# DEFORMING ENDOMORPHISMS OF SUPERSINGULAR BARSOTTI-TATE GROUPS

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Abstract. The formal deformation space of a supersingular $p$-Barsotti-Tate group over $\mathbb{F}_p^{\mathrm{alg}}$ of dimension two equipped with an action of $\mathbb{Z}_{p^2}$ is known to be isomorphic to the formal spectrum of a power series ring in two variables over the Witt ring of $\mathbb{F}_p^{\mathrm{alg}}$. If one chooses an extra $\mathbb{Z}_{p^2}$-linear endomorphism of the $p$-divisible group then the locus in the formal deformation space formed by those deformations for which the extra endomorphism lifts is a closed formal subscheme of codimension two. We give a complete description of the irreducible components of this formal subscheme, compute the multiplicities of these components, and compute the intersection numbers of the components with a distinguished closed formal subscheme of codimension one. These calculations, which extend the Gross-Keating theory of quasi- canonical lifts, are used in the companion article *Intersection theory on Shimura surfaces II* to compute global intersection numbers of special cycles on the integral model of a Shimura surface.

## 0. Introduction

Let $\mathfrak{g}_0$ be a connected $p$-Barsotti-Tate group over $\mathbb{F} = \mathbb{F}_p^{\mathrm{alg}}$ of height two and dimension one, i.e. $\mathfrak{g}_0$ is the $p$-power torsion of a supersingular elliptic curve over $\mathbb{F}$. Fix a quadratic field extension $E_0/\mathbb{Q}_p$. As the endomorphism ring of $\mathfrak{g}_0$ is the maximal order in a quaternion division algebra over $\mathbb{Q}_p$, we may fix an action of $\mathcal{O}_{E_0}$ on $\mathfrak{g}_0$. Let $\mathfrak{M}_0$ be the formal scheme over the Witt ring

$$\mathbb{Z}_p^\circ = W(\mathbb{F})$$

which represents the functor of deformations of $\mathfrak{g}_0$ to complete local Noetherian $\mathbb{Z}_p^\circ$-algebras with residue field $\mathbb{F}$. Suppose now that we fix some $\gamma_0 \in \mathcal{O}_{E_0}$ not contained in $\mathbb{Z}_p$ and ask for the locus in $\mathfrak{M}_0$ of deformations for which the endomorphism $\gamma_0 \in \mathrm{End}(\mathfrak{g}_0)$ lifts. This locus, a closed formal subscheme of $\mathfrak{M}_0$ denoted $\mathfrak{Y}_0$, depends only on the integer $c_0$ defined by

$$\mathbb{Z}_p[\gamma_0] = \mathbb{Z}_p + p^{c_0}\mathcal{O}_{E_0}$$

and not on $\gamma_0$ itself. One knows that $\mathfrak{M}_0 \cong \mathrm{Spf}(\mathbb{Z}_p^\circ[[x]])$, and the Gross-Keating theory of quasi-canonical lifts recalled in §3.1 gives a complete understanding of the closed formal subscheme $\mathfrak{Y}_0$. For example, one knows that $\mathfrak{Y}_0$ has $c_0+1$ irreducible components which we label as $\mathfrak{C}(0), \ldots, \mathfrak{C}(c_0)$. The component $\mathfrak{C}(s)$ is isomorphic to $\mathrm{Spf}(W_s)$ where $W_s$ is the ring of integers in a finite abelian extension of the fraction field of

$$W_0 = \begin{cases} \mathbb{Z}_p^\circ & \text{if } E_0/\mathbb{Q}_p \text{ is unramified} \\ \mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^\circ & \text{if } E_0/\mathbb{Q}_p \text{ is ramified} \end{cases}$$

with Galois group isomorphic to $\mathcal{O}_{E_0}^\times/(\mathbb{Z}_p + p^s\mathcal{O}_{E_0})^\times$. The restriction of the universal deformation of $\mathfrak{g}_0$ to the component $\mathfrak{C}(s)$, a deformation of $\mathfrak{g}_0$ to $W_s$, is the *quasi-canonical lift of level $s$*, and has endomorphism ring $\mathbb{Z}_p + p^s\mathcal{O}_{E_0}$.

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This general type of problem, in which one begins with the formal deformation space of a $p$-Barsotti-Tate group and studies the locus in the deformation space obtained by imposing additional endomorphisms, is central to Kudla's program [13, 14, 16] to relate intersection multiplicities of so-called *special cycles* on Shimura varieties to Fourier coefficients of modular forms. In such generality it is very difficult to determine the structure of this locus (e.g. determine its irreducible components and their multiplicities). In this article we extend the one dimensional theory of Gross and Keating to the case of two dimensional $p$-Barsotti-Tate groups. More specifically, we will study those deformation spaces which arise by formally completing a Hilbert-Blumenthal surface (or Shimura surface) at a closed point in characteristic $p$. The determination of these deformation spaces is used in the companion article [10] to relate the intersection multiplicities of special cycles on a Shimura surface to Fourier coefficients of a Hilbert modular form.

Let $\mathbb{Z}_{p^2}$ be the integer ring in the unramified quadratic extension of $\mathbb{Q}_p$ and define

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{Z}_{p^2},$$

a $p$-Barsotti-Tate group of height four and dimension two which is equipped with an action of

$$\mathcal{O}_E = \mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}.$$

Let $\mathfrak{M}$ be the formal $\mathbb{Z}_p^\circ$-scheme classifying deformations of $\mathfrak{g}$, with its $\mathbb{Z}_{p^2}$-action, to complete local Noetherian $\mathbb{Z}_p^\circ$-algebras with residue field $\mathbb{F}$. Let $\mathfrak{Y} \to \mathfrak{M}$ be the closed formal subscheme classifying those deformations for which the endomorphism $\gamma = \gamma_0 \otimes 1 \in \mathcal{O}_E$ lifts (equivalently, for which the action of $\mathbb{Z}_{p^2}[\gamma] \cong \mathbb{Z}_{p^2} + p^{c_0}\mathcal{O}_E$ lifts). One knows that $\mathfrak{M} \cong \mathrm{Spf}(\mathbb{Z}_p^\circ[[x_1, x_2]])$, and the problem is to determine the structure of $\mathfrak{Y}$. To get started, one may note that there is cartesian diagram

$$\begin{array}{ccc} \mathfrak{Y}_0 & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \mathfrak{M}_0 & \longrightarrow & \mathfrak{M} \end{array}$$

in which all arrows are closed immersions and the horizontal arrows represent the functor which takes a deformation $\mathfrak{G}_0$ of $\mathfrak{g}_0$ to the deformation $\mathfrak{G}_0 \otimes \mathbb{Z}_{p^2}$ of $\mathfrak{g}$. Using the closed immersion $\mathfrak{Y}_0 \to \mathfrak{Y}$ above, we view $\mathfrak{C}(s)$ also as a closed formal subscheme of $\mathfrak{Y}$. There is a natural action of $\mathrm{Aut}_{\mathbb{Z}_{p^2}}(\mathfrak{g})$ on the deformation space $\mathfrak{M}$, and this action restricts to an action of $\mathcal{O}_E^\times$ on $\mathfrak{M}$ and on $\mathfrak{Y}$. In particular $\mathcal{O}_E^\times$ acts on the set of all closed formal subschemes of $\mathfrak{Y}$, and for each $\xi \in \mathcal{O}_E^\times$ we set

$$\mathfrak{C}(s, \xi) = \xi * \mathfrak{C}(s).$$

Our first theorem describes the irreducible components of $\mathfrak{Y}$ when $E_0/\mathbb{Q}_p$ is unramified. The horizontal components arise by taking $\mathcal{O}_E^\times$-orbits of components of $\mathfrak{Y}_0$. The main difference between the dimension two case under consideration and the dimension one case considered by Gross and Keating is the presence of two vertical components. Define a subgroup of $\mathcal{O}_E^\times$

$$H_s = \mathcal{O}_{E_0}^\times \cdot (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times.$$

**Theorem A.** *Assume that $E_0/\mathbb{Q}_p$ is unramified.*

(a) *Each closed formal subscheme $\mathfrak{C}(s, \xi)$ constructed above is an irreducible component of $\mathfrak{Y}$, and every horizontal irreducible component of $\mathfrak{Y}$ has this form for a unique $0 \le s \le c_0$ and a unique $\xi \in \mathcal{O}_E^\times / H_s$.*

(b) *If* $c_0 > 0$ *then* $\mathfrak{Y}$ *has two vertical irreducible components; if* $c_0 = 0$ *there are no such components.*

*Proof.* The first claim is Proposition 3.2.5, the second is Proposition 2.2.1. □

If $E_0/\mathbb{Q}_p$ is ramified then the situation is similar, but slightly more complicated. Let $E = E_0 \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}$, the fraction field of $\mathcal{O}_E$. As $E/\mathbb{Q}_p$ is biquadratic there is a unique quadratic subfield $E_0' \subset E$ which is neither $E_0$ nor $\mathbb{Q}_{p^2}$, and we may fix an action of $\mathcal{O}_{E_0'}$ on $\mathfrak{g}_0$. As $\mathcal{O}_E \cong \mathcal{O}_{E_0'} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$ canonically this choice determines a new action of $\mathcal{O}_E$ on $\mathfrak{g} \cong \mathfrak{g}_0 \otimes \mathbb{Z}_{p^2}$ which is conjugate by some $w \in \mathrm{Aut}_{\mathbb{Z}_{p^2}}(\mathfrak{g})$ to the action used in the definition of the deformation space $\mathfrak{Y}$. Let $\mathfrak{Y}_0'$ be the locus in $\mathfrak{M}_0$ of deformations for which the new action of $\mathbb{Z}_p + p^{c_0}\mathcal{O}_{E_0'}$ lifts, so that the irreducible components of $\mathfrak{Y}_0'$ are indexed, again by the theory of quasi-canonical lifts, as $\mathfrak{C}'(0), \ldots, \mathfrak{C}'(c_0)$. Viewing each $\mathfrak{C}'(s)$ as a closed formal subscheme of $\mathfrak{M}$, we then set

$$\mathfrak{C}'(s,\xi) = (\xi \circ w) * \mathfrak{C}'(s)$$

for every $\xi \in \mathcal{O}_E^\times$ and $0 \le s \le c_0$. One can show that $\mathfrak{C}'(s,\xi)$ is contained in $\mathfrak{Y}$ (although $\mathfrak{C}'(s)$ itself is not). Define

$$H_s' = \mathcal{O}_{E_0'}^\times \cdot (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times.$$

**Theorem B.** *Assume that* $E_0/\mathbb{Q}_p$ *is ramified.*

(a) *Each closed formal subscheme* $\mathfrak{C}(s,\xi)$ *and* $\mathfrak{C}'(s,\xi)$ *constructed above is an irreducible component of* $\mathfrak{Y}$. *Every horizontal irreducible component of* $\mathfrak{Y}$ *is either equal to* $\mathfrak{C}(s,\xi)$ *for a unique* $0 \le s \le c_0$ *and a unique* $\xi \in \mathcal{O}_E^\times/H_s$, *or is equal to* $\mathfrak{C}'(s,\xi)$ *for a unique* $0 \le s \le c_0$ *and a unique* $\xi \in \mathcal{O}_E^\times/H_s'$.

