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Special cohomology classes for modular Galois representations

Abstract

Building on ideas of Vatsal [23], Cornut [5] proved a conjecture of Mazur asserting the generic nonvanishing of Heegner points on an elliptic curve E/\mathbb{Q} as one ascends the anticyclotomic \mathbb{Z}_p -extension of a quadratic imaginary extension K/\mathbb{Q} . In the present article Cornut's result is extended by replacing the elliptic curve E with the Galois cohomology of Deligne's 2-dimensional ℓ -adic representation attached to a modular form of weight $2k > 2$, and replacing the family of Heegner points with an analogous family of special cohomology classes.

0 Introduction

0.1 Statement of the main result

Let $f \in S_{2k}(\Gamma_0(N), \mathbb{C})$ be a normalized newform of weight $2k > 2$ and level $N \geq 4$. Fix a rational prime ℓ and embeddings of algebraic closures $\mathbb{Q}^{\text{al}} \hookrightarrow \mathbb{Q}_\ell^{\text{al}}$, $\mathbb{Q}^{\text{al}} \hookrightarrow \mathbb{C}$. Let $\Phi \subset \mathbb{Q}_\ell^{\text{al}}$ be a finite extension of \mathbb{Q}_ℓ containing all Fourier coefficients of f and let W_f be the 2-dimensional Φ vector space with $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ -action constructed by Deligne [6], so that the geometric Frobenius of a prime $q \nmid \ell N$ acts on W_f with characteristic polynomial $X^2 - a_q(f)X + q^{2k-1}$. Let K be a quadratic imaginary field satisfying the *Heegner hypothesis* that all prime divisors of N are split in K , fix a prime $p \nmid N \cdot \text{disc}(K)$, let $H[p^s]$ denote the ring class field of K of conductor p^s , set $H[p^\infty] = \cup_s H[p^s]$, and define $\mathcal{G} = \text{Gal}(H[p^\infty]/K)$. The torsion subgroup $G_0 \subset \mathcal{G}$ satisfies $\mathcal{G}/G_0 \cong \mathbb{Z}_p$.

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In §5.1 we define for every $s \geq 0$ a subspace

$$\text{Heeg}_s(f) \subset H^1(H[p^s], W_f(k)).$$

This subspace is the higher weight analogue of the subspace generated by the Kummer images of Heegner points in the case $k = 1$, in which case

$$W_f(1) \cong \text{Ta}_\ell(A_f) \otimes \mathbb{Q}_\ell$$

for A_f the modular abelian variety attached to f by Eichler-Shimura theory. Such higher weight Heegner objects have been studied earlier by Brylinski [3], Nekovář [15,16], and Zhang [25], and our construction of $\text{Heeg}_s(f)$ follows Nekovář's [16] construction very closely. The main result (Theorem 5.1.1) extends the results of Cornut [5] and Vatsal [23] from the case $k = 1$, and is as follows:

Theorem A *Fix a character $\chi : G_0 \rightarrow \Phi^\times$ and let*

$$\pi_\chi = \sum_{\sigma \in G_0} \chi(\sigma)\sigma \in \Phi[G_0].$$

Suppose $\ell \nmid p \cdot N \cdot \varphi(N) \cdot \text{disc}(K) \cdot (2k - 2)!$ (φ is Euler's function). As $s \rightarrow \infty$ the Φ -dimension of $\pi_\chi \text{Heeg}_s(f)$ grows without bound.

Let $X(N)_{/\mathbb{Q}}$ be the usual (geometrically disconnected) moduli space of generalized elliptic curves over \mathbb{Q} with full level N structure, and let $\mathcal{V}_{/\mathbb{Q}} \rightarrow X(N)_{/\mathbb{Q}}$ be the Kuga-Sato variety considered in [20,21]. Thus $\mathcal{V}_{/\mathbb{Q}}$ is a desingularization of the $(2k - 2)$ -fold fiber product over $X(N)_{/\mathbb{Q}}$ of the universal generalized elliptic curve. By work of Scholl [21], Deligne's ℓ -adic representation W_f occurs as a summand of $H^{2k-1}(\mathcal{V}_{/\mathbb{Q}^{\text{al}}}, \mathbb{Q}_\ell)$. Combining this with the ℓ -adic Abel-Jacobi map of [17] yields a map [16, §0.3]

$$\Psi_f : \text{CH}_0^k(\mathcal{V}_F) \rightarrow H^{2k}(\mathcal{V}_F, \mathbb{Q}_\ell(k)) \rightarrow H^1(F, W_f(k))$$

for any number field F , where CH_0^k denotes the Chow group of homologically trivial cycles of codimension k , modulo rational equivalence. Nekovář [16,17] shows that the image of Ψ_f is contained in the Bloch-Kato Selmer group

$$\text{Sel}(F, W_f(k)) \subset H^1(F, W_f(k)).$$

Taking $F = H[p^s]$, the subspace $\text{Heeg}_s(f)$ lies in the image of Ψ_f .

As in [23] we may write $H[p^\infty]$ as the compositum of linearly disjoint (over K) fields F and K_∞ where F/K is tamely ramified at p with Galois group G_0 , and K_∞/K is the anticyclotomic \mathbb{Z}_p -extension. By Theorem A (and under the hypotheses of that theorem), the dimension of the χ -component of $\text{Sel}(\mathbb{Q}^{\text{al}}/H[p^s], W_f(k))$ grows without bound. This provides some evidence for

the standard conjecture predicting that for each character χ of G_0

$$\dim_{\mathbb{F}} \pi_{\chi} \text{Sel}(H[p^s], W_f(k)) = \text{ord}_{s=k} \prod_{\psi} L(f \otimes K, \chi^{-1}\psi, s) \quad (1)$$

where the product is over all characters ψ of $\text{Gal}(K_{\infty}/K)$ of conductor $\leq p^s$ and $L(f \otimes K, \chi^{-1}\psi, s)$ is the twisted L -function defined as in [16, §0.5]. Indeed, the Heegner hypothesis and the functional equation force $L(f \otimes K, \chi^{-1}\psi, k) = 0$ for each such ψ , and so the right hand is $\geq p^s$. One might hope to extend Kolyvagin's theory of Euler systems so as to prove that the left hand side is $p^s + O(1)$. Work of Nekovář [15] and of Bertolini and Darmon [2] give evidence that this is accessible.

It is conjectured that the kernel of Ψ_f is independent of the choice of prime ℓ . A proof would allow one to remove the undesirable hypothesis that $\ell \neq p$ in Theorem A, leading to higher weight generalizations of the Iwasawa theoretic results of [1,8,18]. It seems difficult to adapt the methods of the present article to treat the (most interesting) case $\ell = p$; instead that case is treated in the forthcoming work [9] using a completely different construction of Heegner cohomology classes in $H^1(H[p^s], W_f(k))$. The constructions and results of [9] hold only for $\ell = p$ and f ordinary at p , but allow modular forms of odd weight (which seem inaccessible using the methods of the present article).

Zhang [25] has proved a higher weight form of the Gross-Zagier theorem relating the height pairings of certain Heegner cycles in $\text{CH}_0^k(\mathcal{V}_{/H[1]})$ to the derivatives $L'(f \otimes K, \chi^{-1}\psi, k)$ for characters ψ of trivial conductor. The images of these Heegner cycles under Ψ_f generate our $\text{Heeg}_0(f)$, and thus Theorem A would yield nonvanishing results for $L'(f \otimes K, \chi^{-1}\psi, k)$ if Zhang's formula were extended to ramified characters, and (harder) if one knew the nondegeneracy of the height pairing on the Chow group $\text{CH}_0^k(\mathcal{V})$.

0.2 Notation and conventions

Throughout this article we use k , N , and M to denote positive integers with $k > 1$, $N \geq 4$, M squarefree, and $(M, N) = 1$. We will be ultimately be concerned with the case $M = 1$, but must allow more general M for technical reasons (M will eventually be a divisor of $\text{disc}(K)$). We frequently abbreviate $\mathbf{N} = NM$. The letters ℓ and p denote rational primes with $(\ell p, \mathbf{N}) = 1$. We allow $\ell = p$ unless stated otherwise (more precisely, we allow $\ell = p$ except in Sections 3 and 5).

The letter Λ always denotes a \mathbb{Z}_{ℓ} -algebra. If S is a scheme on which \mathbf{N} is invertible we let $Y_0(\mathbf{N})_{/S}$ (resp. $Y_1(N, M)_{/S}$) be the coarse (resp. fine) moduli space of elliptic curves with $\Gamma_0(\mathbf{N})$ level structure (resp. $\Gamma_1(N, M) = \Gamma_1(N) \cap \Gamma_0(M)$).

level structure). If $M = 1$ we omit it from the notation, and we sometimes omit S if it is clear from the context. For a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ we will sometimes refer to a pair (E, x) consisting of an elliptic curve E together with a Γ level structure x on E simply as a Γ structure. Set $\Delta = (\mathbb{Z}/N\mathbb{Z})^\times$ and let φ be Euler's function.

1 Augmented elliptic curves

Throughout §1, $\Lambda = \mathbb{Z}/m\mathbb{Z}$ for some fixed ℓ -power m , and $S = \mathrm{Spec}(F)$ for F a perfect field of characteristic prime to ℓN with algebraic closure F^{al} .

1.1 Sheaves

If L/F is an algebraic extension, let $\pi^{\mathrm{univ}} : E^{\mathrm{univ}} \rightarrow Y_1(N)_{/L}$ be the universal elliptic curve, and define a locally constant constructible sheaf on $Y_1(N)_{/L}$

$$\mathcal{L}_\Lambda = \mathrm{Sym}^{2k-2}(R^1\pi_*^{\mathrm{univ}}\underline{\Delta}). \quad (2)$$

The formation of this sheaf is compatible with base change in L , by the proper base change theorem. There are isomorphisms of étale sheaves on $Y_1(N)_{/L}$

$$R^1\pi_*^{\mathrm{univ}}\underline{\mu}_m \cong \underline{\mathrm{Hom}}(\underline{E}^{\mathrm{univ}}[m], \underline{\mu}_m) \cong \underline{E}^{\mathrm{univ}}[m]$$

where $\underline{E}^{\mathrm{univ}}[m]$ is the étale sheaf on $Y_1(N)_{/L}$ associated to the group scheme $E^{\mathrm{univ}}[m]$ and $\underline{\mathrm{Hom}}$ is sheaf Hom . Taking symmetric powers, there is a canonical isomorphism

$$\mathcal{L}_\Lambda(2k-2) \cong \mathrm{Sym}^{2k-2}(\underline{E}^{\mathrm{univ}}[m]). \quad (3)$$

If we let $Y_{/L}$ be a connected component of the open modular curve parameterizing elliptic curves over L with $\Gamma_1(N) \cap \Gamma(m)$ level structure and fix a geometric point $\bar{z} \rightarrow Y_1(N)_{/L}$, then the forgetful covering map $Y_{/L} \rightarrow Y_1(N)_{/L}$ cuts out a quotient of the fundamental group $\pi_1 = \pi_1(Y_1(N)_{/L}, \bar{z})$. The group of $Y_{/L}$ -valued m -torsion points of the universal elliptic curve over $Y_{/L}$ is canonically isomorphic to Λ^2 (via the universal $\Gamma(m)$ level structure), and the action of π_1 on this group identifies the aforementioned quotient with a subgroup of $\mathrm{GL}_2(\Lambda)$ containing $\mathrm{SL}_2(\Lambda)$. We thus obtain an action of π_1 on Λ^2 and so also on $\mathrm{Sym}^{2k-2}\Lambda^2$. It is immediate from (3) that the locally constant sheaf associated to this action is isomorphic to $\mathcal{L}_\Lambda(2k-2)$. From the discussion following [7, §2 Lemma 2] we see that there is a perfect symmetric pairing of étale sheaves

$$\mathcal{L}_\Lambda(k-1) \otimes \mathcal{L}_\Lambda(k-1) \rightarrow \underline{\Delta}. \quad (4)$$

For any étale sheaf \mathcal{F} on $Y_1(N)_{/L}$ define

$$\tilde{H}^*(Y_1(N)_{/L}, \mathcal{F}) = \text{Image}\left(H_c^*(Y_1(N)_{/L}, \mathcal{F}) \rightarrow H^*(Y_1(N)_{/L}, \mathcal{F})\right). \quad (5)$$

1.2 Augmentations

Let L/F be an algebraic extension and let Γ be any one of $\Gamma_0(N)$, $\Gamma_0(\mathbf{N})$, $\Gamma_1(N)$, or $\Gamma_1(N, M)$. If E is an elliptic curve over L , define a $\text{Gal}(F^{\text{al}}/L)$ -module

$$\mathcal{A}_\Lambda(E) = (\text{Sym}^{2k-2} E[m])(1-k)$$

(where $E[m] = E(F^{\text{al}})[m]$) and set $\mathcal{A}_\Lambda^\circ(E) = \mathcal{A}_\Lambda(E)^{\text{Gal}(F^{\text{al}}/L)}$. Note that \mathcal{A}_Λ and $\mathcal{A}_\Lambda^\circ$ are naturally covariant functors on the category of elliptic curves over L . The construction of $\mathcal{A}_\Lambda(E)$ depends on the embedding $L \hookrightarrow F^{\text{al}}$, but that of $\mathcal{A}_\Lambda^\circ(E)$ does not, in the sense that the Λ -modules defined by two different choices are canonically isomorphic.

Definition 1.2.1 *By a Λ -augmented Γ structure over an algebraic extension L/F we mean a triple (E, x, Θ) in which (E, x) is an elliptic curve with Γ structure over L and $\Theta \in \mathcal{A}_\Lambda^\circ(E)$.*

Two Λ -augmented Γ structures over L , (E_0, x_0, Θ_0) and (E_1, x_1, Θ_1) , are *isomorphic* if there is an isomorphism (over L) of elliptic curves $\phi : E_0 \rightarrow E_1$ such that ϕ identifies x_0 with x_1 and $\phi(\Theta_0) = \Theta_1$. If (E, x, Θ) is a Λ -augmented Γ structure over F^{al} and σ is an automorphism of F^{al} , there is an evident notion of the *conjugate* Λ -augmented Γ structure $(E, x, \Theta)^\sigma = (E^\sigma, x^\sigma, \Theta^\sigma)$.

