb{Z}} F_{\mathfrak{p}}$$

is trivial if and only if $L'(1, \chi_{\mathfrak{p}}^t, f^t) = 0$. This completes the proof of the proposition. \square

It seems likely that the implication $L'(1, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t) = 0 \implies \mathfrak{Z}_{\mathfrak{p}} = 0$ in Proposition 4 holds as well, although we are unable to provide a proof. There are two situations in which a weight 2 arithmetic prime fails to be generic for θ , and in neither case does it seem that one should expect $\mathfrak{Z}_{\mathfrak{p}}$ to be related to the derivative $L'(1, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t)$. The first case is when $g_{\mathfrak{p}}$ is newform of level Mp and trivial nebentype and $\theta_{\mathfrak{p}}$ is trivial. Letting $\mathbf{1}_K$ denote the trivial character of \mathbb{A}_K^\times , the sign in the functional equation of

$$L(s, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t) = L(s, \mathbf{1}_K, g_{\mathfrak{p}}^t)$$

is $-\epsilon_K(p)$ and there are two sub-cases to consider. If p is inert in K then $L(s, \mathbf{1}_K, g_{\mathfrak{p}}^t)$ vanishes to even order at $s = 1$. If p splits in K then the Gross-Zagier theorem [5] relates $L'(1, \mathbf{1}_K, g_{\mathfrak{p}}^t)$ to the point on $X_0(Mp)$ corresponding to the isogeny

$$\mathbb{C}/\mathcal{O}_K \rightarrow \mathbb{C}/(\mathfrak{m}\mathfrak{P})^{-1}$$

where \mathfrak{P} is a prime of K above p . In particular, both the source and target of the isogeny have complex multiplication by the maximal order \mathcal{O}_K . On the other hand our cohomology class $\mathfrak{Z}_{\mathfrak{p}}$ is constructed from a point on $X(\Gamma_0(M) \cap \Gamma_1(p))$ whose image in $X_0(Mp)$ corresponds to the isogeny

$$\mathbb{C}/\mathcal{O}_p \rightarrow \mathbb{C}/\mathfrak{m}^{-1}$$

in which the source and target have different CM orders. Thus in this sub-case our Heegner class $\mathfrak{Z}_{\mathfrak{p}}$ is the wrong object to look at. The other situation in which \mathfrak{p} is not generic for θ is when $\theta_{\mathfrak{p}} = \omega^{\frac{p-1}{2}}$ and g is the p -stabilization of a newform $f \in S_2(\Gamma_0(M), F_{\mathfrak{p}})$. We then have

$$L(s, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t) = L(s, \chi_{\mathfrak{p}}^t, f^t) = L(s, \mathbf{1}_K, f^t \otimes \omega^{\frac{p-1}{2}}).$$

The sign in the functional equation of $L(s, \mathbf{1}_K, f^t \otimes \omega^{\frac{p-1}{2}})$ is again $-\epsilon_K(p)$, and exactly as in the case considered above one does not expect the central derivative of this L -function to be related to the class $\mathfrak{Z}_{\mathfrak{p}}$.

Suppose $\mathfrak{p} \subset R$ is arithmetic of weight two. If

$$(8) \quad \text{ord}_{s=1} L(s, \chi_{\mathfrak{p}}^t, g_{\mathfrak{p}}^t) = 1$$

for every $\iota \in \text{Emb}(\mathfrak{p})$ then we will say that $(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has *analytic rank one*. If we assume that $\mathfrak{p} \in \text{Gen}_2(\theta)$ then it suffices to verify (8) for a single $\iota \in \text{Emb}(\mathfrak{p})$. Indeed

if \mathfrak{p} satisfies condition (b) or (c) in Definition 2 then this is clear from Proposition 3. If \mathfrak{p} satisfies condition (a) in Definition 2 then, in the notation of the proof of Proposition 4,

$$\text{ord}_{s=1} L(s, \chi_{\mathfrak{p}}^{\iota}, g_{\mathfrak{p}}^{\iota}) = \text{ord}_{s=1} L(s, \chi_{\mathfrak{p}}^{\iota}, f^{\iota})$$

where $f \in S_2(\Gamma_0(M), F_{\mathfrak{p}})$ and $\chi_{\mathfrak{p}}$ is the trivial character. It follows from the Gross-Zagier theorem [5] that if the order of vanishing of the L -function on the right is 1 for some ι , then the order of vanishing is 1 for all ι .