(b) *If* $c_0 > 0$ *then* $\mathfrak{Y}$ *has two vertical irreducible components; if* $c_0 = 0$ *there are no such components.*

*Proof.* The first claim is Proposition 3.3.6, the second is Proposition 2.2.1. □

Having determined all irreducible components of $\mathfrak{Y}$, we next turn to the problem of determining the multiplicity $\mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C})$ (in the sense of Definition 1.1.1) of each such component $\mathfrak{C}$. For the horizontal components this will be done using global techniques, i.e. by identifying $\mathfrak{Y}$ with the formal completion at a point of a Hilbert-Blumenthal surface. For the vertical components the argument uses global techniques (the determination of the *Hasse-Witt locus* of $\mathfrak{M}$ in the sense of §1.2) and a heavy dose of explicit calculations using Zink's theory of *windows* for $p$-divisible groups.

**Theorem C.** *Every horizontal irreducible component of* $\mathfrak{Y}$ *appears with multiplicity one. If* $c_0 > 0$ *and* $p$ *is odd then each of the two vertical irreducible components (called* $\mathfrak{C}_1^{\mathrm{ver}}$ *and* $\mathfrak{C}_2^{\mathrm{ver}}$*) of* $\mathfrak{Y}$ *appears with multiplicity*

$$\mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C}_i^{\mathrm{ver}}) = 2p^{c_0-1} + 4p^{c_0-2} + 6p^{c_0-3} + 8p^{c_0-4} + \cdots + (2c_0)p^0.$$

*Proof.* If $E_0/\mathbb{Q}_p$ is unramified this is Proposition 2.3.4, if $E_0/\mathbb{Q}_p$ is ramified this is Proposition 2.4.1. □

The final problem is to determine the relative position of $\mathfrak{M}_0$ and $\mathfrak{Y}$ inside of $\mathfrak{M}$, or, more precisely, the intersection number of $\mathfrak{M}_0$ with each irreducible component of $\mathfrak{Y}$. Let us say that an irreducible component $\mathfrak{C}$ of $\mathfrak{Y}$ is *proper* if it is not contained in $\mathfrak{M}_0$ (and hence meets $\mathfrak{M}_0$ properly). We wish to compute the intersection multiplicity $I_{\mathfrak{M}}(\mathfrak{C},\mathfrak{M}_0)$ of $\mathfrak{C}$ and $\mathfrak{M}_0$ in the sense of Definition 1.1.1. Our notion of intersection does not take multiplicities into account, so really it is the value $\mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C})\cdot I_{\mathfrak{M}}(\mathfrak{C},\mathfrak{M}_0)$ in which we are interested. We have done this for every proper component of $\mathfrak{Y}$, under the hypothesis $p>2$. Summing the results over all proper components yields the following results on the *total proper intersection* of $\mathfrak{Y}$ and $\mathfrak{M}_0$.

**Theorem D.** *Assume that $p$ is odd. If $E_0/\mathbb{Q}_p$ is unramified then*

$$\sum_{\mathfrak{C}\ \mathrm{proper}} \mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C})\cdot I_{\mathfrak{M}}(\mathfrak{C},\mathfrak{M}_0) = c_0\cdot\left(\frac{p^{c_0+1}-1}{p-1}+\frac{p^{c_0}-1}{p-1}\right).$$

*If $E_0/\mathbb{Q}_p$ is ramified then*

$$\sum_{\mathfrak{C}\ \mathrm{proper}} \mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C})\cdot I_{\mathfrak{M}}(\mathfrak{C},\mathfrak{M}_0) = (2c_0+1)\frac{p^{c_0+1}-1}{p-1}.$$

*In both cases the sum is over all proper irreducible components of $\mathfrak{Y}$.*

*Proof.* If $E_0/\mathbb{Q}_p$ is unramified the contribution of all proper horizontal irreducible components is computed in Corollary 1.2.6 (for the multiplicities) and Corollary 3.2.7 (for the intersection numbers), while the contribution of the vertical components is computed in Proposition 2.2.1 (for the intersection numbers) and Proposition 2.3.4 (for the multiplicities).

If $E_0/\mathbb{Q}_p$ is ramified the contribution of all proper horizontal irreducible components is computed in Corollary 1.2.6 (for the multiplicities) and Corollary 3.3.9 (for the intersection numbers), while the contribution of the vertical components is computed in Proposition 2.2.1 (for the intersection numbers) and Proposition 2.4.1 (for the multiplicities). □

It is in fact the calculation of the total proper intersection found in Theorem D which motivated this project. This calculation of local intersection multiplicities is one part of a larger global calculation which relates the intersection multiplicities of special cycles on the integral model of a Shimura surface to the Fourier coefficients of a Hilbert modular form of weight 3/2. This global application of the local calculations carried out herein is the content of [10], which contains a higher-dimensional version of the Kudla-Rapoport-Yang [16] intersection theory on integral models of Shimura curves.

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0.2. **Notation.** The following notation will be used throughout the article. Let $\mathbb{F}=\mathbb{F}_p^{\mathrm{alg}}$ be an algebraic closure of the field of $p$ elements and let $\mathbb{Z}_p^\circ = W(\mathbb{F})$ denote the ring of Witt vectors of $\mathbb{F}$. Equivalently, $\mathbb{Z}_p^\circ$ is the completion of the strict Henselization of $\mathbb{Z}_p$ with respect to the unique $\mathbb{Z}_p$-algebra homomorphism $\mathbb{Z}_p\to\mathbb{F}$. Let $\mathbb{Q}_p^\circ$ be the fraction field of $\mathbb{Z}_p^\circ$ and let $\mathbb{Q}_{p^2}$ be the unique quadratic extension of $\mathbb{Q}_p$ contained in $\mathbb{Q}_p^\circ$. Denote by $\mathbb{Z}_{p^2}$ the ring of integers of $\mathbb{Q}_{p^2}$, and let $\sigma$ be the nontrivial Galois automorphism of $\mathbb{Q}_{p^2}/\mathbb{Q}_p$.

When $R$ is a local ring we denote by $\mathfrak{m}_R$ the maximal ideal of $R$. Denote by **ProArt** the category of complete (i.e. complete and separated) local Noetherian $\mathbb{Z}_p^\circ$-algebras $R$ equipped with an isomorphism $R/\mathfrak{m}_R\to\mathbb{F}$. Morphisms in **ProArt** are local $\mathbb{Z}_p^\circ$-algebra homomorphisms. Let **Art** be the full subcategory of **ProArt** whose objects are Artinian local $\mathbb{Z}_p^\circ$-algebras. Any set-valued functor $\mathcal{F}$ on **Art** extends naturally to a functor on **ProArt** by

$$\mathcal{F}(R) \stackrel{\mathrm{def}}{=} \varprojlim \mathcal{F}(R/\mathfrak{m}_R^k).$$

We say that $\mathcal{F}$ is *pro-representable* if there is an isomorphism of functors

$$\mathcal{F}(-) \cong \mathrm{Hom}_{\mathbf{ProArt}}(R_{\mathcal{F}}, -)$$

for some object $R_{\mathcal{F}}$ of **ProArt**. When this is the case we often confuse $\mathcal{F}$ with the formal $\mathbb{Z}_p^{\circ}$-scheme $\mathrm{Spf}(R_{\mathcal{F}})$.

## 1. Preliminaries

In §1.1 we define formal schemes over $\mathrm{Spf}(\mathbb{Z}_p^{\circ})$ which represent certain deformation functors on **Art**. In §1.2 we relate these form schemes to the completed local rings of Hilbert-Blumenthal surfaces and derive some consequences.