Definition 1.2.2 *Given a Λ -augmented Γ structure (E, x, Θ) over F^{al} the field of moduli, L , of (E, x, Θ) is the extension of F characterized by the property that $\sigma \in \text{Gal}(F^{\text{al}}/F)$ fixes L if and only if $(E, x, \Theta)^\sigma$ is isomorphic (over F^{al}) to (E, x, Θ) .*

Remark 1.2.3 *We will often use (E, \mathbf{C}, Θ) to denote a Λ -augmented $\Gamma_0(\mathbf{N})$ structure, and write $C \subset \mathbf{C}$ for the $\Gamma_0(N)$ structure obtained by forgetting the $\Gamma_0(M)$ structure.*

If $z \in Y_1(N, M)_{/L}$ is a closed point we let E_z^{univ} be the pullback of the universal elliptic curve over $Y_1(N, M)_{/L}$ to $k(z)$. Define the module of Λ -augmented cycles on $Y_1(N, M)_{/L}$

$$\mathcal{A}_\Lambda^\circ(Y_1(N, M)_{/L}) = \bigoplus_z \mathcal{A}_\Lambda^\circ(E_z^{\text{univ}}),$$

where the sum is over all closed points of $Y_1(N, M)_{/L}$ (note that $\mathcal{A}_\Lambda^\circ(E_z^{\text{univ}})$ means the points of $\mathcal{A}_\Lambda(E_z^{\text{univ}})$ defined over the field of definition of E_z^{univ} ,

$k(z)$, *not* over L). For any set of closed points $Z \subset Y_1(N, M)_{/L}$ define

$$\mathcal{A}_\Lambda^\circ(Z; Y_1(N, M)_{/L})$$

in the same way, but with the sum restricted to $z \in Z$. We also define

$$\mathcal{A}_\Lambda(\Gamma_1(N, M)) = \bigoplus_{(E, x)} \mathcal{A}_\Lambda(E), \quad (6)$$

where the sum is over isomorphism classes of $\Gamma_1(N, M)$ structures over F^{al} . A Λ -augmented $\Gamma_0(N)$ -structure (E, x, Θ) over F^{al} defines an element of the modular (6), denoted the same way, by taking the element Θ in the summand attached to (E, x) and 0 in the other summands. The module $\mathcal{A}_\Lambda(\Gamma_1(N, M))$ has a natural action of $\text{Gal}(F^{\text{al}}/L)$, and

$$\mathcal{A}_\Lambda^\circ(Y_1(N, M)_{/L}) \cong \mathcal{A}_\Lambda(\Gamma_1(N, M))^{\text{Gal}(F^{\text{al}}/L)}. \quad (7)$$

Indeed, a closed point $z \in Y_1(N, M)_{/L}$ and a $\Theta \in \mathcal{A}_\Lambda^\circ(E_z^{\text{univ}})$ determine a Λ -augmented $\Gamma_1(N, M)$ structure $(E_z^{\text{univ}}, x_z^{\text{univ}}, \Theta)$ over $k(z)$, where $(E_z^{\text{univ}}, x_z^{\text{univ}})$ is the pullback to $k(z)$ of the universal $\Gamma_1(N, M)$ structure. Each embedding of L -algebras $k(z) \hookrightarrow F^{\text{al}}$ then determines a Λ -augmented $\Gamma_1(N, M)$ structure over F^{al} , and summing over all embeddings $k(z) \hookrightarrow F^{\text{al}}$ determines an element of the right hand side of (7). Extending linearly over all z and Θ gives the desired map. The construction of the inverse is similar and easy.

1.3 A higher weight Kummer map

In this subsection $M = 1$. Let $L \subset F^{\text{al}}$ be an algebraic extension of F . Fix a closed point $z \in Y_1(N)_{/L}$ and write i_z for the closed immersion $\text{Spec}(k(z)) \rightarrow Y_1(N)_{/L}$. Denote by $j : Y_1(N)_{/F^{\text{al}}} \hookrightarrow X_1(N)_{/F^{\text{al}}}$ the usual compactification.

Lemma 1.3.1 *There are canonical isomorphisms*

$$\mathcal{A}_\Lambda^\circ(E_z^{\text{univ}}) \cong H^0(z, i_z^* \mathcal{L}_\Lambda(k-1)) \cong H_z^2(Y_1(N)_{/L}, \mathcal{L}_\Lambda(k)).$$

PROOF. The first isomorphism is induced by the isomorphism (3), and the second is a consequence of cohomological purity as in [13, Chapter VI §5]. \square

Lemma 1.3.2 *There is a canonical isomorphism*

$$H^2(X_1(N)_{/L}, j_* \mathcal{L}_\Lambda(k)) \cong H^1(F^{\text{al}}/L, \tilde{H}^1(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda(k))).$$

PROOF. One checks directly that $\text{Sym}^{2k-2}\Lambda^2$ has no $\text{SL}_2(\Lambda)$ -invariants, and hence, by the discussion of §1.1, $H^0(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda) = 0$. Using Poincaré duality we see also that the group

$$H^2(X_1(N)_{/F^{\text{al}}}, j_*\mathcal{L}) \cong H_c^2(Y_1(N)_{/F^{\text{al}}}, \mathcal{L})$$

is trivial, and so

$$H^i(X_1(N)_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) = 0 \quad (8)$$

for $i \neq 1$. Thus the Hochschild-Serre spectral sequence and the identification

$$H^1(X_1(N)_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) \cong \tilde{H}^1(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda) \quad (9)$$

yield the desired isomorphism. \square

Definition 1.3.3 *Combining Lemmas 1.3.1 and 1.3.2 with the homomorphism*

$$H_z^2(Y_1(N)_{/L}, \mathcal{L}_\Lambda(k)) \rightarrow H^2(X_1(N)_{/L}, j_*\mathcal{L}_\Lambda(k))$$

we obtain a map

$$\mathcal{A}_\Lambda^\circ(E_z^{\text{univ}}) \rightarrow H^1(F^{\text{al}}/L, \tilde{H}^1(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda)(k))$$

for each closed point $z \in Y_1(N)_{/L}$. This map extends linearly to define the Λ -augmented Kummer map

$$\mathcal{A}_\Lambda^\circ(Y_1(N)_{/L}) \rightarrow H^1(F^{\text{al}}/L, \tilde{H}^1(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda)(k)).$$

We now give an alternate definition of the Λ -augmented Kummer map. The proof of the equivalence of the two definitions requires only minor modification of [11, Lemma 9.4] and is omitted. Given a closed point $z \in Y_1(N)_{/L}$, let $U = U_{/F^{\text{al}}}$ be the open complement of $z \times_L F^{\text{al}}$ in $X_1(N)_{/F^{\text{al}}}$. Excision and the relative cohomology sequence give the exact sequence

$$0 \rightarrow H^1(X_1(N)_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) \rightarrow H^1(U, j_*\mathcal{L}_\Lambda) \rightarrow H_{z \times_L F^{\text{al}}}^2(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda) \rightarrow 0 \quad (10)$$

where the initial and terminating zeros are justified by cohomological purity and (8), respectively. Using Lemma 1.3.1 we may identify $\mathcal{A}_\Lambda^\circ(E_z^{\text{univ}})$ with the $\text{Gal}(F^{\text{al}}/L)$ -invariants of

$$\bigoplus_{w \in z \times_L F^{\text{al}}} \mathcal{A}_\Lambda(E_w^{\text{univ}}) \cong H_{z \times_L F^{\text{al}}}^2(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda)(k),$$

and the connecting homomorphism

$$\mathcal{A}_\Lambda^\circ(E_z^{\text{univ}}) \rightarrow H^1(F^{\text{al}}/L, H^1(X_1(N)_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda)(k)) \quad (11)$$

then agrees with Definition 1.3.3, using the identification of (9).

For L/F any algebraic extension, the group Δ acts on $Y_1(N, M)_{/L}$ through the diamond automorphisms. There is a similar action of Δ on $\mathcal{A}_\Lambda(\Gamma_1(N, M))$ commuting with the $\text{Gal}(F^{\text{al}}/L)$ -action, and so Δ also acts on $\mathcal{A}_\Lambda^\circ(Y_1(N, M)_{/L})$ by (7), and on $\mathcal{A}_\Lambda^\circ(Z; Y_1(N, M)_{/L})$ for any subset $Z \subset Y_1(N, M)_{/L}$ stable under Δ . There is also a familiar action of Δ on the cohomology $H^i(Y_1(N)_{/L}, \mathcal{L}_\Lambda(j))$ for any i and j , on compactly supported cohomology, and on the cohomology supported on Z for any closed set $Z \subset Y_1(N)_{/L}$ stable under Δ . The action of Δ is compatible with the Λ -augmented Kummer map of Definition 1.3.3.

1.4 Augmented $\Gamma_0(N)$ structures

Now fix a Λ -augmented $\Gamma_0(\mathbf{N})$ structure (E, \mathbf{C}, Θ) over F^{al} and suppose L is a finite extension of F containing the field of moduli of (E, \mathbf{C}, Θ) . In particular L contains the field of moduli (in the usual sense) of the pair (E, \mathbf{C}) , and so determines a closed point $y \in Y_0(\mathbf{N})_{/L}$ with residue field L . Let $Z \subset Y_1(N, M)_{/L}$ denote the set of closed points lying above y under the forgetful degeneracy map

$$F_{N,M} : Y_1(N, M)_{/L} \rightarrow Y_0(\mathbf{N})_{/L}.$$

Let $P_1, \dots, P_{\varphi(N)}$ be the generators of C (using the convention of Remark 1.2.3) and let x_i be the $\Gamma_1(N, M)$ structure on E determined by P_i and the $\Gamma_0(M)$ structure underlying \mathbf{C} . Define, using (7),

$$F_{N,M}^*(E, \mathbf{C}, \Theta) = \sum_{i=1}^{\varphi(N)} (E, x_i, \Theta) \in \mathcal{A}_\Lambda^\circ(Z; Y_1(N, M)_{/L})^\Delta. \quad (12)$$

Taking $M = 1$ for the moment, we denote by

$$\Omega_L(E, C, \Theta) \in H^1(F^{\text{al}}/L, \tilde{H}^1(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda)(k))^\Delta \quad (13)$$

the image of $F_N^*(E, C, \Theta)$ under the Λ -augmented Kummer map of Definition 1.3.3. Allowing L to vary over all finite extensions of F containing the field of moduli of (E, C, Θ) , the formation of $\Omega_L(E, C, \Theta)$ is compatible with the restriction maps on Galois cohomology.

1.5 Reduction and ramification

In this subsection we assume that $M = 1$ and F is a finite extension of \mathbb{Q}_q for some prime $q \nmid \ell N$. Let \mathbb{F}^{al} and \mathbb{F} denote the residue fields of F^{al} and F , respectively, so that

$$W_\Lambda \stackrel{\text{def}}{=} \tilde{H}^1(Y_1(N)_{/F^{\text{al}}}, \mathcal{L}_\Lambda) \cong \tilde{H}^1(Y_1(N)_{/\mathbb{F}^{\text{al}}}, \mathcal{L}_\Lambda) \quad (14)$$

is an unramified $\text{Gal}(F^{\text{al}}/F)$ -module. Let (E, C, Θ) be a Λ -augmented $\Gamma_0(N)$ structure over F^{al} , and assume that E has good reduction. The reduction of (E, C) , a $\Gamma_0(N)$ -structure over the field \mathbb{F}^{al} , is denoted $(\text{red}(E), \text{red}(C))$, and we identify $E(F^{\text{al}})[m]$ with $\text{red}(E)(\mathbb{F}^{\text{al}})[m]$ as Λ -modules. This determines an isomorphism $\text{red} : \mathcal{A}_\Lambda(E) \cong \mathcal{A}_\Lambda(\text{red}(E))$, and so we obtain a Λ -augmented $\Gamma_0(N)$ structure

$$\text{red}(E, C, \Theta) = (\text{red}(E), \text{red}(C), \text{red}(\Theta))$$

over \mathbb{F}^{al} . If $L \subset F^{\text{al}}$ is a finite extension of F containing the field of moduli of (E, C, Θ) then the residue field of L , \mathbb{L} , contains the field of moduli of $(\text{red}(E), \text{red}(C), \text{red}(\Theta))$, and so we may form

$$\Omega_{\mathbb{L}}(\text{red}(E), \text{red}(C), \text{red}(\Theta)) \in H^1(\mathbb{F}^{\text{al}}/\mathbb{L}, W_\Lambda(k))^\Delta.$$

Proposition 1.5.1 *Suppose $\ell \nmid \varphi(N)$. In the notation above, $\Omega_L(E, C, \Theta)$ is equal to the image of $\Omega_{\mathbb{L}}(\text{red}(E), \text{red}(C), \text{red}(\Theta))$ under the inflation map*

$$H^1(\mathbb{F}^{\text{al}}/\mathbb{L}, W_\Lambda(k)) \cong H^1(L^{\text{unr}}/L, W_\Lambda(k)) \rightarrow H^1(F^{\text{al}}/L, W_\Lambda(k)),$$

where $L^{\text{unr}} \subset F^{\text{al}}$ is the maximal unramified extension of L .

PROOF. It is clear from the definition that the construction

$$(E, C, \Theta) \mapsto F_N^*(E, C, \Theta)$$

is compatible with reduction, so the proof of the proposition amounts to verifying that the constructions of §1.3 extend across integral models. Let $Z \subset Y_1(N)_{/L}$ be as in §1.4 and suppose for the moment that every $z \in Z$ has residue field L . Denoting the integer ring of L by \mathcal{O}_L , each $z \in Y_1(N)_{/L}$ extends to a smooth section $\underline{z} : \text{Spec}(\mathcal{O}_L) \rightarrow X_1(N)_{/\mathcal{O}_L}$ of the canonical integral model of $X_1(N)$ over \mathcal{O}_L . Since E_z^{univ} has potentially good reduction, this section does not meet the cusps in the special fiber, and so factors through the affine subscheme $Y_1(N)_{/\mathcal{O}_L}$. The sequence (10) extends across integral models over the integer ring of the maximal unramified extension of L , denoted R , to give the middle row of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(X_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) & \longrightarrow & H^1(U_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) & \longrightarrow & H^2_{\underline{z} \times F^{\text{al}}}(Y_{/F^{\text{al}}}, \mathcal{L}_\Lambda) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H^1(X_{/R}, j_*\mathcal{L}_\Lambda) & \longrightarrow & H^1(U_{/R}, j_*\mathcal{L}_\Lambda) & \longrightarrow & H^2_{\underline{z} \times R}(Y_{/R}, \mathcal{L}_\Lambda) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(X_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) & \longrightarrow & H^1(U_{/F^{\text{al}}}, j_*\mathcal{L}_\Lambda) & \longrightarrow & H^2_{\underline{z} \times F^{\text{al}}}(Y_{/F^{\text{al}}}, \mathcal{L}_\Lambda) \longrightarrow 0 \end{array}$$

where we abbreviate $X = X_1(N)$ and $Y = Y_1(N)$, and write $U_{/R}$ for the complement of $\underline{z} \times R$ in $X_{/R}$. Lemma 1.3.1 implies that the rightmost vertical

arrows are isomorphisms (the pullback of \mathcal{L}_Λ to $\underline{z} \times R$ is constant, so its global sections can be computed in either geometric fiber). The vertical arrows on the left are isomorphisms, by (9) and the isomorphism of (14). By the five lemma the arrows in the middle column are isomorphisms as well, and from this it follows that the map (11) is compatible with reduction.