Corollary 5. *The following are equivalent:*

- (a) *there is a $\mathfrak{p} \in \text{Gen}_2(\theta)$ such that $(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one,*
- (b) *there is an arithmetic prime \mathfrak{p} such that $\mathfrak{Z}_{\mathfrak{p}} \neq 0$,*
- (c) *\mathfrak{Z} is not R -torsion,*
- (d) *$\mathfrak{Z}_{\mathfrak{p}} \neq 0$ for all but finitely many arithmetic primes \mathfrak{p} ,*
- (e) *$(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one for all but finitely many $\mathfrak{p} \in \text{Gen}_2(\theta)$.*

Proof. The implication (a) \implies (b) is immediate from Propositions 3 and 4. Assume (b) holds. It follows from [6, Proposition 2.4.5] that $\mathfrak{Z} \in \text{Sel}_{\text{Gr}}(K, \mathbf{T}^{\dagger})$, and (b) implies that \mathfrak{Z} has nontrivial image in the localization $\text{Sel}_{\text{Gr}}(K, \mathbf{T}^{\dagger})_{\mathfrak{p}}$ for some arithmetic prime \mathfrak{p} . According to [14, Proposition 12.7.13.4(iii)] (and using [6, (21)] to compare the strict Greenberg Selmer group with Nekovář's extended Selmer group) the localization $\text{Sel}_{\text{Gr}}(K, \mathbf{T}^{\dagger})_{\mathfrak{p}}$ is free of finite rank over $R_{\mathfrak{p}}$. Thus \mathfrak{Z} has non-torsion image after localizing at \mathfrak{p} , and so was non-torsion to begin with. Thus (b) \implies (c). Let \mathcal{H} denote the maximal extension of H unramified outside Mp and set $\mathfrak{G} = \text{Gal}(\mathcal{H}/H)$. For any arithmetic prime \mathfrak{p} we may pick a uniformizing parameter $\varpi_{\mathfrak{p}}$ of the discrete valuation ring (by [14, §12.7.5]) $R_{\mathfrak{p}}$. The \mathfrak{G} -cohomology of the exact sequence

$$0 \rightarrow \mathbf{T}_{\mathfrak{p}}^{\dagger} \xrightarrow{\varpi_{\mathfrak{p}}} \mathbf{T}_{\mathfrak{p}}^{\dagger} \rightarrow V_{\mathfrak{p}}^{\dagger} \rightarrow 0$$

shows that the natural map

$$H^1(\mathfrak{G}, \mathbf{T}^{\dagger})_{\mathfrak{p}} / \mathfrak{p} H^1(\mathfrak{G}, \mathbf{T}^{\dagger})_{\mathfrak{p}} \rightarrow H^1(\mathfrak{G}, V_{\mathfrak{p}}^{\dagger})$$

is injective. As $H^1(\mathfrak{G}, \mathbf{T}^{\dagger})$ is finitely generated as an R -module and contains \mathfrak{Z} , the implication (c) \implies (d) follows from [6, Lemma 2.1.7]. The implication (d) \implies (e) follows by another application of Proposition 3, and (e) \implies (a) is obvious. \square

4. RESULTS OVER \mathbb{Q}

For any arithmetic prime \mathfrak{p} let $\mathcal{O}_{\mathfrak{p}}$ denote ring of integers of $F_{\mathfrak{p}}$. The Mazur-Tate-Teitelbaum [10] p -adic L -function $\mathcal{L}(\cdot, g_{\mathfrak{p}}) \in \mathcal{O}_{\mathfrak{p}}[[\mathbb{Z}_{\mathfrak{p}}^{\times}]]$ of $g_{\mathfrak{p}}$, viewed as an $F_{\mathfrak{p}}$ -valued function on characters $\chi : \mathbb{Z}_{\mathfrak{p}}^{\times} \rightarrow F_{\mathfrak{p}}^{\times}$, satisfies the functional equation [6, Proposition 2.3.6]

$$\mathcal{L}(\chi, g_{\mathfrak{p}}) = -w\chi^{-1}(-M)\theta_{\mathfrak{p}}(-M) \cdot \mathcal{L}([\cdot]_{\mathfrak{p}}\chi^{-1}, g_{\mathfrak{p}})$$

where $w = \pm 1$ is independent of \mathfrak{p} (but depends on θ ; see [6, Remark 2.3.4]). Taking $\chi = \theta_{\mathfrak{p}}$ gives

$$(9) \quad \mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}}) = -w\mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}}).$$