1.1. **Deformation functors.** Let $E_0$ be a quadratic field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_{E_0}$ and fix a $\mathbb{Z}_p$-algebra homomorphism

$$\psi : \mathcal{O}_{E_0} \to \mathbb{F}. \tag{1}$$

Let $\mathfrak{g}_0$ be a connected $p$-Barsotti-Tate group of dimension one and height two over $\mathbb{F}$. Up to isomorphism there is a unique such $\mathfrak{g}_0$ [3, p. 93], and $\mathfrak{g}_0$ is isomorphic to the $p$-Barsotti-Tate group of any supersingular elliptic curve over $\mathbb{F}$. The $\mathbb{Z}_p$-algebra $\mathrm{End}(\mathfrak{g}_0)$ is the maximal order in the unique (up to isomorphism) quaternion division algebra over $\mathbb{Q}_p$. We may choose an embedding

$$j_0 : \mathcal{O}_{E_0} \to \mathrm{End}(\mathfrak{g}_0)$$

in such a way that the action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{g}_0)$ is through the homomorphism $\psi$. Such an embedding is unique up to $\mathrm{Aut}(\mathfrak{g}_0)$-conjugacy by Corollary 1.1.3 below. Pick any $\gamma_0 \in \mathcal{O}_{E_0}$ not contained in $\mathbb{Z}_p$, define $c_0$ by

$$\mathbb{Z}_p[\gamma_0] = \mathbb{Z}_p + p^{c_0}\mathcal{O}_{E_0},$$

and view $\gamma_0$ as an endomorphism of $\mathfrak{g}_0$. For an object $R$ of **ProArt**, a *deformation* of $\mathfrak{g}_0$ to $R$ is a pair $(\mathfrak{G}_0, \rho_0)$ in which $\mathfrak{G}_0$ is a $p$-Barsotti-Tate group over $R$ and

$$\rho_0 : \mathfrak{g}_0 \to \mathfrak{G}_{0/\mathbb{F}}$$

is an isomorphism. We denote by $\mathfrak{M}_0$ the functor on **Art** which assigns to an object $R$ the set $\mathfrak{M}_0(R)$ of isomorphism classes of deformations of $\mathfrak{g}_0$ to $R$. Let $\mathfrak{Y}_0$ be the subfunctor of $\mathfrak{M}_0$ which assigns to $R$ the set $\mathfrak{Y}_0(R)$ of isomorphism classes of deformations $(\mathfrak{G}_0, \rho_0)$ of $\mathfrak{g}_0$ to $R$ for which the endomorphism $\gamma_0 \in \mathrm{End}(\mathfrak{g}_0)$ lifts to an endomorphism of $\mathfrak{G}_0$ (uniquely, by the rigidity theorem [25, Theorem 2.1]).

Given a $p$-Barsotti-Tate group $\mathfrak{G}_0$ over an object $R$ of **ProArt** we define $\mathfrak{G}_0 \otimes \mathbb{Z}_{p^2}$ to be the $p$-Barsotti-Tate group representing the functor on **Art** defined by

$$S \mapsto \mathfrak{G}_0(S) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}.$$

The representability of this functor is seen by choosing a $\mathbb{Z}_p$-basis of $\mathbb{Z}_{p^2}$, defining

$$\mathfrak{G}_0 \otimes \mathbb{Z}_{p^2} = \mathfrak{G}_0 \times \mathfrak{G}_0,$$

and letting $\mathbb{Z}_{p^2}$ act on the right hand side through the ring homomorphism $\mathbb{Z}_{p^2} \to M_2(\mathbb{Z}_p)$ induced by our choice of basis. The rule $\mathfrak{G}_0 \mapsto \mathfrak{G}_0 \otimes \mathbb{Z}_{p^2}$ defines a functor from the category of $p$- Barsotti-Tate groups over $R$ to the category of $p$-Barsotti-Tate groups over $R$ with $\mathbb{Z}_{p^2}$-action. In particular we define a $p$-Barsotti-Tate group

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{Z}_{p^2}$$

of height four and dimension two over $\mathbb{F}$. Set

$$E = E_0 \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2},$$

let

$$\mathcal{O}_E = \mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$$

be the maximal order in $E$, and set $\gamma = \gamma_0 \otimes 1 \in \mathcal{O}_E$. The action of $\mathcal{O}_{E_0}$ on $\mathfrak{g}_0$ fixed above determines an action

$$j : \mathcal{O}_E \to \mathrm{End}(\mathfrak{g}),$$

and in particular we may view $\gamma$ as an endomorphism of $\mathfrak{g}$. If $R$ is an object of **ProArt**, a *deformation* of $\mathfrak{g}$ to $R$ is a pair $(\mathfrak{G}, \rho)$ in which $\mathfrak{G}$ is a $p$-Barsotti-Tate group over $R$ equipped with an action $\mathbb{Z}_{p^2} \to \mathrm{End}(\mathfrak{G})$ and $\rho : \mathfrak{g} \to \mathfrak{G}_{/\mathbb{F}}$ is a $\mathbb{Z}_{p^2}$-linear isomorphism. Let $\mathfrak{M}$ be the functor on **Art** which assigns to each object $R$ the set of isomorphism classes of deformations of $\mathfrak{g}$ to $R$, and let $\mathfrak{Y}$ be the subfunctor of deformations of $\mathfrak{g}$ for which the endomorphism $\gamma$ lifts. There is a cartesian diagram of functors

$$\begin{array}{ccc} \mathfrak{Y}_0 & \xrightarrow{\otimes \mathbb{Z}_{p^2}} & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \mathfrak{M}_0 & \xrightarrow[\otimes \mathbb{Z}_{p^2}]{} & \mathfrak{M} \end{array} \tag{2}$$

in which the horizontal arrows are defined by

$$(\mathfrak{G}_0, \rho_0) \mapsto (\mathfrak{G}_0, \rho_0) \otimes \mathbb{Z}_{p^2} \stackrel{\mathrm{def}}{=} (\mathfrak{G}_0 \otimes \mathbb{Z}_{p^2}, \rho_0 \otimes \mathbb{Z}_{p^2}),$$

where $\rho_0 \otimes \mathbb{Z}_{p^2}$ is the isomorphism

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{Z}_{p^2} \xrightarrow{\rho_0 \otimes \mathrm{id}} \mathfrak{G}_{0/\mathbb{F}} \otimes \mathbb{Z}_{p^2} \cong \mathfrak{G}_{/\mathbb{F}}.$$

All functors in (2) are pro-representable by objects of **ProArt** and all arrows correspond to surjections between the pro-representing objects (this follows from [21, Proposition 2.9]). Expressed differently, the diagram (2) can be identified with a cartesian diagram of formal schemes over $\mathbb{Z}_p^\circ$

$$\begin{array}{ccc} \mathrm{Spf}(R_{\mathfrak{Y}_0}) & \xrightarrow{\otimes \mathbb{Z}_{p^2}} & \mathrm{Spf}(R_{\mathfrak{Y}}) \\ \downarrow & & \downarrow \\ \mathrm{Spf}(R_{\mathfrak{M}_0}) & \xrightarrow[\otimes \mathbb{Z}_{p^2}]{} & \mathrm{Spf}(R_{\mathfrak{M}}) \end{array}$$

in which all arrows are closed immersions.

For any $\xi \in \mathcal{O}_E^\times$, viewed as an element of $\mathrm{Aut}(\mathfrak{g})$, and any $(\mathfrak{G}, \rho) \in \mathfrak{Y}(R)$ we define a new deformation

$$\xi * (\mathfrak{G}, \rho) \stackrel{\mathrm{def}}{=} (\mathfrak{G}, \rho \circ \xi^{-1}) \in \mathfrak{Y}(R).$$

This defines an action of $\mathcal{O}_E^\times$ on the functor $\mathfrak{Y}$. Using the isomorphism

$$\mathfrak{Y}(-) \cong \mathrm{Hom}_{\mathbf{ProArt}}(R_{\mathfrak{Y}}, -)$$

we find an action of $\mathcal{O}_E^\times$ on $\mathrm{Hom}_{\mathbf{ProArt}}(R_{\mathfrak{Y}}, -)$ of the form $\xi * f = f \circ \xi^{-1}$ for some homomorphism

$$\mathcal{O}_E^\times \to \mathrm{Aut}_{\mathbf{ProArt}}(R_{\mathfrak{Y}}).$$

The action of $\mathcal{O}_{E_0}^{\times}$ preserves the subfunctor $\mathfrak{Y}_0$ and similarly determines a homomorphism

$$\mathcal{O}_{E_0}^{\times} \to \mathrm{Aut}_{\mathbf{ProArt}}(R_{\mathfrak{Y}_0}). \tag{3}$$

**Definition 1.1.1.** A *component* of $\mathfrak{Y}$ is a closed formal subscheme $\mathfrak{C} \to \mathfrak{Y}$ of the form

$$\mathfrak{C} = \mathrm{Spf}(R_{\mathfrak{Y}}/\mathfrak{p}) \to \mathrm{Spf}(R_{\mathfrak{Y}})$$

for some minimal prime ideal $\mathfrak{p} \subset R_{\mathfrak{Y}}$. The component $\mathfrak{C}$ is *horizontal* if $p \notin \mathfrak{p}$, and is *vertical* if $p \in \mathfrak{p}$. The *multiplicity* of a component $\mathfrak{C}$ is the length of the local ring at $\mathfrak{p}$

$$\mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C}) = \mathrm{length}_{R_{\mathfrak{Y},\mathfrak{p}}}(R_{\mathfrak{Y},\mathfrak{p}})$$

and the *intersection number* is defined as the (possibly infinite) length

$$I_{\mathfrak{M}}(\mathfrak{C}, \mathfrak{M}_0) = \mathrm{length}_{R_{\mathfrak{M}}}(R_{\mathfrak{Y}}/\mathfrak{p} \otimes_{R_{\mathfrak{M}}} R_{\mathfrak{M}_0}).$$

The component $\mathfrak{C}$ is *improper* if the closed immersion $\mathfrak{C} \to \mathfrak{Y}$ factors as $\mathfrak{C} \to \mathfrak{Y}_0 \to \mathfrak{Y}$ and is *proper* otherwise. Intuitively, the phrase "$\mathfrak{C}$ is proper" should be interpreted as shorthand for "$\mathfrak{C}$ meets $\mathfrak{M}_0$ properly in $\mathfrak{M}$."

Given any ring homomorphism $\phi : \mathcal{O}_{E_0} \to R$ we denote by $\overline{\phi} : \mathcal{O}_{E_0} \to R$ the homomorphism obtained by precomposing $\phi$ with the nontrivial Galois automorphism of $E_0/\mathbb{Q}_p$.