For the general case, choose a $z \in Z$ and an embedding $k(z) \hookrightarrow F^{\text{al}}$. Let L' be the image of this embedding. It is easily seen that L' does not depend on the point z or the choice of embedding, that L'/L is a Galois extension of degree dividing $\varphi(N)$, and that every point in $Z \times_L L' \hookrightarrow Y_1(N)_{/L'}$ has residue field L' . The proposition follows from the bijectivity of the restriction map

$$H^1(F^{\text{al}}/L, W_\Lambda(k)) \cong H^1(F^{\text{al}}/L', W_\Lambda(k))^{\text{Gal}(L'/L)},$$

and of the analogous map on the level of residue fields, together with the special case considered above. \square

1.6 Degeneracy maps

Recall that M is squarefree. Given a divisor $M' \mid M$ we define a degeneracy map

$$\alpha_{M'}^M : \mathcal{A}_\Lambda(\Gamma_1(N, M)) \rightarrow \mathcal{A}_\Lambda(\Gamma_1(N, M'))$$

as follows. Given a Λ -augmented $\Gamma_1(N, M)$ structure (E, x, Θ) over F^{al} , we define

$$\alpha_{M'}^M(E, x, \Theta) = (E, x', \Theta)$$

where x' is the $\Gamma_1(N, M')$ structure on E underlying x . Extend this Λ -linearly to a map on $\mathcal{A}_\Lambda(\Gamma_1(N, M))$. We also define a degeneracy map

$$\beta_{M'}^M : \mathcal{A}_\Lambda(\Gamma_1(N, M)) \rightarrow \mathcal{A}_\Lambda(\Gamma_1(N, M'))$$

as follows. Given a Λ -augmented $\Gamma_1(N, M)$ structure (E, x, Θ) over F^{al} let P and D be the $\Gamma_1(N)$ and $\Gamma_0(M)$ structures underlying x . Let $D_0 \subset D$ be the subgroup of order M/M' , let $E' = E/D_0$, and let P' and D' be the images of P and D under $E \rightarrow E'$. Let Θ' be the image of Θ under $\mathcal{A}_\Lambda(E) \rightarrow \mathcal{A}_\Lambda(E')$. Write x' for the $\Gamma_1(N, M')$ structure (P', D') on E' . Now define

$$\beta_{M'}^M(E, x, \Theta) = (E', x', \Theta')$$

and again extend linearly. The maps $\alpha_{M'}^M$ and $\beta_{M'}^M$ respect the $\text{Gal}(F^{\text{al}}/F)$ action, and so induce maps

$$\alpha_{M'}^M, \beta_{M'}^M : \mathcal{A}_\Lambda^\circ(Y_1(N, M)_{/L}) \rightarrow \mathcal{A}_\Lambda^\circ(Y_1(N, M')_{/L})$$

for any algebraic extension L/F .

2 Families of augmented CM points

Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$ for an ℓ -power m . Fix a quadratic imaginary field $K \subset \mathbb{Q}^{\text{al}}$, assume that all prime divisors of N are split in K , and fix an ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. Fix an elliptic curve E_1 over \mathbb{Q}^{al} with complex multiplication by the maximal order \mathcal{O}_K (there are $\#\text{Pic}(\mathcal{O}_K)$ such curves). Let $j : \mathcal{O}_K \hookrightarrow \text{End}_{\mathbb{Q}^{\text{al}}}(E_1)$ be normalized so that pullback by $j(\alpha)$ acts as multiplication by α on the cotangent space of $E_1(\mathbb{C})$ for every $\alpha \in K$. Set $C_1 = E_1[\mathfrak{N}]$, a cyclic subgroup of order N .

Definition 2.0.1 *An element $c \in \text{GL}_2(\mathbb{Q}_p)$ is cyclic if $(c^{-1}\mathbb{Z}_p^2)$ contains \mathbb{Z}_p^2 with cyclic quotient. The degree of a cyclic c is*

$$\deg(c) = [c^{-1}\mathbb{Z}_p^2 : \mathbb{Z}_p^2] = p^{\text{ord}_p(\det(c))}.$$

2.1 A parametrized family of Heegner points

A choice of isomorphism of \mathbb{Z}_p -modules $\text{Ta}_p(E_1) \cong \mathbb{Z}_p^2$ (which we now fix) determines a family of elliptic curves over \mathbb{Q}^{al} parametrized by

$$\mathcal{T} = \mathbb{Q}_p^\times \text{GL}_2(\mathbb{Z}_p) \backslash \text{GL}_2(\mathbb{Q}_p)$$

as follows. For each cyclic subgroup $X \subset E_1[p^\infty] \cong (\mathbb{Q}_p/\mathbb{Z}_p)^2$ there is a cyclic $c_X \in \text{GL}_2(\mathbb{Q}_p)$ such that $X = (c_X^{-1}\mathbb{Z}_p^2)/\mathbb{Z}_p^2$. The assignment $X \mapsto c_X$ establishes a bijection between the set of such subgroups and the cyclic elements of $\text{GL}_2(\mathbb{Q}_p)$, modulo left multiplication by $\text{GL}_2(\mathbb{Z}_p)$. We denote the inverse by $c \mapsto X_c$. The projection map $\text{GL}_2(\mathbb{Z}_p) \backslash \text{GL}_2(\mathbb{Q}_p) \rightarrow \mathcal{T}$ establishes a bijection between the left $\text{GL}_2(\mathbb{Z}_p)$ -orbits of cyclic elements and the set \mathcal{T} . To each $g \in \mathcal{T}$ we then define $X_g \subset E_1[p^\infty]$ to be X_c for any cyclic $c \in \text{GL}_2(\mathbb{Q}_p)$ lifting g , and define a cyclic p -power isogeny

$$f_g : E_1 \rightarrow E_g = E_1/X_g$$

of degree $\deg(c)$. Define the *degree* of g by $\deg(g) = \deg(c) = \deg(f_g)$. The elliptic curve E_g over \mathbb{Q}^{al} inherits a $\Gamma_0(N)$ structure $C_g = f_g(C_1) = E_g[\mathfrak{N} \cap \mathcal{O}_g]$, where $\mathcal{O}_g \subset \mathcal{O}_K$ is the largest order which leaves the subgroup $X_g \subset E_1[p^\infty]$ stable. The conductor of \mathcal{O}_g is a power of p .

For each $g \in \mathcal{T}$ let H_g be the ring class field of \mathcal{O}_g , thus H_g/K is Galois with Galois group canonically identified with $\text{Pic}(\mathcal{O}_g)$. The Weil pairing on $E_1[m]$ provides a canonical (up to ± 1) isomorphism between $(\text{Sym}^2 E_1[m])(-1)$ and the traceless Λ -module endomorphisms of $E_1[m]$. In particular there is a canonical (up to sign) element $\vartheta_1 \in (\text{Sym}^2 E_1[m])(-1)$ corresponding to

$\sqrt{D} \in \text{End}_{\mathbb{Q}^{\text{al}}}(E_1)$. Let $\Theta_1 \in \mathcal{A}_\Lambda(E_1)$ be the image of ϑ_1^{k-1} under the natural projection

$$\text{Sym}^{k-1}\left((\text{Sym}^2 E_1[m])(-1)\right) \rightarrow \left(\text{Sym}^{2k-2} E_1[m]\right)(1-k),$$

and define $\Theta_g = f_g(\Theta_1) \in \mathcal{A}_\Lambda(E_g)$. The data K, E_1, \mathfrak{N} , and $\text{Ta}_p(E_1) \cong \mathbb{Z}_p^2$ thus determine a family $g \mapsto (E_g, C_g, \Theta_g)$ of Λ -augmented $\Gamma_0(N)$ structures over \mathbb{Q}^{al} parametrized by \mathcal{T} . *This data is to remain fixed throughout the remainder of the article.* Define a $\text{Gal}(\mathbb{Q}^{\text{al}}/K)$ -module

$$W_\Lambda = \tilde{H}^1(Y_1(N)_{/\mathbb{Q}^{\text{al}}}, \mathcal{L}_\Lambda). \quad (15)$$

Using the theory of complex multiplication it is easily seen that the field of moduli of (E_g, C_g, Θ_g) is H_g , and so the construction (13) yields a family of cohomology classes parametrized by $g \in \mathcal{T}$

$$g \mapsto \Omega_{H_g}(E_g, C_g, \Theta_g) \in H^1(\mathbb{Q}^{\text{al}}/H_g, W_\Lambda(k))^\Delta. \quad (16)$$

Let $H[p^s]$ denote the ring class field of conductor p^s of K . For each $s \geq 0$ define $\mathcal{T}_s \subset \mathcal{T}$ to be the subset consisting of all g such that $H_g \subset H[p^s]$. For $g \in \mathcal{T}_s$ we let

$$\Omega_s(g) = \Omega_{H[p^s]}(E_g, C_g, \Theta_g) \in H^1(\mathbb{Q}^{\text{al}}/H[p^s], W_\Lambda(k))^\Delta \quad (17)$$

be the restriction of the cohomology class of (16) to $\text{Gal}(\mathbb{Q}^{\text{al}}/H[p^s])$.

2.2 Level M structure

Suppose that the integer M of §0.2 is a divisor of $\text{disc}(K)$ and let \mathfrak{M} be the unique \mathcal{O}_K -ideal of norm M . Although we assumed in §2.1 that all prime divisors of N are split in K , the constructions of §2.1 work equally well with N replaced by $\mathbf{N} = NM$ and \mathfrak{N} replaced by $\mathfrak{N}\mathfrak{M}$. This has the effect of endowing each (E_g, C_g) with the extra $\Gamma_0(\mathbf{N})$ structure $\mathbf{C}_g = E_g[\mathfrak{N}\mathfrak{M} \cap \mathcal{O}_g]$, so that $g \mapsto (E_g, \mathbf{C}_g, \Theta_g)$ is a parametrized family of Λ -augmented $\Gamma_0(\mathbf{N})$ structures.

3 Reduction of the family

In this section we prove Theorem 3.4.3, which is our analogue of [5, Theorem 3.1]. Suppose $\ell \neq p$ and take M to be a divisor of $\text{disc}(K)$ as in §2.2. Assume $\ell \nmid \varphi(N)$ and $\ell > 2k - 2$. Let Λ be a finite quotient of \mathbb{Z}_ℓ . Let \mathfrak{Q} be a finite set of rational primes inert in K , all prime to $\ell p \mathbf{N}$. For each $q \in \mathfrak{Q}$, let \mathfrak{q} denote

the prime of K above q and fix an extension of \mathfrak{q} to a place of \mathbb{Q}^{al} . We will abusively use \mathfrak{Q} to refer to the set of rational primes, the set of primes of K above them, and also the set of chosen places of \mathbb{Q}^{al} . Let $\mathbb{F}_{\mathfrak{q}}^{\text{al}}$ and $\mathbb{F}_{\mathfrak{q}}$ denote the residue fields of \mathbb{Q}^{al} and K at $\mathfrak{q} \in \mathfrak{Q}$, respectively, so that $\mathbb{F}_{\mathfrak{q}}$ has q^2 elements and $\mathbb{F}_{\mathfrak{q}}^{\text{al}}$ is algebraically closed. For each $\mathfrak{q} \in \mathfrak{Q}$ let

$$Z_0(\mathbf{N})_{\mathfrak{q}} \subset Y_0(\mathbf{N})_{/\mathbb{F}_{\mathfrak{q}}} \quad Z_1(N, M)_{\mathfrak{q}} \subset Y_1(N, M)_{/\mathbb{F}_{\mathfrak{q}}}$$

denote the subsets of supersingular points. The points of $Z_0(\mathbf{N})_{\mathfrak{q}}$ all have residue field $\mathbb{F}_{\mathfrak{q}}$, but this need not be true of $Z_1(N, M)_{\mathfrak{q}}$.