For each arithmetic prime $\mathfrak{p} \subset R$ of even weight r the character $\theta_{\mathfrak{p}}$ has the form

$$\theta_{\mathfrak{p}}(z) = z^{\frac{r-2}{2}} \vartheta_{\mathfrak{p}}(z)$$

with $\vartheta_{\mathfrak{p}}$ of finite order. Define a p -adic eigenform of trivial nebentype

$$f_{\mathfrak{p}} = g_{\mathfrak{p}} \otimes \vartheta_{\mathfrak{p}}^{-1} \in S_r(\Gamma_0(Mp^{2t}), F_{\mathfrak{p}})$$

where $t = t(\mathfrak{p})$ as in §2. For any $\iota \in \text{Emb}(\mathfrak{p})$ define

$$f_{\mathfrak{p}}^{\iota} \in S_r(\Gamma_0(Mp^{2t}), \mathbb{Q}^{\text{alg}})$$

to be the classical eigenform obtained by applying ι to the Fourier coefficients of $f_{\mathfrak{p}}$ (all of which are algebraic over \mathbb{Q}). Let us say that \mathfrak{p} has a *trivial zero* if $r = 2$, $\theta_{\mathfrak{p}}$ is trivial, and $a_{\mathfrak{p}}(p) = 1$. We write $\Omega(\theta)$ for the set of arithmetic primes $\mathfrak{p} \subset R$ of even weight which do *not* have a trivial zero, and note that $\text{Gen}_2(\theta) \subset \Omega(\theta)$.

Lemma 6. *If $\mathfrak{p} \in \Omega(\theta)$ has weight r and $\iota \in \text{Emb}(\mathfrak{p})$ then*

$$(10) \quad \mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}}) = 0 \iff L(r/2, f_{\mathfrak{p}}^{\iota}) = 0.$$

Furthermore the functional equation of $L(s, f_{\mathfrak{p}}^{\iota})$ has sign $-w$.

Proof. The equivalence (10) follows from the interpolation formula of Mazur-Tate-Teitelbaum [10, §I.14]. The assumption that \mathfrak{p} does not have a trivial zero ensures that the p -adic multiplier appearing in the interpolation formula is nonzero, so that there is no extra zero in the sense of [10, §I.15]. According to [14, Proposition 12.7.14.4(i)] the sign in the functional equation of $L(s, f_{\mathfrak{p}}^{\iota})$ is independent of $\mathfrak{p} \in \Omega(\theta)$, and according to [10, §I.18] the sign is equal to $-w$ for any $\mathfrak{p} \in \Omega(\theta)$ such that $g_{\mathfrak{p}}$ has trivial nebentype. \square

Given $\mathfrak{p} \in \Omega(\theta)$ of weight r we say that $f_{\mathfrak{p}}$ has *analytic rank zero* if $L(r/2, f_{\mathfrak{p}}^{\iota}) \neq 0$. It follows from (10) that this condition is independent of the choice of ι . The following result is an easy consequence of work of Kato, Kitagawa, and Mazur; we state it for the purpose of comparison with Theorem 8 below.

Theorem 7. *The following are equivalent:*

- (a) *there is a $\mathfrak{p} \in \Omega(\theta)$ such that $f_{\mathfrak{p}}$ analytic rank zero,*
- (b) *$f_{\mathfrak{p}}$ has analytic rank zero for all but finitely many $\mathfrak{p} \in \Omega(\theta)$.*

Furthermore, when these conditions hold $\text{Sel}_{\text{Gr}}(\mathbb{Q}, \mathbf{T}^{\dagger})$ is a rank zero R -module.

Proof. We give a quick sketch of the proof. In [3] one finds the construction of a two-variable p -adic L -function (originally constructed by Kitagawa [9] following unpublished work of Mazur) $\mathcal{L} \in R[[\mathbb{Z}_p^{\times}]]$ whose image $\mathcal{L}(\theta, \cdot)$ under the map $R[[\mathbb{Z}_p^{\times}]] \rightarrow R$ induced by θ has the property that $R \rightarrow F_{\mathfrak{p}}$ takes $\mathcal{L}(\theta, \cdot)$ to $\mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}})$ for any arithmetic prime \mathfrak{p} . If $\mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}}) \neq 0$ for some arithmetic prime \mathfrak{p} then $\mathcal{L}(\theta, \cdot) \neq 0$, and hence by [6, Lemma 2.1.7] $\mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}}) \neq 0$ for *all but finitely many* arithmetic primes \mathfrak{p} . Thus the equivalence of (a) and (b) follows from (10).