**Lemma 1.1.2.** *Let $\Delta_0$ be the maximal order in a ramified quaternion algebra over $\mathbb{Q}_p$ with uniformizing parameter $\Pi$. Two embeddings $i_1, i_2 : \mathcal{O}_{E_0} \to \Delta_0$ are $\Delta_0^{\times}$-conjugate if and only if they reduce to the same $\mathbb{Z}_p$-algebra homomorphism $\mathcal{O}_{E_0} \to \Delta_0/\Pi\Delta_0 \cong \mathbb{F}_{p^2}$.*

*Proof.* Easy exercise. □

**Corollary 1.1.3.** *Let $\mathfrak{G}$ and $\mathfrak{G}'$ be $p$-Barsotti-Tate groups of dimension one and height two over $\mathbb{F}$, each equipped with an action of $\mathcal{O}_{E_0}$. Then $\mathfrak{G}$ and $\mathfrak{G}'$ are $\mathcal{O}_{E_0}$-linearly isomorphic if and only if their Lie algebras are isomorphic as $\mathcal{O}_{E_0}$-modules.*

*Proof.* By [3, p. 93] $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic as $p$-Barsotti-Tate groups, and so we may assume that $\mathfrak{G}' = \mathfrak{G}$. The two actions of $\mathcal{O}_{E_0}$ on $\mathfrak{G}$ are determined by homomorphisms $\phi_1, \phi_2 : \mathcal{O}_{E_0} \to \Delta_0$ where $\Delta_0 = \mathrm{End}(\mathfrak{G})$ is the maximal order in a ramified quaternion algebra over $\mathbb{Q}_p$. The action of $\Delta_0$ on $\mathrm{Lie}(\mathfrak{G})$ is through some embedding $\Delta_0/\Pi\Delta_0 \to \mathbb{F}$, and the claim reduces to Lemma 1.1.2. □

1.2. **Hilbert-Blumenthal surfaces.** It will be useful to identify $R_{\mathfrak{M}}$ with a completed local ring of a Hilbert-Blumenthal surface. Choose number fields $\mathcal{E}_0$, $\mathcal{F}$, and $\mathcal{E}$ such that the following properties hold:

(a) $\mathcal{F}$ is real quadratic, $\mathcal{E}_0$ is imaginary quadratic, and $\mathcal{E} \cong \mathcal{E}_0 \otimes_{\mathbb{Q}} \mathcal{F}$,
(b) there are isomorphisms (which we now fix)

$$\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_{p^2} \qquad \mathcal{O}_{\mathcal{E}_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}_{E_0}$$

Set $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}} = \mathcal{O}_{\mathcal{E}_0} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{F}}$ and let $\mathcal{R} \subset \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$ be the $\mathcal{O}_{\mathcal{F}}$-order defined by

$$\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \begin{cases} \mathbb{Z}_{p^2}[\gamma] & \text{if } \ell = p \\ \mathcal{O}_{\mathcal{E}_0,\mathcal{F}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \text{if } \ell \neq p \end{cases}$$

for every rational prime $\ell$. Choose an elliptic curve $\mathfrak{a}_0$ over $\mathbb{F}$ and an action $\mathcal{O}_{\mathcal{E}_0} \to \mathrm{End}(\mathfrak{a}_0)$ in such a way that the $p$-Barsotti-Tate group of $\mathfrak{a}_0$ is $\mathcal{O}_{E_0}$- linearly isomorphic to $\mathfrak{g}_0$ (this is possible by Corollary 1.1.3). The abelian surface $\mathfrak{a} = \mathfrak{a}_0 \otimes \mathcal{O}_{\mathcal{F}}$ then carries an action of $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$ and the $p$-Barsotti-Tate group of $\mathfrak{a}$ is $\mathcal{O}_E$-linearly isomorphic to $\mathfrak{g}$.

Fix a Zariski open neighborhood $U \to \mathrm{Spec}(\mathbb{Z})$ of $p$.

**Definition 1.2.1.** An *abelian surface with real multiplication* (RM) over a $U$-scheme $S$ is an abelian scheme $A \to S$ with an action $\mathcal{O}_\mathcal{F} \to \mathrm{End}(A)$ satisfying the *Rapoport condition*: every $s \in S$ has an open affine neighborhood over which the coherent sheaf $\mathrm{Lie}(A)$ is a free $\mathcal{O}_\mathcal{F} \otimes_\mathbb{Z} \mathcal{O}_S$-module of rank one.

**Definition 1.2.2.** Let $\mathcal{D}_\mathcal{F}$ be the different of $\mathcal{F}/\mathbb{Q}$. Suppose $S$ is a $U$-scheme and $A \to S$ is an RM abelian surface. A *Deligne-Pappas* polarization of $A$ is an $\mathcal{O}_\mathcal{F}$-linear polarization $\lambda : A \to A^\vee$ whose kernel is $A[\mathcal{D}_\mathcal{F}]$.

**Definition 1.2.3.** A *polarized RM abelian surface* over a $U$-scheme $S$ is a pair $(A, \lambda)$ in which $A$ is an RM abelian surface over $S$, and $\lambda : A \to A^\vee$ is a Deligne-Pappas polarization.

*Remark* 1.2.4. See [23] for an extensive discussion of Deligne-Pappas polarizations and the Rapoport condition. What we have called a Deligne-Pappas polarization is equivalent to what Vollaard calls a $\mathcal{D}_\mathcal{F}^{-1}$-polarization.

Let $\mathcal{M}$ be the Deligne-Mumford stack over $U$ classifying polarized RM abelian surfaces over $U$-schemes. Let $\mathcal{Y}$ be the Deligne-Mumford stack classifying triples $(A, \lambda, i)$ where $(A, \lambda)$ is a polarized RM abelian surface over a $U$-scheme and $i : \mathcal{R} \to \mathrm{End}(A)$ is an action of $\mathcal{R}$ which extends the action of $\mathcal{O}_\mathcal{F}$ on $A$. The stack $\mathcal{M}$ is smooth of relative dimension two over $U$. See [2, 20, 23]. After shrinking $U$ and adding rigidifying étale level structure at primes not in $U$ to these moduli problems the resulting stacks are schemes. From now on we assume that such étale level structure has been imposed, but make no explicit mention of it.

Suppose $A_0$ is any elliptic curve over a scheme $S$ and set $A = A_0 \otimes \mathcal{O}_\mathcal{F}$. Fix a basis $\{x, y\}$ of $\mathcal{O}_\mathcal{F}$ as a $\mathbb{Z}$-module and let $\{x^\vee, y^\vee\}$ be the dual basis (relative to the trace form) of the inverse different $\mathcal{D}_\mathcal{F}^{-1}$. These bases determine two homomorphisms $i, i^\vee : \mathcal{O}_\mathcal{F} \to M_2(\mathbb{Z})$ which are interchanged by transposition in $M_2(\mathbb{Z})$. Thus if we identify $A \cong A_0 \times A_0$ in such a way that the action of $\mathcal{O}_\mathcal{F}$ on the right hand side is through $i$, the induced action of $\mathcal{O}_\mathcal{F}$ on $A^\vee \cong A_0^\vee \times A_0^\vee$ is through $i^\vee$. In other words there is an $\mathcal{O}_\mathcal{F}$-linear isomorphism $A^\vee \cong A_0^\vee \otimes \mathcal{D}_F^{-1}$. Moreover, if $\lambda_0 : A_0 \to A_0^\vee$ is the unique principal polarization of $A_0$ then as in [11, §2.1] the isogeny

$$A \cong A_0 \otimes \mathcal{O}_\mathcal{F} \xrightarrow{\lambda_0 \otimes \iota} A_0^\vee \otimes \mathcal{D}_\mathcal{F}^{-1} \cong A^\vee$$

is a polarization with kernel $A[\mathcal{D}_\mathcal{F}]$, and does not depend on the choice of basis $\{x, y\}$. Here we have used $\iota$ to denote the inclusion $\mathcal{O}_\mathcal{F} \to \mathcal{D}_\mathcal{F}^{-1}$. The above construction equips every abelian surface of the form $A_0 \otimes \mathcal{O}_\mathcal{F}$ with a canonical Deligne-Pappas polarization. In particular the abelian surface $\mathfrak{a} = \mathfrak{a}_0 \otimes \mathcal{O}_\mathcal{F}$ defined above has a Deligne-Pappas polarization $\lambda : \mathfrak{a} \to \mathfrak{a}^\vee$, and, as $\mathrm{Lie}(\mathfrak{a}) \cong \mathrm{Lie}(\mathfrak{a}_0) \otimes_\mathbb{Z} \mathcal{O}_\mathcal{F}$, the abelian surface $\mathfrak{a}$ satisfies the Rapoport condition. Thus the pair $(\mathfrak{a}, \lambda)$ determines an $\mathbb{F}$-valued point of $\mathcal{M}$. Our chosen action of $\mathcal{O}_{\mathcal{E}_0}$ on $\mathfrak{a}_0$ determines an action $i : \mathcal{O}_{\mathcal{E}_0, \mathcal{F}} \to \mathrm{End}_{\mathcal{O}_\mathcal{F}}(\mathfrak{a})$, and the triple $(\mathfrak{a}, \lambda, i)$ defines a point of $\mathcal{Y}(\mathbb{F})$.