Definition 3.0.1 *A subset $\mathcal{S} \subset \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ is chaotic if for any distinct $\sigma, \tau \in \mathcal{S}$, the restriction of $\sigma\tau^{-1}$ to $\text{Gal}(H[p^\infty]/K)$ is not the Artin symbol of any idele with trivial p -component.*

3.1 Simultaneous reduction

As E_1 has complex multiplication, any model of E_1 over a number field has everywhere potentially good reduction. Fix a finite Galois extension F_1/K over which E_1 has a model with good reduction at every prime above every rational prime $q \in \mathfrak{Q}$, and fix such a model. All endomorphisms of E_1 are defined over F_1 , and hence so is the subgroup $\mathbf{C}_1 = E_1[\mathfrak{NM}]$. For each $g \in \mathcal{T}$ let F_g be a finite extension of F_1 , Galois over K , over which the subgroup X_g is defined. We may then view E_g , \mathbf{C}_g , and the isogeny f_g all as being defined over F_g . Fixing these choices, we may reduce everything at $w_{\mathfrak{q}}$ to obtain a family of Λ -augmented $\Gamma_0(\mathbf{N})$ structures over $\mathbb{F}_{\mathfrak{q}}^{\text{al}}$

$$\text{red}_{\mathfrak{q}}(E_g, \mathbf{C}_g, \Theta_g) = (\text{red}_{\mathfrak{q}}(E_g), \text{red}_{\mathfrak{q}}(\mathbf{C}_g), \text{red}_{\mathfrak{q}}(\Theta_g)).$$

We also denote by $\text{red}_{\mathfrak{q}}(f_g)$ the reduction of the isogeny f_g . Given an element $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ we may also form

$$\text{red}_{\mathfrak{q}}(E_g^\sigma, \mathbf{C}_g^\sigma, \Theta_g^\sigma) = \left(\text{red}_{\mathfrak{q}}(E_g^\sigma), \text{red}_{\mathfrak{q}}(\mathbf{C}_g^\sigma), \text{red}_{\mathfrak{q}}(\Theta_g^\sigma) \right) \quad (18)$$

and

$$\text{red}_{\mathfrak{q}}(f_g^\sigma) : \text{red}_{\mathfrak{q}}(E_1^\sigma) \rightarrow \text{red}_{\mathfrak{q}}(E_g^\sigma).$$

To emphasize, we regard these as Λ -augmented $\Gamma_0(\mathbf{N})$ structures over $\mathbb{F}_{\mathfrak{q}}^{\text{al}}$, regardless of the residue field of F_g at \mathfrak{q} . The field of moduli of (18) is $\mathbb{F}_{\mathfrak{q}}$; a fact (Lemma 4.1.1) whose proof we postpone until the next section. Abbreviate

$$\mathcal{Z}_{\mathfrak{q}}(M) = \mathcal{A}_{\Lambda}^{\circ}(Z_1(N, M)_{\mathfrak{q}}; Y_1(N, M)_{/\mathbb{F}_{\mathfrak{q}}})^{\Delta}. \quad (19)$$

Let $\Lambda[\mathcal{T}]$ denote the free Λ -module on the set \mathcal{T} . For each $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ and each $\mathfrak{q} \in \mathfrak{Q}$ define the reduction map $\text{Red}_{\sigma, \mathfrak{q}} : \Lambda[\mathcal{T}] \rightarrow \mathcal{Z}_{\mathfrak{q}}(M)$ by taking

$L = \mathbb{F}_q$ in (12) and linearly extending

$$\text{Red}_{\sigma, \mathfrak{q}}(g) = F_{N, M}^* \left(\text{red}_{\mathfrak{q}}(E_g^\sigma), \text{red}_{\mathfrak{q}}(\mathbf{C}_g^\sigma), \text{red}_{\mathfrak{q}}(\Theta_g^\sigma) \right).$$

For any subset $\mathcal{S} \subset \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ define the simultaneous reduction map

$$\text{Red}_{\mathcal{S}, \Omega} : \Lambda[\mathcal{T}] \rightarrow \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \Omega} \mathcal{Z}_{\mathfrak{q}}(M) \quad (20)$$

by linearly extending $\text{Red}_{\mathcal{S}, \Omega}(g) = \bigoplus_{\sigma, \mathfrak{q}} \text{Red}_{\sigma, \mathfrak{q}}(g)$. The reader may wish to skip directly to Theorem 3.4.3, the main result of §3.

3.2 Reduction at \mathfrak{q}

Fix a $\mathfrak{q} \in \Omega$. Define $S = \text{End}_{\mathbb{F}_q^{\text{al}}}(\text{red}_{\mathfrak{q}}(E_1))$ and $B = S \otimes \mathbb{Q}$ so that B is a quaternion algebra ramified exactly at q and ∞ , and S is a maximal order in B . Let $R \subset S$ be the subring of endomorphisms which leave $\text{red}_{\mathfrak{q}}(\mathbf{C}_1)$ stable, so that R is a level \mathbf{N} -Eichler order in B . The embedding $j : \mathcal{O}_K \rightarrow \text{End}_{F_1}(E_1)$ determines an embedding which we again denote by j

$$j : K \cong \text{End}_{F_1}(E_1) \otimes \mathbb{Q} \hookrightarrow \text{End}_{\mathbb{F}_q^{\text{al}}}(\text{red}_{\mathfrak{q}}(E_1)) \otimes \mathbb{Q} \cong B,$$

with $j(\mathcal{O}_K) \subset R$. For any rational prime r and any \mathbb{Z} (resp. \mathbb{Q}) algebra A , set $A_r = A \otimes_{\mathbb{Z}} \mathbb{Z}_r$ (resp. $A_r = A \otimes_{\mathbb{Q}} \mathbb{Q}_r$). Let \hat{B} be the restricted topological product $\prod'_r B_r$ with respect to the local orders $R_r \subset B_r$, and define \hat{K}, \hat{R}, \dots similarly. The embedding j induces embeddings $\hat{K} \hookrightarrow \hat{B}$ and $K_r \hookrightarrow B_r$ at every r . We denote all of these again by j .

Recall that we have fixed an isomorphism of \mathbb{Z}_p -modules $\text{Ta}_p(E_1) \cong \mathbb{Z}_p^2$. As the p -adic Tate modules of E_1 and $\text{red}_{\mathfrak{q}}(E_1)$ are canonically identified as \mathbb{Z}_p -modules (and $R_p \subset B_p$ is a maximal order), this induces isomorphisms

$$R_p \cong M_2(\mathbb{Z}_p) \quad B_p \cong M_2(\mathbb{Q}_p). \quad (21)$$

We henceforth identify $R_p^\times \cong \text{GL}_2(\mathbb{Z}_p)$ and $B_p^\times \cong \text{GL}_2(\mathbb{Q}_p)$ using *these* isomorphisms, and in particular identify \mathcal{T} with $\mathbb{Q}_p^\times R_p^\times \backslash B_p^\times$. This gives a right action of B_p^\times (and hence also of B^\times) on \mathcal{T} . The group R_ℓ^\times acts on $\text{Ta}_\ell(\text{red}_{\mathfrak{q}}(E_1))$ on the left, almost by definition, and we denote by $\rho_{\mathfrak{q}}$ the action of R_ℓ^\times on $\mathcal{A}_\Lambda(\text{red}_{\mathfrak{q}}(E_1))$ obtained by taking symmetric powers, with the understanding that R_ℓ^\times acts trivially on the twist $\Lambda(1-k)$. Writing \det for the reduced norm on B^\times , B_ℓ^\times , and so on, we also define $\rho_{\mathfrak{q}}^* = \rho_{\mathfrak{q}} \otimes \det^{1-k}$, and note that the center $\mathbb{Z}_\ell^\times \subset B_\ell^\times$ acts trivially under $\rho_{\mathfrak{q}}^*$. The group $\Gamma_{\mathfrak{q}} = R[1/p]^\times$ acts on $\mathcal{A}_\Lambda(\text{red}_{\mathfrak{q}}(E_1))$ through $\rho_{\mathfrak{q}}$ or $\rho_{\mathfrak{q}}^*$ by the inclusion $\Gamma_{\mathfrak{q}} \hookrightarrow R_\ell^\times$.

Fix a $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ whose restriction to K^{ab} (the maximal abelian extension of K) is equal to the Artin symbol of a finite idele $\hat{\sigma} \in \hat{K}^\times$. Let $b_{\sigma, \mathfrak{q}} \in B^\times$

be such that the r -component of $j(\hat{\sigma})b_{\sigma,q}$ lies in R_r^\times for all primes $r \neq p$, and let $\alpha_{\sigma,q} \in R_\ell^\times$ and $\beta_{\sigma,q} \in B_p^\times$ be the ℓ and p components, respectively, of $j(\hat{\sigma})b_{\sigma,q} \in \hat{B}^\times$.

Proposition 3.2.1 *Fix $g, h \in \mathcal{T}$. There is a $\gamma \in \Gamma_q$ such that $g\beta_{\sigma,q} = h\gamma \in \mathcal{T}$, if and only if there is an isomorphism of $\Gamma_0(\mathbf{N})$ structures over \mathbb{F}_q^{al}*

$$\phi : \text{red}_q(E_g^\sigma, \mathbf{C}_g^\sigma) \cong \text{red}_q(E_h, \mathbf{C}_h).$$

If these equivalent conditions hold then ϕ may be chosen so that

$$\phi(\text{red}_q(\Theta_g^\sigma)) = \omega_{\text{cyc}}^{1-k}(\sigma) \cdot \text{red}_q(f_h) \left(\rho_q(\gamma \alpha_{\sigma,q}^{-1}) \text{red}_q(\Theta_1) \right),$$

where ω_{cyc} is the ℓ -adic cyclotomic character and $\gamma \in \Gamma_q$ has the property that there are cyclic (in the sense of Definition 2.0.1) lifts $c(g), c(h) \in B_p^\times$ of g and h satisfying $c(g)\beta_{\sigma,q} = c(h)\gamma$ in $R_p^\times \setminus B_p^\times$.

PROOF. For any $g \in \mathcal{T}$ there is an isomorphism of $\Gamma_0(\mathbf{N})$ structures over \mathbb{F}_q^{al}

$$\text{red}_q(E_g^\sigma, \mathbf{C}_g^\sigma) \cong \text{red}_q(E_{g\beta_{\sigma,q}}, \mathbf{C}_{g\beta_{\sigma,q}}). \quad (22)$$

This is exactly the calculation performed in [5, §3.3]. On the other hand, by the parametrization of $Z_0(\mathbf{N})_q$ given in [5, §2.3] there is an isomorphism

$$\text{red}_q(E_{g\beta_{\sigma,q}}, \mathbf{C}_{g\beta_{\sigma,q}}) \cong \text{red}_q(E_h, \mathbf{C}_h) \quad (23)$$

if and only if $g\beta_{\sigma,q}$ and h lie in the same orbit under the right action of Γ_q on \mathcal{T} . This proves the first claim. The proof of the second claim follows from an examination of the isomorphisms (22) and (23), and we give a sketch. The isomorphisms (22) and (23), disregarding the $\Gamma_0(\mathbf{N})$ structure, arise from isomorphisms (again, see [5, §3.3])

$$\text{red}_q(E_g^\sigma) \cong \text{Hom}_R(R \cdot c(g)j(\hat{\sigma}), \text{red}_q(E_1)) \quad (24)$$

$$\text{red}_q(E_h) \cong \text{Hom}_R(R \cdot c(h), \text{red}_q(E_1)) \quad (25)$$

of functors on \mathbb{F}_q^{al} -schemes, where Hom_R means homomorphisms of left R -modules, and $c(g)$ and $c(h)$ are viewed as elements of \hat{B}^\times with trivial components away from p . The map $x \mapsto xb_{\sigma,q}\gamma^{-1}$ induces an isomorphism of left R -submodules of \hat{B}

$$R \cdot c(g)j(\hat{\sigma}) \xrightarrow{b_{\sigma,q}} R \cdot c(g)j(\hat{\sigma})b_{\sigma,q} = R \cdot c(g)\beta_{\sigma,q} = R \cdot c(h)\gamma \xrightarrow{\gamma^{-1}} R \cdot c(h)$$

and so identifies $\text{red}_q(E_g^\sigma) \cong \text{red}_q(E_h)$ and

$$\text{Hom}_{R_\ell}(R_\ell \cdot j(\hat{\sigma})_\ell, \text{Ta}_\ell(\text{red}_q(E_1))) \cong \text{Hom}_{R_\ell}(R_\ell, \text{Ta}_\ell(\text{red}_q(E_1))) \quad (26)$$

By the main theorem of complex multiplication, the isomorphism (24) may be chosen so that the induced isomorphism

$$\mathrm{Ta}_\ell(E_1) \xrightarrow{f_g} \mathrm{Ta}_\ell(E_g) \xrightarrow{\sigma} \mathrm{Ta}_\ell(E_g^\sigma) \cong \mathrm{Hom}_{R_\ell}(R_\ell \cdot j(\hat{\sigma})_\ell, \mathrm{Ta}_\ell(\mathrm{red}_q(E_1)))$$

takes $t \in \mathrm{Ta}_\ell(E_1) \cong \mathrm{Ta}_\ell(\mathrm{red}_q(E_1))$ to the R_ℓ -linear map determined by $j(\hat{\sigma})_\ell \mapsto t$. The isomorphism (26) takes $j(\hat{\sigma})_\ell \mapsto t$ to $\alpha_{\sigma,q} \gamma^{-1} \mapsto t$. Under (25) this latter map corresponds to $\mathrm{red}_q(f_h)(\gamma \alpha_{\sigma,q}^{-1} t) \in \mathrm{Ta}_\ell(\mathrm{red}_q(E_h))$. This shows that the composition

$$\mathrm{Ta}_\ell(E_1) \xrightarrow{f_g} \mathrm{Ta}_\ell(E_g) \xrightarrow{\sigma} \mathrm{Ta}_\ell(E_g^\sigma) \cong \mathrm{Ta}_\ell(\mathrm{red}_q(E_g^\sigma)) \cong \mathrm{Ta}_\ell(\mathrm{red}_q(E_h))$$

is given by $t \mapsto \mathrm{red}_q(f_h)(\gamma \alpha_{\sigma,q}^{-1} t)$. The proposition now follows by taking symmetric powers and twisting by $\Lambda(1-k)$. \square

Corollary 3.2.2 *Let σ and $\beta_{\sigma,q}$ be as in Proposition 3.2.1. For each $h \in \mathcal{T}$ there is a $\varpi_{\sigma,q,h} \in \mathcal{A}_\Lambda(\mathrm{red}_q(E_1))$ with the property that for any $g \in h\Gamma_q \beta_{\sigma,q}^{-1} \subset \mathcal{T}$ there exists an isomorphism of Λ -augmented $\Gamma_0(N)$ structures over $\mathbb{F}_q^{\mathrm{al}}$*

$$\mathrm{red}_q(E_g^\sigma, \mathbf{C}_g^\sigma, \deg(g)^{1-k} \cdot \Theta_g^\sigma) \cong \left(\mathrm{red}_q(E_h), \mathrm{red}_q(\mathbf{C}_h), \mathrm{red}_q(f_h)(\rho_q^*(\gamma) \varpi_{\sigma,q,h}) \right) \quad (27)$$

where $\gamma \in \Gamma_q$ is any element with $g\beta_{\sigma,q} = h\gamma$ in \mathcal{T} .