Choosing a $\mathfrak{p} \in \Omega(\theta)$ such that $\mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}}) \neq 0$, the work of Kato [8] shows that the Bloch-Kato Selmer group $H_f^1(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger})$ is trivial (alternatively, one could choose \mathfrak{p} to have weight 2 and then deduce the triviality of the Bloch-Kato Selmer group using the results of Gross-Zagier [5] and the methods of Kolyvagin, as extended by Nekovář [13] to include modular forms with coefficients in number fields). It follows from [14, Proposition 12.7.13.4(1)] that the natural map

$$\text{Sel}_{\text{Gr}}(\mathbb{Q}, \mathbf{T}^{\dagger}) \otimes_R F_{\mathfrak{p}} \rightarrow H_f^1(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger})$$

is injective, and therefore the rank of $\text{Sel}_{\text{Gr}}(\mathbb{Q}, \mathbf{T}^{\dagger})$ is zero. \square

If $w = -1$ then it is a conjecture of Greenberg [4] that condition (b) always holds in Theorem 7, and condition (a) gives an effective method of verifying this conjecture for a given Hida family. Of course if $w = 1$ then $\mathcal{L}(\theta_{\mathfrak{p}}, g_{\mathfrak{p}})$ vanishes for every arithmetic prime \mathfrak{p} by (9) and Theorem 7 is vacuous; one should instead look at the behavior of $L'(r/2, f_{\mathfrak{p}}^{\iota})$ as $\mathfrak{p} \in \Omega(\theta)$ varies. As it is not known that the order of vanishing of $L(s, f_{\mathfrak{p}}^{\iota})$ at $s = r/2$ is equal to the order of vanishing of $\mathcal{L}(\chi, g_{\mathfrak{p}})$ at $\chi = \theta_{\mathfrak{p}}$, studying the two-variable p -adic L -function \mathcal{L} yields little information about $L'(r/2, f_{\mathfrak{p}})$. We instead use the big Heegner point as a substitute for the element $\mathcal{L}(\theta, \cdot)$ in order to obtain information about the generic behavior of the central derivatives, and substitute the Gross-Zagier type results of §3 for the interpolation formula (10). As noted earlier, the cost is that we lose all information about forms of weight greater than two.

We now assume that $w = 1$. Fix a $\mathfrak{p} \in \text{Gen}_2(\theta)$ and $\iota \in \text{Emb}(\mathfrak{p})$. The sign in the functional equation of $L(s, f_{\mathfrak{p}}^{\iota})$ is equal to -1 , and we further assume that $L'(1, f_{\mathfrak{p}}^{\iota}) \neq 0$. By a result of Bump-Friedberg-Hoffstein [1] one may choose a quadratic imaginary field K of discriminant prime to Mp in which all prime divisors of M are split, and such that $L(1, f_{\mathfrak{p}}^{\iota} \otimes \epsilon_K) \neq 0$. We now apply the results of §3. Using the factorization of L -functions

$$(11) \quad L(s, \chi_{\mathfrak{p}}^{\iota}, g_{\mathfrak{p}}^{\iota}) = L(s, f_{\mathfrak{p}}^{\iota})L(s, f_{\mathfrak{p}}^{\iota} \otimes \epsilon_K)$$

we find that $(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one, and it follows that $L'(1, f_{\mathfrak{p}}^{\iota}) \neq 0$ for every choice of ι . In this situation we say that $f_{\mathfrak{p}}$ has *analytic rank one*. We now come to our main result.

Theorem 8. *Suppose that $w = 1$. The following are equivalent:*

- (a) *there exists a $\mathfrak{p} \in \text{Gen}_2(\theta)$ such that $f_{\mathfrak{p}}$ has analytic rank one;*
- (b) *$f_{\mathfrak{p}}$ has analytic rank one for all but finitely many $\mathfrak{p} \in \text{Gen}_2(\theta)$.*

Furthermore, when these conditions are satisfied $\text{Sel}_{\text{Gr}}(\mathbb{Q}, \mathbf{T}^{\dagger})$ is a rank one R -module.

Proof. Suppose $\mathfrak{p} \in \text{Gen}_2(\theta)$ is such that $f_{\mathfrak{p}}$ has analytic rank one. The argument above shows that we may choose the quadratic imaginary field K of §3 in such a way that $\mathfrak{Z}_{\mathfrak{p}} \neq 0$. By Corollary 5 we see that $(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one for all but finitely many $\mathfrak{p} \in \text{Gen}_2(\theta)$. The factorization (11), together with the odd order of vanishing of $L(s, f_{\mathfrak{p}}^{\iota})$ shows that $f_{\mathfrak{p}}$ has analytic rank one whenever $(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one, and hence (a) \implies (b). The other implication is obvious, and the final claim follows from Corollary 5, [6, Corollary 3.4.3], and [6, (21)]. \square

Once again it is a conjecture of Greenberg [4] that condition (b) holds in Theorem 8, and condition (a) gives an effective method for verifying this for a given Hida family. We point out that the implication

$$(a) \implies \text{rank}_R \text{Sel}_{\text{Gr}}(\mathbb{Q}, \mathbf{T}^{\dagger}) = 1$$

in Theorem 8 can also be deduced using Nekovář's parity results [14, §12] (although this implication is never stated explicitly there), and so it is only the implication (a) \implies (b) which is new.

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