For an object $R$ of **Art** let $\mathfrak{M}^\mathfrak{a}(R)$ denote the set of isomorphism classes of deformations $(A, \rho)$ of $\mathfrak{a}$, with its $\mathcal{O}_\mathcal{F}$-action, to $R$. Thus $A$ is an abelian surface over $R$ equipped with an action of $\mathcal{O}_\mathcal{F}$, and $\rho : \mathfrak{a} \to A_{/\mathbb{F}}$ is an $\mathcal{O}_\mathcal{F}$-linear isomorphism. The deformation $A$ automatically satisfies the Rapoport condition, and by the corollary to [23, Theorem 3] the Deligne-Pappas polarization of $\mathfrak{a}$ lifts uniquely to $A$. Similarly let $\mathfrak{Y}^\mathfrak{a}(R)$ denote the set of isomorphism classes of deformations $(A, \rho)$ of $\mathfrak{a}$ for which the action $\mathcal{R} \to \mathrm{End}(A)$ lifts (necessarily uniquely, by [19, Corollary 6.2]) to an action of $\mathcal{R}$ on $A$. For any abelian scheme $A$ over a base scheme $S$ let $A_{p^\infty}$ be the $p$-Barsotti-Tate group of $A$. If we let $x \in \mathcal{M}(\mathbb{F})$

be the geometric point corresponding to the polarized RM abelian surface $(\mathfrak{a}, \lambda)$ then it follows from the discussion above and the Serre-Tate theorem that there are isomorphisms of functors on **Art**

$$\mathrm{Hom}_{\mathbf{ProArt}}(\mathcal{O}^{\circ}_{\mathcal{M},x}, -) \cong \mathfrak{M}^{\mathfrak{a}}(-) \cong \mathfrak{M}(-)$$

in which $\mathcal{O}^{\circ}_{\mathcal{M},x}$ is the completion of the strictly Henselian local ring of $\mathcal{M}$ at $x$, and the second arrow is defined by passage to $p$-Barsotti-Tate groups $(A, \rho) \mapsto (A_{p^\infty}, \rho)$. Similarly, if we let $y \in \mathcal{Y}(\mathbb{F})$ be the point corresponding to $(\mathfrak{a}, \lambda)$ with its above $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$-action then there are isomorphisms

$$\mathrm{Hom}_{\mathbf{ProArt}}(\mathcal{O}^{\circ}_{\mathcal{Y},y}, -) \cong \mathfrak{Y}^{\mathfrak{a}}(-) \cong \mathfrak{Y}(-).$$

In particular there is a commutative diagram in **ProArt**

$$\begin{array}{ccc} \mathcal{O}^{\circ}_{\mathcal{Y},y} & \longrightarrow & R_{\mathfrak{Y}} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\circ}_{\mathcal{M},x} & \longrightarrow & R_{\mathfrak{M}} \end{array}$$

in which the horizontal arrows are isomorphisms.

By definition of $\mathfrak{M}$, for an object $R$ of **ProArt** an element of

$$\mathfrak{M}(R) = \varprojlim \mathfrak{M}(R/\mathfrak{m}_R^k)$$

is a compatible family $(\mathfrak{G}^{(k)}, \rho^{(k)})$ of deformations of $\mathfrak{g}$ to $R/\mathfrak{m}_R^k$. One would like to know that such a family comes from a single deformation $(\mathfrak{G}, \rho)$ of $\mathfrak{g}$ to $R$. This is true in great generality (see [1, Lemma 2.4.4]), but in this particular case one can use the bijection

$$\mathrm{Hom}_{\mathbf{ProArt}}(\mathcal{O}^{\circ}_{\mathcal{M},x}, R) \cong \mathfrak{M}(R)$$

to see that the $p$-divisible group of the pullback of the universal Hilbert-Blumenthal moduli via

$$\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}^{\circ}_{\mathcal{M},x}) \to \mathcal{M}$$

gives the desired deformation of $\mathfrak{g}$ to $R$. Similarly any element of $\mathfrak{Y}(R)$, $\mathfrak{M}^{\mathfrak{a}}(R)$, or $\mathfrak{Y}^{\mathfrak{a}}(R)$, *a priori* defined as a compatible family of deformations to Artinian quotients of $R$, in fact determines a deformation to $R$ (necessarily unique by [5, Corollary 8.4.6]).

We will exploit the isomorphism $\mathcal{O}^{\circ}_{\mathcal{Y},y} \cong R_{\mathfrak{Y}}$ to deduce properties about the deformation space $\mathfrak{Y}$ which seem difficult to obtain by working purely in the context of $p$-Barsotti-Tate groups. The following lemma is a good example of this.

**Lemma 1.2.5.** *The $\mathbb{Q}_p^{\circ}$-algebra $R_{\mathfrak{Y}}[1/p]$ is a finite product of field extensions of finite degree.*

*Proof.* Exactly as in [11, Lemma 2.1.3] one can choose a prime $\ell \in U$ for which $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathbb{Z}_\ell^4$, and use the Serre-Tate deformation theory of ordinary abelian varieties to prove that $\mathcal{Y} \times_U \mathrm{Spec}(W(\mathbb{F}_\ell^{\mathrm{alg}}))$ is isomorphic to a disjoint union of copies of $\mathrm{Spec}(W(\mathbb{F}_\ell^{\mathrm{alg}}))$. It follows that $\mathcal{Y}_{/\mathbb{Q}}$ is a disjoint union of spectra of number fields.

Let $\mathrm{Spec}(R) \to \mathcal{Y}_{/\mathbb{Z}_p^{\circ}}$ be an open affine neighborhood of the point $y \in \mathcal{Y}_{/\mathbb{Z}_p^{\circ}}(\mathbb{F})$ corresponding to the triple $(\mathfrak{a}, \lambda, i)$. By the previous paragraph $R \otimes_{\mathbb{Z}_p^{\circ}} \mathbb{Q}_p^{\circ}$ is a finite product of field extensions of $\mathbb{Q}_p^{\circ}$ of finite degree. Let $R_y$ be the local ring of $R$ at $y$ and let $\widehat{R}_y$ be the completion of $R_y$ with respect to the topology induced by its maximal ideal. We let $I \subset R$

be the ideal of $\mathbb{Z}_p^\circ$-torsion elements and set $S = R/I$. The local ring $S$ is then free of finite rank as a $\mathbb{Z}_p^\circ$-module, and so admits a decomposition

$$S \cong \prod_{\mathfrak{m}} S_{\mathfrak{m}}$$

as a product of complete and separated local rings, where $\mathfrak{m}$ runs over the finitely many maximal ideals of $S$. There are two possibilities to consider: either the $\mathbb{Z}_p^\circ$-algebra map $R \to \mathbb{F}$ determined by $y$ factors through $S$, or it does not. If $R \to \mathbb{F}$ does not factor through $S$ then the local ring $R_y$ contains an invertible $\mathbb{Z}_p^\circ$-torsion element, and hence $\widehat{R}_y$ is $\mathbb{Z}_p^\circ$-torsion. If $R \to \mathbb{F}$ does factor through $S$ then there is a unique factor $S_{\mathfrak{m}}$ in the above decomposition for which the composition $R \to S_{\mathfrak{m}}$ extends to a (necessarily surjective) homomorphism of local rings $R_y \to S_{\mathfrak{m}}$ with $\mathbb{Z}_p^\circ$-torsion kernel $I \otimes_R R_y$. As $S_{\mathfrak{m}}$ is complete, this map extends uniquely to a homomorphism $\widehat{R}_y \to S_{\mathfrak{m}}$, still with $\mathbb{Z}_p^\circ$-torsion kernel. We deduce that

$$\widehat{R}_y \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ \cong S_{\mathfrak{m}} \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$$

is a direct factor of the product of fields

$$R \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ \cong S \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ.$$

In either case we see that $\widehat{R}_y \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ is a finite product of field extensions of $\mathbb{Q}_p^\circ$ of finite degree. Using $\widehat{R}_y \cong \mathcal{O}_{Y,y}^\circ \cong R_{\mathfrak{Y}}$ completes the proof. □

**Corollary 1.2.6.** *Let $\mathfrak{p}$ be a prime ideal of $R_{\mathfrak{Y}}$ with $p \notin \mathfrak{p}$. Then the local ring $R_{\mathfrak{Y},\mathfrak{p}}$ is a field extension of $\mathbb{Q}_p^\circ$ of finite degree. In particular, every horizontal component of $\mathfrak{Y}$ has multiplicity one.*

*Proof.* This is immediate from Lemma 1.2.5. □

For each $\xi \in \mathcal{O}_E^\times$ let $I_\xi \subset \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$ be the proper fractional $\mathcal{R}$-ideal defined by

$$I_\xi \otimes_{\mathbb{Z}} \mathbb{Z}_\ell = \begin{cases} \xi \cdot (\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p) & \text{if } \ell = p \\ \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \text{if } \ell \neq p \end{cases}$$

for all primes $\ell$. Let $(A, \rho) \in \mathfrak{Y}^{\mathfrak{a}}(R)$ be a deformation of $\mathfrak{a}$ to some object $R$ of **Art** for which the action of $\mathcal{R}$ lifts. Denoting by $A \mapsto A^\sim$ the reduction from $R$ to $\mathbb{F}$, there are canonical isomorphisms

$$(A \otimes_{\mathcal{R}} I_\xi)^\sim \cong \mathfrak{a} \otimes_{\mathcal{R}} I_\xi \cong \mathfrak{a} \tag{4}$$

in which the first isomorphism is

$$(A \otimes_{\mathcal{R}} I_\xi)^\sim \cong A^\sim \otimes_{\mathcal{R}} I_\xi \xrightarrow{\rho^{-1} \otimes \mathrm{id}} \mathfrak{a} \otimes_{\mathcal{R}} I_\xi$$

and the second is given on points by

$$\mathfrak{a}(S) \otimes_{\mathcal{R}} I_\xi \to \mathfrak{a}(S) \qquad P \otimes \alpha \mapsto \alpha \cdot P$$

for any $\mathbb{F}$-scheme $S$ (the latter makes sense because the action of $\mathcal{R}$ on $\mathfrak{a}$ extends to an action of $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$). Expressed differently, we are making use of the canonical isomorphism

$$\mathfrak{a} \otimes_{\mathcal{R}} I_\xi \cong \mathfrak{a} \otimes_{\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}} (I_\xi \otimes_{\mathcal{R}} \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}) \cong \mathfrak{a} \otimes_{\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}} \mathcal{O}_{\mathcal{E}_0,\mathcal{F}} \cong \mathfrak{a}$$

arising from $I_\xi \mathcal{O}_{\mathcal{E}_0,\mathcal{F}} = \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$. Denote by $\rho \otimes_{\mathcal{R}} I_\xi$ the inverse of the composition (4) and note that while $A \otimes_{\mathcal{R}} I_\xi$ depends only on the class of $I_\xi$ in $\mathrm{Pic}(\mathcal{R})$ the isomorphism $\rho \otimes_{\mathcal{R}} I_\xi$

depends on the fractional ideal $I_\xi$ itself. We obtain a new deformation of $\mathfrak{a}$, with its $\mathcal{R}$-action, to $R$

$$(A,\rho)\otimes_{\mathcal{R}} I_\xi \stackrel{\text{def}}{=} (A\otimes_{\mathcal{R}} I_\xi, \rho\otimes_{\mathcal{R}} I_\xi)\in \mathfrak{Y}^{\mathfrak{a}}(R)$$

and the operation $\otimes_{\mathcal{R}} I_\xi$ defines an automorphism of the functor $\mathfrak{Y}^{\mathfrak{a}}$.