PROOF. Suppose we have an equality $g = h\gamma\beta_{\sigma,q}^{-1}$ in \mathcal{T} with $g, h \in \mathcal{T}$ and $\gamma \in \Gamma_q$. Fix cyclic lifts $c(g)$ and $c(h)$ of g and h , respectively, to B_p^\times , and choose $\gamma_0 \in \gamma \cdot \mathbb{Z}[1/p]^\times$ so that $c(g)\beta_{\sigma,q} = c(h)\gamma_0$ in $R_p^\times \setminus B_p^\times$. Using Proposition 3.2.1 and the fact that $\rho_q^*(\gamma_0) = \rho_q^*(\gamma)$, one checks directly that (27) holds with

$$\varpi_{\sigma,q,h} = \left(\deg(g) \omega_{\mathrm{cyc}}(\sigma) \det(\gamma_0^{-1}) \det(\alpha_{\sigma,q}) \right)^{1-k} \rho_q^*(\alpha_{\sigma,q}^{-1}) \mathrm{red}_q(\Theta).$$

As $\deg(g) \det(\gamma_0)^{-1} = \deg(h) p^{-\mathrm{ord}_p \det(\beta_{\sigma,q})}$ depends on h but not on g , the same is true of $\varpi_{\sigma,q,h}$. \square

3.3 Vatsal's lemma

Fix a subset $\mathcal{S} \subset \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/K)$. For each $\mathfrak{q} \in \mathfrak{Q}$ and each $\sigma \in \mathcal{S}$ let $\beta_{\sigma,q} \in B_p^\times$ be as in Proposition 3.2.1. The quaternion algebra B depends on \mathfrak{q} , but using the isomorphisms of (21) we identify $B_p^\times \cong \mathrm{GL}_2(\mathbb{Q}_p)$ and view both $\beta_{\sigma,q}$ and Γ_q as living in $\mathrm{GL}_2(\mathbb{Q}_p)$ under this identification.

Lemma 3.3.1 *For each $\mathfrak{q} \in \mathfrak{Q}$ there is a finite index subgroup $\Gamma_q^* \subset \Gamma_q$ containing $\mathbb{Z}[1/p]^\times$ such that $\det(\Gamma_q^*) = p^{\mathbb{Z}}$ and the restriction of ρ_q^* to Γ_q^* is trivial.*

PROOF. Define a subgroup $U = \prod U_r \subset \hat{B}^\times$ by

$$U_r = \begin{cases} \text{Ker}(\rho_{\mathfrak{q}}^* : R_\ell^\times \rightarrow \text{Aut}(\mathcal{A}_\Lambda(\text{red}_{\mathfrak{q}}(E_1)))) & \text{if } r = \ell \\ B_p^\times & \text{if } r = p \\ R_r^\times & \text{else} \end{cases}$$

and let $\Gamma_{\mathfrak{q}}^* = B^\times \cap U \subset \hat{B}^\times$. Then $\Gamma_{\mathfrak{q}}^* \subset \Gamma_{\mathfrak{q}}$ is exactly the kernel of $\rho_{\mathfrak{q}}^*$ restricted to $\Gamma_{\mathfrak{q}}$. We must show that $\Gamma_{\mathfrak{q}}^*$ contains an element of norm p . By [24, Theoreme III.4.1] there is a $b_0 \in B^\times$ of norm p . Let $x = (x_r) \in U$ be an element of norm $p \in \hat{\mathbb{Q}}^\times$. By strong approximation [24, Theoreme III.4.3] the norm one element $b_0^{-1}x \in \hat{B}^\times$ may be written in the form $b_1yu = b_0^{-1}x$ for some norm one elements $b_1 \in B^\times$, $y \in B_p^\times$, and $u \in U$. Then b_0b_1 has norm p and is contained in $\Gamma_{\mathfrak{q}}^*$. \square

Proposition 3.3.2 *For each $\mathfrak{q} \in \mathfrak{Q}$ let $\Gamma_{\mathfrak{q}}^*$ be as in Lemma 3.3.1, and for each $(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}$ set*

$$\Gamma_{\sigma, \mathfrak{q}}^* = \beta_{\sigma, \mathfrak{q}} \Gamma_{\mathfrak{q}}^* \beta_{\sigma, \mathfrak{q}}^{-1} \subset \text{GL}_2(\mathbb{Q}_p).$$

If \mathcal{S} is chaotic then the quotient map $\mathcal{T} \rightarrow \prod_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}} \mathcal{T} / \Gamma_{\sigma, \mathfrak{q}}^$ is surjective.*

PROOF. Let $\tilde{\Gamma}_{\sigma, \mathfrak{q}}^*$ be the image of $\Gamma_{\sigma, \mathfrak{q}}^*$ in $\text{PGL}_2(\mathbb{Q}_p)$ and let $\tilde{\Gamma}_{\sigma, \mathfrak{q}}^{*,1}$ the intersection of $\tilde{\Gamma}_{\sigma, \mathfrak{q}}^*$ with $\text{PSL}_2(\mathbb{Q}_p)$. Then $\tilde{\Gamma}_{\sigma, \mathfrak{q}}^{*,1}$ is discrete and cocompact by [24, p.104], and these subgroups are pairwise non-commensurable as (σ, \mathfrak{q}) varies by [5, Proposition 3.7]. By Vatsal's application of a theorem of Ratner (see [5, Proposition 3.11] or [23, Lemma 5.10]), the natural map

$$\text{PSL}_2(\mathbb{Q}_p) \rightarrow \prod_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}} \text{PSL}_2(\mathbb{Z}_p) \backslash \text{PSL}_2(\mathbb{Q}_p) / \tilde{\Gamma}_{\sigma, \mathfrak{q}}^{*,1}$$

is surjective, and the proposition follows as in [5, Proposition 3.4]. \square

3.4 Surjectivity of the reduction map

Assume $\mathcal{S} \subset \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ is finite and chaotic.

Proposition 3.4.1 *Fix $(\sigma', \mathfrak{q}') \in \mathcal{S} \times \mathfrak{Q}$, $\gamma_0, \gamma_1 \in \Gamma_{\mathfrak{q}}$, and $h \in \mathcal{T}$. There exist $g_0, g_1 \in \mathcal{T}$ such that*

$$\deg(g_0)^{1-k} \text{Red}_{\mathcal{S}, \mathfrak{Q}}(g_0) - \deg(g_1)^{1-k} \text{Red}_{\mathcal{S}, \mathfrak{Q}}(g_1) \in \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}} \mathcal{Z}_{\mathfrak{q}}(M)$$

has trivial components except at the summand $(\sigma, \mathfrak{q}) = (\sigma', \mathfrak{q}')$, at which the component is equal to

$$F_{N,M}^* \left(\text{red}_{\mathfrak{q}}(E_h), \text{red}_{\mathfrak{q}}(\mathbf{C}_h), \text{red}_{\mathfrak{q}}(f_h) \left(\rho_{\mathfrak{q}}^*(\gamma_0) \varpi_{\sigma, \mathfrak{q}, h} - \rho_{\mathfrak{q}}^*(\gamma_1) \varpi_{\sigma, \mathfrak{q}, h} \right) \right),$$

where $\varpi_{\sigma, \mathfrak{q}, h} \in \mathcal{A}_{\Lambda}(\text{red}_{\mathfrak{q}}(E_1))$ is the element of Corollary 3.2.2.

PROOF. For each $i \in \{0, 1\}$ Proposition 3.3.2 allows us to choose a $g_i \in \mathcal{T}$ such that the reduction map $\mathcal{T} \rightarrow \mathcal{T}/\Gamma_{\sigma, \mathfrak{q}}^*$ takes

$$g_i \mapsto \begin{cases} h\gamma_i\beta_{\sigma, \mathfrak{q}}^{-1}\Gamma_{\sigma, \mathfrak{q}}^* = h\gamma_i\Gamma_{\mathfrak{q}}^*\beta_{\sigma, \mathfrak{q}}^{-1} & \text{if } (\sigma, \mathfrak{q}) = (\sigma', \mathfrak{q}') \\ h\beta_{\sigma, \mathfrak{q}}^{-1}\Gamma_{\sigma, \mathfrak{q}}^* = h\Gamma_{\mathfrak{q}}^*\beta_{\sigma, \mathfrak{q}}^{-1} & \text{if } (\sigma, \mathfrak{q}) \neq (\sigma', \mathfrak{q}') \end{cases}$$

for every $(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}$. By Corollary 3.2.2 we have

$$\text{red}_{\mathfrak{q}}(E_{g_0}^{\sigma}, \mathbf{C}_{g_0}^{\sigma}, \text{deg}(g_0)^{1-k}\Theta_{g_0}^{\sigma}) \cong \text{red}_{\mathfrak{q}}(E_{g_1}^{\sigma}, \mathbf{C}_{g_1}^{\sigma}, \text{deg}(g_1)^{1-k}\Theta_{g_1}^{\sigma})$$

as a Λ -augmented $\Gamma_0(\mathbf{N})$ structure over $\mathbb{F}_{\mathfrak{q}}^{\text{al}}$ whenever $(\sigma, \mathfrak{q}) \neq (\sigma', \mathfrak{q}')$, while

$$\text{red}_{\mathfrak{q}}(E_{g_i}^{\sigma}, \mathbf{C}_{g_i}^{\sigma}, \text{deg}(g_i)^{1-k}\Theta_{g_i}^{\sigma}) \cong \left(\text{red}_{\mathfrak{q}}(E_h), \text{red}_{\mathfrak{q}}(\mathbf{C}_h), \text{red}_{\mathfrak{q}}(f_h) (\rho_{\mathfrak{q}}^*(\gamma_i) \varpi_{\sigma, \mathfrak{q}, h}) \right)$$

if $(\sigma, \mathfrak{q}) = (\sigma', \mathfrak{q}')$. The proposition is now immediate from the definition (20) of $\text{Red}_{\mathcal{S}, \mathfrak{Q}}$. \square

Lemma 3.4.2 *Fix $\mathfrak{q} \in \mathfrak{Q}$ and suppose $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. Then $\mathcal{A}_{\Lambda}(\text{red}_{\mathfrak{q}}(E_1))$ has no proper, nonzero Λ -submodules which are stable under $\rho_{\mathfrak{q}}^*(\Gamma_{\mathfrak{q}})$.*

PROOF. Fix a \mathbb{Z}_{ℓ} -basis for the ℓ -adic Tate module of $\text{red}_{\mathfrak{q}}(E_1)$, so that R_{ℓ}^{\times} is identified with $\text{GL}_2(\mathbb{Z}_{\ell})$. Let $\Gamma_{\mathfrak{q}}^1$ and $R_{\ell}^{\times, 1}$ denote the norm one elements of $\Gamma_{\mathfrak{q}}$ and R_{ℓ}^{\times} , respectively. Then $\mathcal{A}_{\Lambda}(\text{red}_{\mathfrak{q}}(E_1))$ is identified with $\text{Sym}^{2k-2}\Lambda^2$ and the action of $\rho_{\mathfrak{q}}^*$ restricted to $\Gamma_{\mathfrak{q}}^1$ is through

$$\Gamma_{\mathfrak{q}}^1 \rightarrow R_{\ell}^{\times, 1} \rightarrow \text{SL}_2(\mathbb{Z}_{\ell}) \rightarrow \text{SL}_2(\Lambda).$$

Using strong approximation [24, Theoreme III.4.3] one may show that the first arrow has dense image, and so the composition is surjective. By the assumption $\ell > 2k - 2$, $\text{Sym}^{2k-2}\Lambda^2$ has no proper, nonzero submodules stable under the action of $\text{SL}_2(\Lambda)$. \square

Theorem 3.4.3 *Let $\mathcal{S} \subset \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ be finite and chaotic, and suppose $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$. Then the simultaneous reduction map (20) is surjective.*

PROOF. Fix $(\sigma', \mathfrak{q}') \in \mathcal{S} \times \Omega$ and a supersingular point $z \in Z_0(\mathbf{N})_{\mathfrak{q}'}$. Let $Z \subset Z_1(N, M)_{\mathfrak{q}'}$ be the set of closed points lying above z . We will show that the image of (20) contains the submodule

$$\mathcal{A}_\Lambda^\circ(Z; Y_1(N, M)_{/\mathbb{F}_q'})^\Delta \subset \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \Omega} \mathcal{A}_\Lambda^\circ(Z_1(N, M)_{\mathfrak{q}}; Y_1(N, M)_{/\mathbb{F}_q})^\Delta \quad (28)$$

supported in the (σ', \mathfrak{q}') component. The parametrization [5, §2.3] shows that the map $\mathcal{T} \mapsto Z_0(\mathbf{N})_{\mathfrak{q}'}$ defined by $h \mapsto \text{red}_{\mathfrak{q}'}(E_h, \mathbf{C}_h)$ establishes a bijection $\mathcal{T}/\Gamma_{\mathfrak{q}'} \cong Z_0(\mathbf{N})_{\mathfrak{q}'}$. Thus we may fix an $h \in \mathcal{T}$ such that the supersingular $\Gamma_0(\mathbf{N})$ structure $\text{red}_{\mathfrak{q}'}(E_h, \mathbf{C}_h)$ corresponds to the point z . For any $g \in \mathcal{T}$, $\text{red}_{\mathfrak{q}'}(\Theta_g^{\sigma'}) \neq 0$ (from the construction one sees that $\Theta_1 \neq 0$, and $\ell \neq p$ implies that $f_g : \mathcal{A}_\Lambda(E_1) \rightarrow \mathcal{A}_\Lambda(E_g)$ is an isomorphism). It follows that $\varpi_{\sigma', \mathfrak{q}', h} \neq 0$. By Lemma 3.4.2 we may choose a $\gamma_1 \in \Gamma_{\mathfrak{q}'}$ such that

$$\pi \stackrel{\text{def}}{=} \rho_{\mathfrak{q}'}^*(\gamma_1) \varpi_{\sigma', \mathfrak{q}', h} - \varpi_{\sigma', \mathfrak{q}', h} \in \mathcal{A}_\Lambda(\text{red}_{\mathfrak{q}'}(E_1))$$

is nonzero. Again by Lemma 3.4.2, choose $\gamma^{(0)}, \dots, \gamma^{(n)} \in \Gamma_{\mathfrak{q}'}$ such that the elements $\rho_{\mathfrak{q}'}^*(\gamma^{(i)})\pi$, $0 \leq i \leq n$, generate $\mathcal{A}_\Lambda(\text{red}_{\mathfrak{q}'}(E_1))$. Set $\gamma_0^{(i)} = \gamma^{(i)}\gamma_1$ and let $g_0^{(i)}, g_1$ be as in Proposition 3.4.1, so that

$$\deg(g_0^{(i)})^{1-k} \text{Red}_{\mathcal{S}, \Omega}(g_0^{(i)}) - \deg(g_1)^{1-k} \text{Red}_{\mathcal{S}, \Omega}(g_1)$$

has trivial components except at the summand $(\sigma, \mathfrak{q}) = (\sigma', \mathfrak{q}')$, where the component is equal to

$$F_{N, M}^*(\text{red}_{\mathfrak{q}'}(E_h), \text{red}_{\mathfrak{q}'}(\mathbf{C}_h), \text{red}_{\mathfrak{q}'}(f_h)(\rho_{\mathfrak{q}'}^*(\gamma^{(i)})\pi)). \quad (29)$$

As i varies the elements $\text{red}_{\mathfrak{q}'}(f_h)(\rho_{\mathfrak{q}'}^*(\gamma^{(i)})\pi)$ generate $\mathcal{A}_\Lambda(\text{red}_{\mathfrak{q}'}(E_h))$, and the elements (29) generate the submodule (28). \square

4 Augmented theorems of Deuring, Ihara, and Ribet

Let $q \nmid N$ be a rational prime and let \mathbb{F} ($= F$ when we refer to the notions of §1) be a field of q^2 elements with algebraic closure \mathbb{F}^{al} . Unless specified otherwise, all geometric objects (e.g. $Y_1(N)$, $Y_1(N, M)$, ...) are defined over $\text{Spec}(\mathbb{F})$. Let $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ for some prime ℓ and assume that ℓ does not divide Nq . Let \mathcal{L}_Λ be the locally constant constructible sheaf on $Y_1(N)$ defined by (2). Denote by

$$Z_1(N) \subset Y_1(N) \quad Z_1(N, M) \subset Y_1(N, M)$$

the subsets of supersingular closed points.