**Lemma 1.2.7.** *For any $\xi\in\mathcal{O}_E^\times$ the diagram*

$$\begin{array}{ccc} \mathfrak{Y}^{\mathfrak{a}} & \longrightarrow & \mathfrak{Y} \\ {\scriptstyle \otimes_{\mathcal{R}} I_\xi}\downarrow & & \downarrow{\scriptstyle \xi *} \\ \mathfrak{Y}^{\mathfrak{a}} & \longrightarrow & \mathfrak{Y} \end{array}$$

*commutes (the action on the right is that defined in §1.1).*

*Proof.* Fix an object $R$ of **Art** and a deformation $(A,\rho)\in\mathfrak{Y}^{\mathfrak{a}}(R)$. The $p$-Barsotti-Tate group of $A\otimes_{\mathcal{R}} I_\xi$ represents the functor on $R$-schemes

$$S\mapsto A_{p^\infty}(S)\otimes_{\mathbb{Z}_{p^2}[\gamma]}\xi\mathbb{Z}_{p^2}[\gamma],$$

and so there is an isomorphism of $p$-Barsotti-Tate groups

$$A_{p^\infty}\to (A\otimes_{\mathcal{R}} I_\xi)_{p^\infty} \tag{5}$$

which on $S$-points is given by $P\mapsto P\otimes\xi$. Reducing this isomorphism modulo $\mathfrak{m}_R$ one finds the commutative diagram

$$\begin{array}{ccc} A^\sim_{p^\infty} & \longrightarrow & (A\otimes_{\mathcal{R}} I_\xi)^\sim_{p^\infty} \\ {\scriptstyle \rho}\uparrow & & \uparrow{\scriptstyle \rho\otimes_{\mathcal{R}} I_\xi} \\ \mathfrak{a}_{p^\infty} & \xrightarrow[\xi]{} & \mathfrak{a}_{p^\infty}. \end{array}$$

In other words (5) defines an isomorphism of deformations

$$(A_{p^\infty},\rho\circ\xi^{-1})\cong((A\otimes_{\mathcal{R}} I_\xi)_{p^\infty},\rho\otimes_{\mathcal{R}} I_\xi)$$

of $\mathfrak{a}_{p^\infty}\cong\mathfrak{g}$. □

Suppose $R$ is the ring of integers in a finite extension of $\mathbb{Q}_p^\circ$ and fix some $(\mathfrak{G},\rho)\in\mathfrak{Y}(R)$. Let $\mathrm{Ta}_p(\mathfrak{G})$ be the $p$-adic Tate module of $\mathfrak{G}$. Note that the action of $\mathbb{Z}_{p^2}$ and the endomorphism $\gamma\in\mathcal{O}_E$ make $\mathrm{Ta}_p(\mathfrak{G})$ into a $\mathbb{Z}_{p^2}[\gamma]$-module, and that $\mathrm{Ta}_p(\mathfrak{G})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is free of rank one over $E$.

**Definition 1.2.8.** Define the *geometric CM-order* of $(\mathfrak{G},\rho)$ by

$$\mathcal{O}(\mathfrak{G})=\{x\in E\mid x\cdot\mathrm{Ta}_p(\mathfrak{G})\subset\mathrm{Ta}_p(\mathfrak{G})\}$$

and define the *reflex type* of $(\mathfrak{G},\rho)$ to be the isomorphism class of $\mathrm{Lie}(\mathfrak{G})$ as a module over $\mathbb{Z}_{p^2}[\gamma]\otimes_{\mathbb{Z}_p} R$.

*Remark* 1.2.9. The action of $\mathbb{Z}_p^2\otimes_{\mathbb{Z}_p} R\cong R\times R$ on $\mathrm{Lie}(\mathfrak{G})$ induces a decomposition

$$\mathrm{Lie}(\mathfrak{G})\cong\Lambda_1\oplus\Lambda_2$$

in such a way that each $\Lambda_i$ a free $R$-module of rank one, and so that $\mathbb{Z}_{p^2}$ acts through the embedding $\mathbb{Z}_{p^2}\to\mathbb{Z}_p^\circ\to R$ on $\Lambda_1$ and through the conjugate embedding on $\Lambda_2$. The action of $\mathbb{Z}_p[\gamma_0]$ preserves this decomposition and determines a pair of embeddings $(\phi_1,\phi_2)$ of $\mathbb{Z}_p[\gamma_0]$ into $\mathrm{End}_R(\Lambda_i)\cong R$. This pair of embeddings completely determines the reflex type of the deformation $(\mathfrak{G},\rho)\in\mathfrak{Y}(R)$.

The following proposition will be crucial in the determination of the horizontal components of $\mathfrak{Y}$ carried out in §3.

**Proposition 1.2.10.** *Let $R$ be the integer ring of a finite extension of $\mathbb{Q}_p^{\circ}$. Two deformations*

$$(\mathfrak{G},\rho),(\mathfrak{G}',\rho')\in\mathfrak{Y}(R)$$

*lie in the same $\mathcal{O}_E^{\times}$-orbit if and only if they have the same geometric CM-order and the same reflex type.*

*Proof.* One implication is obvious: if there is a $\xi\in\mathcal{O}_E^{\times}$ for which

$$(\mathfrak{G}',\rho')\cong(\mathfrak{G},\rho\circ\xi^{-1})$$

then in particular there is a $\mathbb{Z}_{p^2}[\gamma]$-linear isomorphism $\mathfrak{G}'\cong\mathfrak{G}$, which implies that the geometric CM-orders and reflex types agree. For the other implication we use Lemma 1.2.7. Fix an embedding $\mathrm{Frac}(R)\to\mathbb{C}$. It suffices to prove that if we are given two deformations

$$(A,\rho),(A',\rho')\in\mathfrak{Y}^{\mathfrak{a}}(R)$$

of $\mathfrak{a}$ for which $\mathcal{O}(A_{p^\infty})=\mathcal{O}(A'_{p^\infty})$ and

$$\mathrm{Lie}(A)\cong\mathrm{Lie}(A')$$

as $\mathbb{Z}_{p^2}[\gamma]\otimes_{\mathbb{Z}_p}R$-modules, then there is a $\xi\in\mathcal{O}_E^{\times}$ such that

$$(A',\rho')\cong(A,\rho)\otimes_{\mathcal{R}}I_\xi$$

as deformations of $\mathfrak{a}$. The pro-representability of $\mathfrak{Y}^{\mathfrak{a}}$ implies that the map $\mathfrak{Y}^{\mathfrak{a}}(R)\to\mathfrak{Y}^{\mathfrak{a}}(R')$ is injective whenever $R'$ is the ring of integers of a finite extension of the fraction field of $R$, thus it suffices to prove the existence of such a $\xi$ after enlarging $R$. Therefore we are free to assume that

$$\mathrm{End}_{\mathcal{R}}(A)=\mathrm{End}_{\mathcal{R}}(A_{/\mathrm{Frac}(R)})=\mathrm{End}_{\mathcal{R}}(A_{/\mathbb{C}}).$$

The theory of complex multiplication implies that $\mathrm{End}_{\mathcal{R}}(A)$ is an $\mathcal{O}_{\mathcal{F}}$-order in $\mathcal{E}$ which satisfies

$$\mathrm{End}_{\mathcal{R}}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell\cong\mathrm{End}_{\mathcal{R}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell}(\mathrm{Ta}_\ell(A))$$

for every prime $\ell$. Using

$$\mathrm{Ta}_\ell(A)\cong\mathrm{Ta}_\ell(\mathfrak{a}_0)\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{F}}\cong(\mathcal{O}_{\mathcal{E}_0}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}(\mathcal{O}_{\mathcal{F}}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell)$$

for $\ell\neq p$ we find that

$$\mathrm{End}_{\mathcal{R}}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell=\begin{cases}\mathcal{O}(A_{p^\infty}) & \text{if } \ell=p\\ \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}\otimes_{\mathbb{Z}}\mathbb{Z}_\ell & \text{if } \ell\neq p.\end{cases}$$

Applying the same reasoning to $A'$ shows that $A_{/\mathbb{C}}$ and $A'_{/\mathbb{C}}$ have the same endomorphism ring. As the CM abelian surfaces $A_{/\mathbb{C}}$ and $A'_{/\mathbb{C}}$ have the same reflex types (in the classical sense) and endomorphism rings, the theory of complex multiplication implies that there is some $I\in\mathrm{Pic}(\mathrm{End}_{\mathcal{R}}(A))$ such that

$$A'_{/\mathbb{C}}\cong A_{/\mathbb{C}}\otimes_{\mathrm{End}_{\mathcal{R}}(A)}I.$$

Replacing $I$ by any one of its preimages under $\mathrm{Pic}(\mathcal{R})\to\mathrm{Pic}(\mathrm{End}_{\mathcal{R}}(A))$ we find an $I\in\mathrm{Pic}(\mathcal{R})$ such that

$$A'_{/\mathbb{C}}\cong A_{/\mathbb{C}}\otimes_{\mathcal{R}}I,$$

and after possibly further enlarging $R$ we may assume that

$$A'_{/\mathrm{Frac}(R)}\cong A_{/\mathrm{Frac}(R)}\otimes_{\mathcal{R}}I. \tag{6}$$

Applying the theory of Néron models over the discrete valuation ring $R$ we find that (6) extends uniquely to an isomorphism

$$A' \cong A \otimes_{\mathcal{R}} I$$

of abelian schemes over $R$. As the reductions of $A'$ and $A$ to $\mathbb{F}$ are isomorphic to $\mathfrak{a}$, we deduce that there exists an isomorphism $\mathfrak{a} \cong \mathfrak{a} \otimes_{\mathcal{R}} I$ of abelian varieties over $\mathbb{F}$. Hence we obtain isomorphisms of $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$-modules

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}}(\mathfrak{a},\mathfrak{a}) &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}}(\mathfrak{a},\mathfrak{a}\otimes_{\mathcal{R}} I) \\ &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}}(\mathfrak{a},\mathfrak{a})\otimes_{\mathcal{R}} I. \end{aligned}$$

Using

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}}(\mathfrak{a},\mathfrak{a}) \cong \mathrm{End}_{\mathcal{O}_{\mathcal{E}_0}}(\mathfrak{a}_0)\otimes \mathcal{O}_{\mathcal{F}} \cong \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$$

we find

$$\mathcal{O}_{\mathcal{E}_0,\mathcal{F}} \cong \mathcal{O}_{\mathcal{E}_0,\mathcal{F}} \otimes_{\mathcal{R}} I$$

as $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$-modules. In other words $I$ lies in the kernel of $\mathrm{Pic}(\mathcal{R}) \to \mathrm{Pic}(\mathcal{O}_{\mathcal{E}_0,\mathcal{F}})$. Every such $I$ has the form $I_\xi$ for some $\xi \in \mathcal{O}_E^\times$, and we have now found an isomorphism $A' \to A \otimes_{\mathcal{R}} I_\xi$ of abelian schemes over $R$. Reducing this isomorphism modulo $\mathfrak{m}_R$ yields a commutative diagram

$$\begin{array}{ccc} (A')^{\sim} & \longrightarrow & (A\otimes_{\mathcal{R}} I_\xi)^{\sim} \\ {\scriptstyle\rho'}\uparrow & & \uparrow{\scriptstyle\rho\otimes_{\mathcal{R}} I_\xi} \\ \mathfrak{a} & \longrightarrow & \mathfrak{a} \end{array}$$

in which the bottom horizontal arrow is some $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$-linear automorphism of $\mathfrak{a}$. Every such automorphism is given by the action of some $\zeta \in \mathcal{O}_{\mathcal{E}_0,\mathcal{F}}^\times$, and if we replace $\xi$ by $\xi\zeta^{-1}$ then the bottom horizontal arrow is the identity automorphism of $\mathfrak{a}$. This shows that there is an isomorphism of deformations

$$(A',\rho') \cong (A,\rho)\otimes_{\mathcal{R}} I_\xi,$$

completing the proof. □

**Lemma 1.2.11.** *All but finitely many triples $(A,\lambda,i)\in\mathcal{Y}(\mathbb{F})$ are supersingular.*

*Proof.* The only possible slope sequences for the $p$-Barsotti-Tate group of an abelian surface over $\mathbb{F}$ are $\{0,\frac{1}{2},1\}$, $\{0,0,1,1\}$, or $\{\frac{1}{2},\frac{1}{2}\}$. A $p$-Barsotti-Tate group with slopes $\{0,\frac{1}{2},1\}$ cannot admit an action of $\mathbb{Z}_{p^2}$, thus every point of $\mathcal{Y}(\mathbb{F})$ is either ordinary or supersingular. If $\mathcal{Y}^{\mathrm{ord}}(\mathbb{F}) \subset \mathcal{Y}(\mathbb{F})$ is the subset of ordinary points then taking Serre-Tate canonical lifts gives an injection $\mathcal{Y}^{\mathrm{ord}}(\mathbb{F}) \to \mathcal{Y}(\mathbb{Z}_p^\circ)$. The proof of Lemma 1.2.5 implies that $\mathcal{Y}(\mathbb{Z}_p^\circ)$ is finite, completing the proof. □

Suppose $R$ is an object of **ProArt** with $pR=0$ and that $\mathfrak{G}$ is a $p$-Barsotti-Tate group of dimension $k$ over $R$. The *Hasse-Witt invariant* of $\mathfrak{G}$ is the homomorphism of free rank one $R$-modules

$$\wedge^k \mathrm{Ver} : \wedge^k \mathrm{Lie}(\mathfrak{G}^{(p)}) \to \wedge^k \mathrm{Lie}(\mathfrak{G})$$

where $\mathrm{Ver} : \mathfrak{G}^{(p)} \to \mathfrak{G}$ is the usual Verschiebung morphism. Consider in particular the universal deformation $(\mathfrak{g}^{\mathrm{univ}},\rho^{\mathrm{univ}}) \in \mathfrak{M}(R_{\mathfrak{M}})$ and its reduction $(\mathfrak{G},\rho)$ to $R = R_{\mathfrak{M}} \otimes_{\mathbb{Z}_p^\circ} \mathbb{F}$. By choosing an isomorphism $R \cong \wedge^2\mathrm{Lie}(\mathfrak{G})$ we may identify the kernel of the Hasse-Witt invariant with an ideal $I \subset R$. The kernel of

$$R_{\mathfrak{M}} \to R \to R/I$$

is denoted $I_{\mathrm{HW}}$, and the closed formal subscheme

$$\mathfrak{M}_{\mathrm{HW}} \stackrel{\mathrm{def}}{=} \mathrm{Spf}(R_{\mathfrak{M}}/I_{\mathrm{HW}}) \to \mathfrak{M}$$

is the *Hasse-Witt locus* of $\mathfrak{M}$.

**Proposition 1.2.12.** *Suppose $\mathfrak{C}$ is a vertical component of $\mathfrak{Y}$ in the sense of Definition 1.1.1. The closed immersion $\mathfrak{C} \to \mathfrak{M}$ factors as*

$$\mathfrak{C} \to \mathfrak{M}_{\mathrm{HW}} \to \mathfrak{M}.$$

*Proof.* Fix a vertical component $\mathfrak{C} = R_{\mathfrak{Y}}/\mathfrak{p}$ and let $(\mathfrak{G}, \rho)$ be the pullback to $R_{\mathfrak{Y}}/\mathfrak{p}$ of the universal deformation of $\mathfrak{g}$. We must show that the Hasse-Witt invariant of $\mathfrak{G}$ is 0. Let $y \in \mathcal{Y}(\mathbb{F})$ be the point corresponding to the polarized RM abelian surface $(\mathfrak{a}, \lambda)$ with the action of $\mathcal{O}_{\mathcal{E}_0,\mathcal{F}}$ induced by our fixed action of $\mathcal{O}_{\mathcal{E}_0}$ on $\mathfrak{a}_0$. Let $V \to \mathcal{Y}_{/\mathbb{F}}$ be an open affine neighborhood of $y$ and let $A$ be the restriction to $V$ of the universal RM abelian surface over $\mathcal{Y}_{/\mathbb{F}}$. After shrinking $V$ we may assume that $\mathrm{Lie}(A)$ is free of rank two over $\Gamma(V)$, and (by Lemma 1.2.11) that the fiber of $A$ at every point of $V(\mathbb{F})$ is supersingular.

After choosing generators for the exterior squares of the Lie algebras of $A$ and $A^{(p)}$ the Verschiebung

$$\wedge^2\mathrm{Ver} : \wedge^2\mathrm{Lie}(A^{(p)}) \to \wedge^2\mathrm{Lie}(A)$$

is given by multiplication by some $\beta \in \Gamma(V)$. At any point $z \in V(\mathbb{F})$ the supersingularity of the fiber $A_z$ implies that

$$\mathrm{Ver} : \mathrm{Lie}(A_z^{(p)}) \to \mathrm{Lie}(A_z)$$

is not invertible, and thus $\beta$ vanishes at every $z \in V(\mathbb{F})$ (in general, an abelian variety over $\mathbb{F}$ is ordinary if and only if the Verschiebung map induces an isomorphism of Lie algebras). As $\Gamma(V)$ is of finite type over $\mathbb{F}$ its Jacobson radical is equal to its nilradical by [17, Theorem 5.5], and we deduce that $\beta^n = 0$ for some positive integer $n$. If we let $\mathcal{O}^{\circ}_{V,y}$ denote the completion of the local ring of $V$ at $y$ then $\mathcal{O}^{\circ}_{V,y} \cong R_{\mathfrak{Y}} \otimes_{\mathbb{Z}_p^{\circ}} \mathbb{F}$, the $p$-Barsotti-Tate group $\mathfrak{G}$ is the base change of the $p$-Barsotti-Tate group $A_{p^\infty}$ via

$$\Gamma(V) \to \mathcal{O}^{\circ}_{V,y} \to R_{\mathfrak{Y}}/\mathfrak{p},$$

and the Hasse-Witt invariant of $\mathfrak{G}$ is (for appropriate choice of generators of the exterior squares of the Lie algebras of $\mathfrak{G}$ and $\mathfrak{G}^{(p)}$) multiplication by the image of $\beta$ in $R_{\mathfrak{Y}}/\mathfrak{p}$. As the image of $\beta$ is 0, we are done. □

## 2. Vertical components

In this section we will determine all vertical components of $\mathfrak{Y}$. The final answer is simple: if $c_0 = 0$ then there are no vertical components, while if $c_0 > 0$ there are exactly two, each meeting $\mathfrak{M}_0$ transversely (i.e. the intersection number of Definition 1.1.1 is equal to 1). More difficult is the problem of determining the multiplicities of these components. We will accomplish this (at least for $p > 2$, a hypothesis which seems essential to the method) by making heavy use of Zink's theory of *windows* [27], an alternative to the theory of displays [18, 28] which is more amenable to explicit calculation. The calculations are inspired by a method used by Kudla-Rapoport for studying special cycles on unitary Shimura varieties [15].