4.1 Fields of moduli

We need a slight generalization of the well-known theorem of Deuring that all supersingular points on $Y_0(N)$ have residue degree one.

Lemma 4.1.1 *Let E be a supersingular elliptic curve over \mathbb{F}^{al} , let $C \subset E[N]$ be a cyclic subgroup of order N , and let Θ be any element of $\mathcal{A}_\Lambda(E)$. The field of moduli of the Λ -augmented $\Gamma_0(N)$ structure (E, C, Θ) is \mathbb{F} .*

PROOF. As E is supersingular, its j -invariant lies in \mathbb{F} . Let A be an elliptic curve over \mathbb{F} with the same j -invariant as E , and let $\text{Fr} \in \text{End}_{\mathbb{F}}(A)$ be the degree q^2 (relative) Frobenius. If $\text{Fr} \in \mathbb{Z}$, then Fr commutes with all elements of $\text{End}_{\mathbb{F}^{\text{al}}}(A)$, and so

$$\text{End}_{\mathbb{F}}(A) = \text{End}_{\mathbb{F}^{\text{al}}}(A). \quad (30)$$

If $\text{Fr} \notin \mathbb{Z}$ then Fr generates a quadratic imaginary subfield L of the definite quaternion algebra (ramified exactly at q and ∞) $\text{End}_{\mathbb{F}^{\text{al}}}(A) \otimes \mathbb{Q}$, and q is nonsplit in L . As Fr has degree q^2 we must have $\text{Fr} = \zeta^{-1}q$ for some root of unity $\zeta \in L$, and in fact ζ belongs to $L \cap \text{End}_{\mathbb{F}}(A)$ (this follows from the fact [12, Corollary 12.3.5] that Fr and $[q]$ have the same scheme-theoretic kernel, and so there is a factorization $[q] = \zeta \circ \text{Fr}$ for some automorphism ζ of A). Replacing A by its twisted form corresponding to the cocycle sending the relative Frobenius $\sigma \in \text{Gal}(\mathbb{F}^{\text{al}}/\mathbb{F})$ to $\zeta \in \text{Aut}_{\mathbb{F}}(E)$, a simple calculation shows that (30) holds. Then Fr is a central element of $\text{End}_{\mathbb{F}^{\text{al}}}(A)$, and so $\text{Fr} = [\pm q]$.

With this choice of A , $\text{Gal}(\mathbb{F}^{\text{al}}/\mathbb{F})$ acts trivially on $\mathcal{A}_\Lambda(A)$ and the triple (A, C_A, Θ_A) is defined over \mathbb{F} for *any* cyclic order N subgroup $C_A \subset A(\mathbb{F}^{\text{al}})$ and *any* $\Theta_A \in \mathcal{A}_\Lambda(A)$. Over \mathbb{F}^{al} we may fix an isomorphism $f : E \cong A$ and set $C_A = f(C)$ and $\Theta_A = f(\Theta)$. Then (E, C, Θ) and (A, C_A, Θ_A) are isomorphic (over \mathbb{F}^{al}) and so have the same field of moduli \mathbb{F} . \square

4.2 Ihara's theorem

We now recall a theorem of Ihara [10] and derive some consequences; our exposition of Ihara's theorem is influenced by the discussion of [4, Chaptire 7]. For each integer m prime to q set

$$\underline{\mu}_m^* = \text{Spec}(\mathbb{F}[X]/\Phi_m(X)) \quad \underline{\mu}_m = \text{Spec}(\mathbb{F}[X]/(X^m - 1)),$$

where $\Phi_m(X)$ is the m^{th} cyclotomic polynomial. Let $Y(m)$ be the affine modular curve classifying “naive” level m structures in the sense of [12] on elliptic curves over \mathbb{F} -schemes. Thus $Y(m)$ is a fine moduli space if $m > 2$, and for

all m (prime to q) the Weil pairing provides a canonical map $Y(m) \rightarrow \underline{\mu}_m^*$ of \mathbb{F} -schemes. Fix a topological generator

$$\zeta = \varprojlim_{(m,q)=1} \zeta_m \in \varprojlim_{(m,q)=1} \underline{\mu}_m(\mathbb{F}^{\text{al}}).$$

For each m there is a map $\text{Spec}(\mathbb{F}[\zeta_m]) \rightarrow \underline{\mu}_m^*$ determined by the map $X \mapsto \zeta_m$ on \mathbb{F} -algebras. Define

$$Y_\zeta(m) = Y(m) \times_{\underline{\mu}_m^*} \text{Spec}(\mathbb{F}[\zeta_m]),$$

a smooth curve over \mathbb{F} (geometrically disconnected unless $m \mid q^2 - 1$).

The subgroup $G_\zeta(m) \subset G(m) = \text{GL}_2(\mathbb{Z}/m\mathbb{Z})/\{\pm 1\}$ defined by

$$G_\zeta(m) = \{A \in G(m) \mid \det(A) \in q^{2\mathbb{Z}} \subset (\mathbb{Z}/m\mathbb{Z})^\times\}$$

acts on both $Y_\zeta(m)$ and $\text{Spec}(\mathbb{F}[\zeta_m])$, and the actions are compatible with the structure map $Y_\zeta(m) \rightarrow \text{Spec}(\mathbb{F}[\zeta_m])$. Set $G^1(m) = \text{PSL}(\mathbb{Z}/m\mathbb{Z})$, let

$$\Gamma_0(m) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset G(m) \quad \Gamma_1(m) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\} \subset \Gamma_0(m)$$

be the habitual congruence subgroups, and let $\Gamma_{\text{Ih}}(m) \subset G(m)$ be the center. For $*$ $\in \{0, 1, \text{Ih}\}$ let $Y_*(m)$ be the quotient of $Y_\zeta(m)$ by the action of

$$\Gamma_*(m) \cap G_\zeta(m).$$

The function field of the curve $Y_{\text{Ih}}(m)$ is the field denoted K_m in [10], and there is a canonical isomorphism of curves over \mathbb{F}^{al} .

$$Y_{\text{Ih}}(m) \times_{\text{Spec}(\mathbb{F})} \text{Spec}(\mathbb{F}^{\text{al}}) \cong Y_\zeta(m) \times_{\text{Spec}(\mathbb{F}[\zeta_m])} \text{Spec}(\mathbb{F}^{\text{al}}).$$

Denote by $K_*(m)$ the function field of $Y_*(m)$ for $*$ $\in \{0, 1, \zeta, \text{Ih}\}$ or for $*$ equal to the empty character, and view these as subfields of some fixed separable closure $K(1)^{\text{sep}}$. Define a Λ -vector space $L_\Lambda = \text{Sym}^{2k-2} \Lambda^2$ and endow L_Λ with an action of $\text{Gal}(K(1)^{\text{sep}}/K(1))$ via

$$\text{Gal}(K(1)^{\text{sep}}/K(1)) \rightarrow \text{Gal}(K_\zeta(\ell)/K(1)) \cong G_\zeta(\ell) \subset \text{GL}_2(\Lambda)/\{\pm 1\}.$$

Remark 4.2.1 *If the action of $\text{GL}_2(\Lambda)$ on L_Λ is twisted by \det , then the Galois action is twisted by the cyclotomic character. In particular $\Gamma_{\text{Ih}}(\ell)$ acts trivially on $L_\Lambda \otimes \det^{1-k}$, and so the Galois action on $L_\Lambda(1-k)$ factors through*

$$\text{Gal}(K(1)^{\text{sep}}/K(1)) \rightarrow \text{Gal}(K_{\text{Ih}}(\ell)/K(1)).$$

Under the bijection between locally constant étale sheaves on $Y_1(N)$ and modules for the absolute Galois group of $K_1(N)$ which are unramified outside of the cusps, $\mathcal{L}_\Lambda(2k-2)$ corresponds to L_Λ .

Definition 4.2.2 *If $M/K(1)$ is a separable extension, a cusp of M is a place lying above the place $J = \infty$ of $K(1)$. A supersingular prime of M is a place lying above a place $J = j$ of $K(1)$ with $j \in \mathbb{F}$ a supersingular j -invariant.*

Theorem 4.2.3 (Ihara) *For any $m > 1$ with $(m, q) = 1$, $K_{\text{Ih}}(m)$ has no non-trivial everywhere unramified extensions in which all supersingular primes are split completely. Furthermore, $K_{\text{Ih}}(\infty) = \cup_{(r,q)=1} K_{\text{Ih}}(r)$ is the maximal Galois extension of $K_{\text{Ih}}(m)$ satisfying*

- (a) *it is tamely ramified, and unramified outside the cusps of $K_{\text{Ih}}(m)$,*
- (b) *the supersingular primes of $K_{\text{Ih}}(m)$ are split completely in $K_{\text{Ih}}(\infty)$.*

PROOF. This is the main result of [10]. \square

Corollary 4.2.4 *Let $M_{\text{Ih}}(N) \supset K_{\text{Ih}}(\infty)$ be the maximal separable extension of $K_{\text{Ih}}(N)$ unramified away from the cusps. The restriction map on Galois cohomology*

$$H^1(M_{\text{Ih}}(N)/K_{\text{Ih}}(N), L_{\Lambda}(1-k)) \rightarrow \left(\bigoplus_v H^1(K_{\text{Ih}}(N)_v, L_{\Lambda}(1-k)) \right) \oplus \left(\bigoplus_w H^1(K_{\text{Ih}}(N)_w^{\text{unr}}, L_{\Lambda}(1-k)) \right) \quad (31)$$

is injective. Here the sum over v is over all supersingular primes, the sum over w is over all cusps, and the superscript unr denotes maximal unramified extension.

PROOF. First consider the restriction map (note Remark 4.2.1)

$$H^1(M_{\text{Ih}}(N)/K_{\text{Ih}}(N\ell), L_{\Lambda}(1-k)) \rightarrow \bigoplus_v \text{Hom}(H_v, L_{\Lambda}(1-k)) \quad (32)$$

where the sum is over all supersingular primes and all cusps, and $H_v \subset \text{Gal}(M_{\text{Ih}}(N)/K_{\text{Ih}}(N))$ is either the decomposition group or inertia group of a fixed place of $M_{\text{Ih}}(N)$ above v , according as v is supersingular or a cusp. Any homomorphism from $\text{Gal}(M_{\text{Ih}}(N)/K_{\text{Ih}}(N\ell))$ to $L_{\Lambda}(1-k)$ which vanishes on all H_w factors through $\text{Gal}(\Phi/K_{\text{Ih}}(N\ell))$ where Φ is the maximal Galois extension of $K_{\text{Ih}}(N\ell)$ which is everywhere unramified and in which all supersingular primes split completely. By Theorem 4.2.3 $\Phi = K_{\text{Ih}}(N\ell)$, and so the map (32) is injective. Thus any class in the kernel of (31) also lies in the kernel of restriction

$$H^1(M_{\text{Ih}}(N)/K_{\text{Ih}}(N), L_{\Lambda}(1-k)) \rightarrow H^1(M_{\text{Ih}}(N)/K_{\text{Ih}}(N\ell), L_{\Lambda}(1-k)),$$

and so is in the image of the inflation map

$$H^1(K_{\text{Ih}}(N\ell)/K_{\text{Ih}}(N), L_\Lambda(1-k)) \rightarrow H^1(M_{\text{Ih}}(N)/K_{\text{Ih}}(N), L_\Lambda(1-k)) \quad (33)$$

and is unramified at the cusps. The inertia subgroup in $\text{Gal}(K_{\text{Ih}}(N\ell)/K_{\text{Ih}}(N))$ of the cusp ∞ is an ℓ -Sylow subgroup (this follows from [10, p. 167]), and so any element in the image of (33) which is unramified at the cusps is trivial by [22, Theorem IX.2.4]. \square

Proposition 4.2.5 *Let $j : Y_1(N) \hookrightarrow X_1(N)$ be the usual compactification and assume $\ell \nmid \varphi(N)$. The natural map*

$$H_{Z_1(N)}^2(Y_1(N), \mathcal{L}_\Lambda(k))^\Delta \rightarrow H^2(X_1(N), j_*\mathcal{L}_\Lambda(k))^\Delta \quad (34)$$

is surjective.

PROOF. Let $i : C_1(N) \hookrightarrow X_1(N)$ denote the subscheme of cusps, i.e. the complement of $Y_1(N)$ in $X_1(N)$. From the exact sequence of sheaves on $X_1(N)$

$$0 \rightarrow j_*\mathcal{L}_\Lambda \rightarrow j_*\mathcal{L}_\Lambda \rightarrow i_*i^*j_*\mathcal{L}_\Lambda \rightarrow 0$$

and [13, Proposition II.2.3] we obtain the exact sequence

$$H^1(C_1(N), i^*j_*\mathcal{L}_\Lambda(k)) \rightarrow H_c^2(Y_1(N), \mathcal{L}_\Lambda(k)) \rightarrow H^2(X_1(N), j_*\mathcal{L}_\Lambda(k)) \rightarrow 0$$

in which the terminating zero is justified by the observation that closed points on $X_1(N)$, having finite residue field, have cohomological dimension 1. It therefore suffices to prove the surjectivity of

$$H_{Z_1(N)}^2(Y_1(N), \mathcal{L}_\Lambda(k))^\Delta \oplus H^1(C_1(N), i^*j_*\mathcal{L}_\Lambda(k))^\Delta \rightarrow H_c^2(Y_1(N), \mathcal{L}_\Lambda(k))^\Delta. \quad (35)$$

We take Λ -duals and translate the problem into the language of Galois cohomology.

Let $M_1(N)$ denote the maximal extension of $K_1(N)$ unramified outside the cusps. By [14, Corollary II.4.13(c)] there is an isomorphism

$$H_c^2(Y_1(N), \mathcal{L}_\Lambda(k)) \cong H^1(M_1(N)/K_1(N), L_\Lambda(1-k))^\vee$$

in which the superscript \vee denotes Λ -dual. If we let U denote the open complement of $Z_1(N)$ in $Y_1(N)$ then the pairing of [14, Corollary II.3.3] identifies the exact sequence [14, Proposition II.2.3(d)]

$$H_c^r(U, \mathcal{L}_\Lambda(k-1)) \rightarrow H_c^r(Y_1(N), \mathcal{L}_\Lambda(k-1)) \rightarrow \bigoplus_{z \in Z_1(N)} H^r(z, i_z^*\mathcal{L}_\Lambda(k-1))$$

with the dual of the relative cohomology sequence

$$H^{3-r}(U, \mathcal{L}_\Lambda(k)) \leftarrow H^{3-r}(Y_1(N), \mathcal{L}_\Lambda(k)) \leftarrow H_{Z_1(N)}^{3-r}(Y_1(N), \mathcal{L}_\Lambda(k)).$$

This gives the first isomorphism of

$$\begin{aligned} H_{Z_1(N)}^2(Y_1(N), \mathcal{L}_\Lambda(k))^\vee &\cong \bigoplus_{z \in Z_1(N)} H^1(z, i_z^* \mathcal{L}_\Lambda(k-1)) \\ &\cong \bigoplus_v H^1(D_v/I_v, L_\Lambda(1-k)), \end{aligned}$$

in which the second sum is over all supersingular primes and $I_v \subset D_v$ are the inertia and decomposition subgroups in $\text{Gal}(M_1(N)/K_1(N))$ of some choice of place above v . Finally local duality gives the second isomorphism of

$$\begin{aligned} H^1(C_1(N), i_* j_* \mathcal{L}_\Lambda(k))^\vee &\cong \bigoplus_w H^1(D_w/I_w, L_\Lambda(2-k)^{I_w})^\vee \\ &\cong \bigoplus_w H^1(I_w, L_\Lambda(1-k))^{D_w/I_w} \end{aligned}$$

where both sums are over all cusps. Thus the cokernel of (35) is isomorphic to the kernel of

$$\begin{aligned} H^1(M_1(N)/K_1(N), L_\Lambda(1-k))^\Delta \rightarrow & \quad (36) \\ \left(\bigoplus_v H^1(K_1(N)_v, L_\Lambda(1-k)) \right) \oplus \left(\bigoplus_w H^1(K_1(N)_w^{\text{unr}}, L_\Lambda(1-k)) \right) \end{aligned}$$

where again the v 's range over supersingular primes and the w range over cusps.

As we assume that ℓ is prime to $\varphi(N)$, the inflation-restriction sequence identifies the kernel of (36) with the kernel of

$$\begin{aligned} H^1(M_1(N)/K_0(N), L_\Lambda(1-k)) \rightarrow & \quad (37) \\ \left(\bigoplus_v H^1(K_0(N)_v, L_\Lambda(1-k)) \right) \oplus \left(\bigoplus_w H^1(K_0(N)_w^{\text{unr}}, L_\Lambda(1-k)) \right). \end{aligned}$$

The fields $K_1(N)$ and $K_{\text{th}}(N)$ have a common extension which is unramified outside the cusps (namely $K_\zeta(N)$) and so $M_1(N) = M_{\text{th}}(N)$. We may therefore consider the restriction map

$$H^1(M_1(N)/K_0(N), L_\Lambda(1-k)) \rightarrow H^1(M_{\text{th}}(N)/K_{\text{th}}(N), L_\Lambda(1-k)),$$

which is injective as $K_{\text{th}}(N)$ and $K_{\text{th}}(\ell)$ are linearly disjoint over $K(1)$, so that $L_\Lambda(1-k)$ has no $\text{Gal}(K(1)^{\text{sep}}/K_{\text{th}}(N))$ invariants. The kernel of (37) therefore

injects into the kernel of (31), which is trivial by Corollary 4.2.4. Thus (36) is injective and the proposition is proved. \square

The following is our analogue of [5, Proposition 4.4].

Corollary 4.2.6 *Assume $\ell \nmid \varphi(N)$. The Λ -augmented Kummer map*

$$\mathcal{A}_\Lambda^\circ(Z_1(N); Y_1(N))^\Delta \rightarrow H^1(\mathbb{F}^{\text{al}}/\mathbb{F}, \tilde{H}^1(Y_1(N)_{/\mathbb{F}^{\text{al}}}, \mathcal{L}_\Lambda(k))^\Delta)$$

of Definition 1.3.3 is surjective.

PROOF. Lemma 1.3.1 gives isomorphisms

$$\mathcal{A}_\Lambda^\circ(Z_1(N); Y_1(N)) \cong \bigoplus_{z \in Z_1(N)} H_z^2(Y_1(N), \mathcal{L}_\Lambda(k)) \cong H_{Z_1(N)}^2(Y_1(N), \mathcal{L}_\Lambda(k))$$

which restrict to isomorphisms of Δ -invariants. The claim is now immediate from Lemma 1.3.2 and Proposition 4.2.5. \square

4.3 Degeneracy maps on supersingular points

Suppose $M = rM'$ for a prime r . The following theorem and its proof are based on work of Ribet [19, Theorem 3.15].

Proposition 4.3.1 *Assume $\ell \nmid \varphi(N)$ and $\ell > 2k - 2$, and abbreviate*

$$\mathcal{Z}(M) = \mathcal{A}_\Lambda^\circ(Z_1(N, M); Y_1(N, M))^\Delta$$

and similarly for M' . The sum of the degeneracy maps of §1.6

$$\alpha_{M'}^M \oplus \beta_{M'}^M : \mathcal{Z}(M) \rightarrow \mathcal{Z}(M') \oplus \mathcal{Z}(M')$$

is surjective.

PROOF. Let N_Δ be the norm element in the group algebra $\Lambda[\Delta]$. Suppose we are given a Λ -augmented $\Gamma_1(N, M')$ structure over \mathbb{F}^{al}

$$(E, x, \Theta) \in \mathcal{A}_\Lambda(\Gamma_1(N, M'))$$

with E supersingular, and a degree r^{2n} endomorphism $f : E \rightarrow E$ preserving the $\Gamma_0(NM')$ structure underlying x . Factor the endomorphism $f : E \rightarrow E$ as

$$E = E_0 \xrightarrow{h_1} E_1 \xrightarrow{h_2} \dots \xrightarrow{h_{2n-1}} E_{2n-1} \xrightarrow{h_{2n}} E_{2n} = E$$

with each h_i of degree r . Set $f_i = h_i \circ \cdots \circ h_1 : E \rightarrow E_i$ and let $x_i = f_i(x)$ be the induced $\Gamma_1(N, M')$ structure on E_i . For $i < 2n$ let y_i be the $\Gamma_1(N, M)$ structure on E_i obtained by adding the $\Gamma_0(r)$ structure $\ker(h_{i+1})$ to x_i , and for $i > 0$ let y_i^\vee be the $\Gamma_1(N, M)$ structure obtained by adding the $\Gamma_0(r)$ structure $\ker(h_i^\vee)$. Define

$$\Theta_i = r^{i(1-k)} f_i(\Theta) \in \mathcal{A}_\Lambda(E_i).$$

A simple calculation of the degeneracy maps of §1.6 shows that the element

$$T = T_{E,x,f,\Theta} \in \mathcal{A}_\Lambda(\Gamma_1(N, M))$$

defined by

$$T = N_\Delta \left[(E_0, y_0, \Theta_0) - (E_2, y_2^\vee, \Theta_2) + (E_2, y_2, \Theta_2) - (E_4, y_4^\vee, \Theta_4) + \right. \\ \left. \cdots + (E_{2n-2}, y_{2n-2}, \Theta_{2n-2}) - (E_{2n}, y_{2n}^\vee, \Theta_{2n}) \right]$$

satisfies $\beta_{M'}^M(T) = 0$ and

$$\alpha_{M'}^M(T) = N_\Delta \cdot (E, x, \Theta - r^{2n(1-k)} f(\Theta)).$$

It follows from Lemma 4.1.1 (with N replaced by $\mathbf{N} = NM$) that T is fixed by the action of $\text{Gal}(\mathbb{F}^{\text{al}}/\mathbb{F})$, and so defines an element of $\mathcal{Z}(M)$.

We pause for a

Lemma 4.3.2 *With (E, x) as above, let D denote the $\Gamma_0(NM')$ structure underlying the $\Gamma_1(N, M')$ structure x . The Λ -module $\mathcal{A}_\Lambda(E)$ has a set of generators $A_{E,x}$ such that each $a \in A_{E,x}$ has the form $a = \Theta_a - \deg(f_a)^{(1-k)} f_a(\Theta_a)$ for some $\Theta_a \in \mathcal{A}_\Lambda(E)$ and some endomorphism $f_a : E \rightarrow E$ such that $f_a(D) = D$ and $\deg(f_a)$ is an even power of r .*

PROOF. Set $R = \text{End}_{\mathbb{F}^{\text{al}}}(E, D)$, a level N Eichler order in a quaternion algebra ramified exactly at q and ∞ , and let $\Gamma = R[1/r]^\times$. Let ρ denote the natural action of R on $\mathcal{A}_\Lambda(E)$, extend ρ to an action of Γ (recall $\ell \nmid M$ so that $r \neq \ell$), and let $\rho^* = \rho \otimes \det^{1-k}$ be the twist such that $\mathbb{Z}[1/r]^\times \subset \Gamma$ acts trivially. All of this notation is exactly as in §3.2 with p replaced by r . As in the proof of Lemma 3.4.2, $\mathcal{A}_\Lambda(E)$ has no submodules stable under the restriction of ρ^* to the subgroup of norm one elements $\Gamma^1 \subset \Gamma$. As the set

$$A_{E,x} = \{\Theta - \rho^*(\gamma)\Theta \mid \Theta \in \mathcal{A}_\Lambda(E), \gamma \in \Gamma^1\}$$

is stable under the action of $\rho^*(\Gamma^1)$, it must generate $\mathcal{A}_\Lambda(E)$. For each

$$\Theta - \rho^*(\gamma)\Theta \in A_{E,x},$$

let $f = r^n \gamma$ for n large enough that $r^n \gamma \in R$. Then f has degree r^{2n} and

$$\Theta - \deg(f)^{1-k} f(\Theta) = \Theta - \rho^*(f)\Theta = \Theta - \rho^*(\gamma)\Theta,$$

so that $A_{E,x}$ has the desired properties. \square

If we let E vary over all supersingular elliptic curves over \mathbb{F}^{al} , x vary over all $\Gamma_1(N, M)$ structures on E , and Θ' vary over the set $A_{E,x}$ of Lemma 4.3.2, the elements

$$N_\Delta \cdot (E, x, \Theta') \in \mathcal{A}_\Delta(\Gamma_1(N, M'))$$

generate the submodule $\mathcal{Z}(M')$. Hence, by the construction of $T_{E,x,f,\Theta}$ above, there is a family $\{T_i\} \subset \mathcal{Z}(M)$ such that $\beta_{M'}^M(T_i) = 0$ for all i and such that $\{\alpha_{M'}^M(T_i)\}$ generates $\mathcal{Z}(M')$. A construction similar to that of T produces a family with the same properties but with the roles of α and β reversed, completing the proof of Proposition 4.3.1. \square

5 Nonvanishing of Heegner classes

Keep K , E_1 , \mathfrak{N} , and $\text{Ta}_p(E_1) \cong \mathbb{Z}_p^2$ as in §2.1, so that K is an imaginary quadratic field in which the prime divisors of N are split, $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$, and E_1 is an elliptic curve over \mathbb{Q}^{al} with complex multiplication by \mathcal{O}_K . Let $D = \text{disc}(K)$ and let $H[p^s]$, \mathcal{G} , and G_0 be as in §0.1. Let $f \in S_{2k}(\Gamma_0(N), \mathbb{C})$, Φ , χ , and π_χ also be as in §0.1. Let \mathbf{T} be the \mathbb{Z} -algebra generated by the Hecke operators $\{T_m \mid (m, N) = 1\}$ and the group of diamond operators Δ acting on $S_{2k}(\Gamma_1(N), \mathbb{C})$, so that f determines an idempotent π_f in the semi-simple Φ -algebra $\mathbf{T} \otimes_{\mathbb{Z}} \Phi$.

5.1 Heegner cohomology classes

For each finite quotient Λ of \mathbb{Z}_ℓ we have the $\text{Gal}(\mathbb{Q}^{\text{al}}/K)$ -module W_Λ of (15) and, for each $s \geq 0$, the family of cohomology classes $\Omega_s(g)$ of (17) parametrized by $g \in \mathcal{T}_s \subset \mathcal{T}$. If we set $W_{\mathbb{Z}_\ell} = \varprojlim W_{\mathbb{Z}/\ell^e \mathbb{Z}}$, then the classes $\Omega_s(g)$ are compatible as $\Lambda = \mathbb{Z}/\ell^e \mathbb{Z}$ varies, and define classes

$$\Omega_s(g) \in H^1(\mathbb{Q}^{\text{al}}/H[p^s], W_{\mathbb{Z}_\ell}(k))^\Delta,$$

and also classes (denoted the same way) in the cohomology of $W_\Lambda = W_{\mathbb{Z}_\ell} \otimes \Lambda$ for any \mathbb{Z}_ℓ -algebra Λ . Other constructions made with $\Lambda = \mathbb{Z}/\ell^e \mathbb{Z}$ extend to any \mathbb{Z}_ℓ -algebra Λ in the same way. We denote by

$$\text{Heeg}_s \subset H^1(\mathbb{Q}^{\text{al}}/H[p^s], W_\Phi(k))^\Delta$$

the Φ -submodule generated by the classes $\Omega_s(g)$ as g ranges over \mathcal{T}_s . By a well known theorem of Deligne, the Hecke algebra $\mathbf{T} \otimes_{\mathbb{Z}} \Phi$ acts on W_Φ , and the Galois representation $W_f = \pi_f W_\Phi$ is a two dimensional Φ -vector space. Set

$$\text{Heeg}_s(f) = \pi_f \text{Heeg}_s.$$

Theorem 5.1.1 *Fix a character $\chi : G_0 \rightarrow \Phi^\times$ and let π_χ be as in §0.1. Suppose ℓ does not divide p , N , $\varphi(N)$, $\text{disc}(K)$, or $(2k-2)!$ As s grows the Φ -dimension of $\pi_\chi \text{Heeg}_s(f)$ grows without bound.*

PROOF. Let r_1, r_2, \dots be the prime divisors of D and let \mathfrak{r}_i denote the unique prime of K above r_i . Let $G_1 \subset \mathcal{G}$ be the subgroup generated by the Frobenius classes of the \mathfrak{r}_i , so that G_1 has exponent 2, and in particular $G_1 \subset G_0$. Reordering the r_i if needed, choose n such that the Frobenius classes of $\mathfrak{r}_1, \dots, \mathfrak{r}_n$ form a basis for the $\mathbb{Z}/2\mathbb{Z}$ -vector space G_1 . Set $M = r_1 \cdots r_n$, so that divisors of M are naturally in bijection with the elements of G_1 . We denote this bijection by $d \mapsto \sigma_d$. Set $\mathfrak{M} = \mathfrak{r}_1 \cdots \mathfrak{r}_k$. For each $\sigma \in G_1$ fix once and for all an extension of σ to $\text{Gal}(\mathbb{Q}^{\text{al}}/K)$, and let \mathcal{S}_1 denote the set of extensions so chosen. Let $\mathcal{S}_0 \subset \text{Gal}(\mathbb{Q}^{\text{al}}/K)$ be chosen so that restriction to $H[p^\infty]$ takes \mathcal{S}_0 injectively into G_0 with image equal to a set of representatives for the cosets G_0/G_1 . Let $\mathcal{S} = \{\sigma\tau \mid \sigma \in \mathcal{S}_1, \tau \in \mathcal{S}_0\}$.

As in §3, let Ω be a finite set of rational primes, all inert in K and all prime to $\ell p \mathbf{N}$, and fix extensions of these places to \mathbb{Q}^{al} . We will continue our practice of writing $\mathfrak{q} \in \Omega$ to indicate that \mathfrak{q} is the prime of K above the rational prime $q \in \Omega$. For each $\mathfrak{q} \in \Omega$ define, using the notation (19), $\lambda_d : \mathcal{Z}_\mathfrak{q}(M) \rightarrow \mathcal{Z}_\mathfrak{q}(1)$ by $\lambda_d = d^{1-k} \cdot (\beta_1^d \circ \alpha_d^M)$ where α_d^M and β_1^d are the degeneracy maps of §1.6. Consider the composition

$$(\mathbb{Z}/\ell\mathbb{Z})[\mathcal{T}] \rightarrow \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S}_0 \times \Omega} \mathcal{Z}_\mathfrak{q}(M) \xrightarrow{\bigoplus_{d|M} \lambda_d} \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S}_0 \times \Omega} \bigoplus_{d|M} \mathcal{Z}_\mathfrak{q}(1) \rightarrow \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \Omega} \mathcal{Z}_\mathfrak{q}(1) \quad (38)$$

in which the first arrow is the map $\text{Red}_{\mathcal{S}_0, \Omega}$ of (20), and the final arrow rearranges the sum, taking the summand $(\sigma, \mathfrak{q}, d)$ to the summand $(\sigma_d \sigma, \mathfrak{q})$.

Lemma 5.1.2 *The composition (38) is surjective, and is equal to the simultaneous reduction map (20) defined with $M = 1$.*

PROOF. By [5, Lemma 4.5] the set \mathcal{S}_0 is chaotic in the sense of Definition 3.0.1, and so Theorem 3.4.3 gives the surjectivity of $\text{Red}_{\mathcal{S}_0, \Omega}$. The surjectivity of $\bigoplus_{d|M} \lambda_d$ is an easy induction using Proposition 4.3.1.

Fix $g \in \mathcal{T}$ and let A be an elliptic curve over \mathbb{Q}^{al} with complex multiplication by $\mathcal{O}_g \subset K$. If $d|M$, let \mathfrak{d} be the unique \mathcal{O}_g -ideal of norm d and set $A' = A/A[\mathfrak{d}]$.

The main theorem of complex multiplication provides an isomorphism $A' \cong A^{\sigma_d}$ such that the composition $A \rightarrow A' \cong A^{\sigma_d}$ agrees with $P \mapsto P^{\sigma_d}$ for all torsion points $P \in A(\mathbb{Q}^{\text{al}})$ of order prime to d . Thus

$$\lambda_d(E_g^\sigma, \mathbf{C}_g^\sigma, \Theta_g^\sigma) = (E_g^\sigma, C_g^\sigma, \Theta_g^\sigma)^{\sigma_d}$$

for any $\sigma \in \text{Gal}(\mathbb{Q}^{\text{al}}/K)$, and the lemma follows. \square

For each $(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}$ and each $g \in \mathcal{T}_s$ the cohomology class $\Omega_s(g)$ is unramified at \mathfrak{q} , and, since the residue field of $H[p^s]$ at \mathfrak{q} is $\mathbb{F}_{\mathfrak{q}}$, the localization of $\Omega_s(g)$ at \mathfrak{q} defines a class

$$\text{loc}_{\sigma, \mathfrak{q}}(g) \in H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, W_{\mathbb{Z}_\ell}(k))^\Delta.$$

Summing over all $(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}$ and extending linearly to the free \mathbb{Z}_ℓ -module on \mathcal{T}_s defines

$$\text{loc}_{\mathcal{S}, \mathfrak{Q}} : \mathbb{Z}_\ell[\mathcal{T}_s] \rightarrow \bigoplus_{(\sigma, \mathfrak{q}) \in \mathcal{S} \times \mathfrak{Q}} H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, W_{\mathbb{Z}_\ell}(k))^\Delta.$$

This map is compatible with the natural inclusions as s varies. Proposition 1.5.1 gives the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_\ell[\mathcal{T}_s] & \xrightarrow{\text{loc}_{\mathcal{S}, \mathfrak{Q}}} & \bigoplus H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, W_{\mathbb{Z}_\ell}(k))^\Delta \\ \text{Red}_{\mathcal{S}, \mathfrak{Q}} \downarrow & & \downarrow \\ \bigoplus \mathcal{A}_{\mathbb{Z}/\ell\mathbb{Z}}^\circ(Z_1(N)_{\mathfrak{q}}; Y_1(N)_{/\mathbb{F}_{\mathfrak{q}}})^\Delta & \longrightarrow & \bigoplus H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, W_{\mathbb{Z}/\ell\mathbb{Z}}(k))^\Delta \end{array} \quad (39)$$

where all sums are over $\mathcal{S} \times \mathfrak{Q}$, $\text{Red}_{\mathcal{S}, \mathfrak{Q}}$ is the restriction of the simultaneous reduction map (20), with $M = 1$, to $\mathbb{Z}_\ell[\mathcal{T}_s]$, and the bottom horizontal arrow is the $\mathbb{Z}/\ell\mathbb{Z}$ -augmented Kummer map of Definition 1.3.3. By Corollary 4.2.6 the bottom horizontal arrow is surjective, and by Lemma 5.1.2 the restriction of $\text{Red}_{\mathcal{S}, \mathfrak{Q}}$ to $\mathbb{Z}_\ell[\mathcal{T}_s]$ is surjective for $s \gg 0$. The same argument as [15, Lemma 2.2] gives the exactness of

$$0 \rightarrow W_{\mathbb{Z}_\ell}(k) \xrightarrow{\ell} W_{\mathbb{Z}_\ell}(k) \rightarrow W_{\mathbb{Z}/\ell\mathbb{Z}}(k) \rightarrow 0,$$

and taking $\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}$ cohomology shows that the right vertical arrow is surjective with kernel equal to the image of multiplication by ℓ . Applying Nakayama's lemma, we have proved

Lemma 5.1.3 *For $s \gg 0$ the restriction of $\text{loc}_{\mathcal{S}, \mathfrak{Q}}$ to $\mathbb{Z}_\ell[\mathcal{T}_s]$ is surjective.*

Let R be the integer ring of Φ , so that W_R is an R lattice in W_Φ and $\pi_f W_R$ is an R lattice in $W_f = \pi_f W_\Phi$. Let

$$\text{Heeg}_{R, s} \subset H^1(\mathbb{Q}^{\text{al}}/H[p^s], W_R(k))^\Delta$$

be the R submodule generated by the classes $\Omega_s(g)$ for $g \in \mathcal{T}_s$, and abbreviate

$$T = \pi_f W_R(k) \subset W_f(k).$$

Lemma 5.1.4 *For $s \gg 0$, the image of the composition*

$$\text{Heeg}_{R,s} \xrightarrow{\pi_\chi \pi_f} H^1(\mathbb{Q}^{\text{al}}/H[p^s], T) \xrightarrow{\oplus_{\mathfrak{q} \in \Omega} \text{loc}_{\mathfrak{q}}} \bigoplus_{\mathfrak{q} \in \Omega} H^1(\mathbb{Q}_{\mathfrak{q}}^{\text{al}}/K_{\mathfrak{q}}, T)$$

is $\bigoplus_{\mathfrak{q} \in \Omega} H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, T)$, the submodule of unramified cohomology classes.

PROOF. Using Proposition 1.5.1 we see that the image of the composition lies in the unramified cohomology, and is equal to the image of

$$\begin{aligned} R[\mathcal{T}_s] &\xrightarrow{\text{loc}_{\mathcal{S}, \Omega}} \bigoplus_{\mathcal{S}} \bigoplus_{\mathfrak{q} \in \Omega} H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, W_R(k))^{\Delta} \\ &\xrightarrow{\chi} \bigoplus_{\mathfrak{q} \in \Omega} H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, W_R(k))^{\Delta} \xrightarrow{\pi_f} \bigoplus_{\mathfrak{q} \in \Omega} H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, T) \end{aligned}$$

where arrow labeled χ takes the element $(x_\sigma)_{\sigma \in \mathcal{S}}$ to $\sum_{\sigma \in \mathcal{S}} \chi(\sigma) x_\sigma$. The first arrow is surjective for $s \gg 0$ by Lemma 5.1.3, the second is obviously surjective, and the third is surjective by the fact that $\text{Gal}(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}})$ has cohomological dimension one. \square

Let \mathfrak{m} denote the maximal ideal of R and set $\overline{T} = T \otimes_R R/\mathfrak{m}$. If $q \nmid \ell ND$ is a rational prime whose absolute Frobenius acts as complex conjugation on $K(\overline{T})$, the extension of \mathbb{Q} cut out by the Galois action on $\overline{T} = T \otimes_R R/\mathfrak{m}$, then clearly q is inert in K and the Frobenius of the unique prime \mathfrak{q} of K above q acts trivially on \overline{T} . By the Chebetarov theorem we may choose Ω as large as we want and containing only primes of this form. For $s \gg 0$, Lemma 5.1.4 gives a surjection from $\pi_\chi \pi_f \text{Heeg}_{R,s}$ to

$$\bigoplus_{\mathfrak{q} \in \Omega} H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, T) \otimes R/\mathfrak{m} \cong H^1(\mathbb{F}_{\mathfrak{q}}^{\text{al}}/\mathbb{F}_{\mathfrak{q}}, \overline{T}) \cong \bigoplus_{\mathfrak{q} \in \Omega} \overline{T}/(\text{Frob}_{\mathfrak{q}} - 1)\overline{T} \cong \bigoplus_{\mathfrak{q} \in \Omega} \overline{T}.$$

Thus the R/\mathfrak{m} dimension of $(\pi_\chi \pi_f \text{Heeg}_{R,s}) \otimes_R R/\mathfrak{m}$ is at least $\#\Omega$ for $s \gg 0$. Enlarging Ω , the R/\mathfrak{m} dimension of $(\pi_\chi \pi_f \text{Heeg}_{R,s}) \otimes_R R/\mathfrak{m}$ grows without bound as s increases.

Lemma 5.1.5 *The R -torsion submodule of $H^1(\mathbb{Q}^{\text{al}}/H[p^s], T)$ is finite and of bounded order as $s \rightarrow \infty$.*

PROOF. The R -torsion submodule of $H^1(\mathbb{Q}^{\text{al}}/H[p^s], T)$ is isomorphic to the quotient of

$$H^0(\mathbb{Q}^{\text{al}}/H[p^s], T \otimes_{\mathbb{Z}_\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)) \quad (40)$$

by its maximal divisible subgroup. Let $\ell \nmid pN$ be a rational prime which is inert in K , and let λ be the prime of K above ℓ . Then λ splits completely in $H[p^\infty]$ and, by Deligne's proof of the Ramanujan conjecture, $\text{Frob}_\lambda = \text{Frob}_\ell^2$ acts on $W_f(k)$ with eigenvalues of (complex) absolute value ℓ^{2k-1} . Hence $\text{Frob}_\lambda - 1$ is invertible on $W_f(k) \cong T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. A snake lemma argument then shows that the order of (40) is bounded by the order of $T/(\text{Frob}_\lambda - 1)T$. \square

The R -torsion submodule of $\pi_\chi \pi_f \text{Heeg}_{R,s}$ is contained in the torsion submodule of $H^1(\mathbb{Q}^{\text{al}}/H[p^s], T)$, and so is finite and bounded as $s \rightarrow \infty$ by Lemma 5.1.5. We have seen that the R/\mathfrak{m} dimension of $(\pi_\chi \pi_f \text{Heeg}_{R,s}) \otimes_R R/\mathfrak{m}$ increases without bound, and it now follows that the R -rank of $\pi_\chi \pi_f \text{Heeg}_{R,s}$ also increases without bound. This completes the proof of Theorem 5.1.1. \square

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