2.1. **Dieudonné theory.** Let $W_0$ denote the completion of the strict henselization of $\mathcal{O}_{E_0}$ with respect to the homomorphism $\psi : \mathcal{O}_{E_0} \to \mathbb{F}$ fixed in (1). More concretely,

$$W_0 \cong \begin{cases} \mathbb{Z}_p^\circ & \text{if } E_0/\mathbb{Q}_p \text{ is unramified} \\ \mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^\circ & \text{if } E_0/\mathbb{Q}_p \text{ is ramified.} \end{cases}$$

In either case $W_0$ is integrally closed in its fraction field $M_0$, and we are given an embedding $\Psi : \mathcal{O}_{E_0} \to W_0$ which lifts $\psi$. For any ring homomorphism $\phi : \mathcal{O}_{E_0} \to R$ we denote by $\overline{\phi}$ the map obtained by precomposing $\phi$ with the nontrivial Galois automorphism of $E_0/\mathbb{Q}_p$. If $R$ is an object of **ProArt** we let $W(R)$ be the ring of Witt vectors of $R$ and let $I_R \subset W(R)$ be the kernel of the natural surjection $W(R) \to R$. For any $r \in R$ denote by $[r] \in W(R)$ the Teichmuller lift. The ring $W(R)$ is equipped with a ring endomorphism $r \mapsto {}^F r$ and an endomorphism of the underlying additive group $r \mapsto {}^V r$.

We will use Zink's theory of *displays* as in [28]; a quick summary of the basics can be found in [6]. A display over an object $R$ of **ProArt** consists of a quadruple $\mathbf{D} = (P, Q, F, V^{-1})$ in which $P$ is finitely generated free $W(R)$-module, $Q \subset P$ is a $W(R)$-submodule, and $F : P \to P$ and $V^{-1} : Q \to P$ are ${}^F$-linear operators. There are addition properties which the quadruple $(P, Q, F, V^{-1})$ is to satisfy; see [28, Definitions 1 and 2]. If $(D, F, V)$ is a Dieudonné module over $\mathbb{F}$ with the property that $V$ is topologically nilpotent then the quadruple $(D, VD, F, V^{-1})$ is a display, and this construction establishes an equivalence of categories between displays over $\mathbb{F}$ and Dieudonné modules over $\mathbb{F}$ on which $V$ is topologically nilpotent. By [28, Theorem 9] for any object $R$ of **Art** there is a (covariant) equivalence of categories between displays over $R$ and $p$-Barsotti-Tate groups over $R$ with connected special fiber. By the comments after [28, Theorem 4] the Lie algebra of a connected $p$-Barsotti-Tate group is isomorphic to the Lie algebra of its display $\mathbf{D} = (P, Q, F, V^{-1})$, defined as the free $R$-module

$$\mathrm{Lie}(\mathbf{D}) \stackrel{\mathrm{def}}{=} P/Q.$$

The *height* of $\mathbf{D}$ is $\mathrm{rank}_{W(R)}(P)$ and the *dimension* of $\mathbf{D}$ is $\mathrm{rank}_R(P/Q)$. The Lie algebra of a Dieudonné module $(D, F, V)$ is defined to be the Lie algebra of its associated display $\mathrm{Lie}(D) = D/VD$.

Consider the *standard supersingular Dieudonné module* $(D_0, F, V)$ over $\mathbb{F}$ whose underlying $\mathbb{Z}_p^\circ$-module is free on two generators $\{e_0, f_0\}$ with the operators $F$ and $V$ defined by

$$Fe_0 = f_0 \qquad Ff_0 = pe_0$$

and

$$Ve_0 = f_0 \qquad Vf_0 = pe_0.$$

If $E_0/\mathbb{Q}_p$ is unramified then $\Psi$ takes values in $W_0 = \mathbb{Z}_p^\circ$, and the *normalized action* of $\mathcal{O}_{E_0}$ on $D_0$ is defined by

$$r * e_0 = \Psi(r)e_0 \qquad r * f_0 = \overline{\Psi}(r)f_0$$

for all $r \in \mathcal{O}_{E_0}$. The *anti-normalized* action of $\mathcal{O}_{E_0}$ on $D_0$ is defined by

$$r * e_0 = \overline{\Psi}(r)e_0 \qquad r * f_0 = \Psi(r)f_0.$$

The action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(D_0)$ induced by the normalized action is through $\psi$, and the action of $\mathcal{O}_{E_0}$ induced by the anti-normalized action is through $\overline{\psi}$. If instead $E_0/\mathbb{Q}_p$ is ramified let $\varpi_{E_0} \in \mathcal{O}_{E_0}$ be a root of an Eisenstein polynomial in $\mathbb{Z}_p[x]$, so that $\mathcal{O}_{E_0} = \mathbb{Z}_p \oplus \mathbb{Z}_p \varpi_{E_0}$. Using the surjectivity of the trace $\mathbb{Z}_{p^2} \to \mathbb{Z}_p$ and norm $\mathbb{Z}_{p^2}^\times \to \mathbb{Z}_p^\times$ this Eisenstein polynomial has the form

$$x^2 - (a + a^\sigma)x + (aa^\sigma - bb^\sigma p)$$

for some $a \in p\mathbb{Z}_{p^2}$ and $b \in \mathbb{Z}_{p^2}^\times$. Having fixed such $a$ and $b$ we now let $\varpi_{E_0}$ act on $D_0$ by

$$\varpi_{E_0} * e_0 = ae_0 + b^\sigma f_0 \qquad \varpi_{E_0} * f_0 = pbe_0 + a^\sigma f_0.$$

This determines an action of $\mathcal{O}_{E_0}$ on $D_0$ which we call the *normalized* action of $\mathcal{O}_{E_0}$. The *anti- normalized* action is obtained by precomposing the normalized action with the nontrivial Galois automorphism of $\mathcal{O}_{E_0}$. Under either action $\mathcal{O}_{E_0}$ acts on $\mathrm{Lie}(D_0)$ through $\psi = \overline{\psi}$ (the unique $\mathbb{Z}_p$-algebra homomorphism $\mathcal{O}_{E_0} \to \mathbb{F}$.)

**Lemma 2.1.1.** *The covariant Dieudonné module of the p-Barsotti-Tate group $\mathfrak{g}_0$ fixed in §1.1 is $\mathcal{O}_{E_0}$-linearly isomorphic to the standard supersingular Dieudonné module $D_0$ with its normalized $\mathcal{O}_{E_0}$-action.*

*Proof.* As the action of $\mathcal{O}_{E_0}$ on the Lie algebra of $D_0$ is through $\psi$, this is a consequence of Corollary 1.1.3. □

**Lemma 2.1.2.** *Suppose $\mathfrak{G}_0'$ and $\mathfrak{G}_0''$ are p-Barsotti-Tate groups of height two and dimension one over $\mathbb{F}$, each equipped with an $\mathcal{O}_{E_0}$-action. Suppose further that there is an $\mathcal{O}_{E_0}$-linear isogeny $f : \mathfrak{G}_0' \to \mathfrak{G}_0''$ of degree $p^k$. Let $\phi : \mathcal{O}_{E_0} \to \mathbb{F}$ be the homomorphism giving the action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{G}_0'')$.*

(a) *If $k$ is even then $\mathcal{O}_{E_0}$ acts on $\mathrm{Lie}(\mathfrak{G}_0')$ through $\phi$.*
(b) *If $k$ is odd then $\mathcal{O}_{E_0}$ acts on $\mathrm{Lie}(\mathfrak{G}_0')$ through $\overline{\phi}$.*

*Proof.* If $E_0/\mathbb{Q}_p$ is ramified then there is a unique $\mathbb{Z}_p$-algebra homomorphism $\mathcal{O}_{E_0} \to \mathbb{F}$, and so there is nothing to prove. Thus we assume that $E_0/\mathbb{Q}_p$ is unramified and let $D_0'$ and $D_0''$ be the covariant Dieudonné modules of $\mathfrak{G}_0'$ and $\mathfrak{G}_0''$. For simplicity we assume $\phi = \psi$ and fix, using Corollary 1.1.3, an $\mathcal{O}_{E_0}$-linear isomorphism from $D_0''$ to the standard supersingular Dieudonné module $D_0$ with its normalized action of $\mathcal{O}_{E_0}$. The isogeny $f$ then identifies $D_0'$ with an $\mathcal{O}_{E_0}$-stable sub-Dieudonné module $D_0' \subset D_0$ with $D_0/D_0'$ of length $k$ as a $\mathbb{Z}_p^\circ$-module. Every such submodule has the form

$$D_0' = p^a \mathbb{Z}_p^\circ e_0 + p^{a+\epsilon}\mathbb{Z}_p^\circ f_0$$

with $a \geq 0$, $\epsilon \in \{0,1\}$, and $k = 2a + \epsilon$. If $k$ is even then $\epsilon = 0$ and $\mathrm{Lie}(\mathfrak{G}_0') = D_0'/VD_0'$ is generated by the image of $p^a e_0$. Hence $\mathcal{O}_{E_0}$ acts on $\mathrm{Lie}(\mathfrak{G}_0')$ through $\psi$. If $k$ is odd then $\epsilon = 1$ and $\mathrm{Lie}(\mathfrak{G}_0') = D_0'/VD_0'$ is generated by the image of $p^{a+1}f_0$. Hence $\mathcal{O}_{E_0}$ acts on $\mathrm{Lie}(\mathfrak{G}_0')$ through $\overline{\psi}$. □

Let $\mathbf{d}_0 = (P_0, Q_0, F, V^{-1})$ be the display associated associated to $(D_0, F, V)$, the *standard supersingular display.* Thus $P_0$ is the free $\mathbb{Z}_p^\circ$-module on $\{e_0, f_0\}$, $Q_0 \subset P_0$ is free on the generators $\{pe_0, f_0\}$, and the operators $F$ and $V^{-1}$ are completely determined by the relations

$$Fe_0 = f_0 \qquad V^{-1}f_0 = e_0.$$

As in [28, (9)] we encode this information in the *displaying matrix* $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ of $\mathbf{d}_0$. Define a display $\mathbf{d}_0^{\mathrm{univ}} = (P_0, Q_0, F, V^{-1})$ over $R = \mathbb{Z}_p^\circ[[x_0]]$ as follows. Let $P_0$ be the free $W(R)$ module on two generators $\{e_0^{\mathrm{univ}}, f_0^{\mathrm{univ}}\}$, let $Q_0 \subset P_0$ be the $W(R)$-submodue

$$Q_0 = I_R e_0^{\mathrm{univ}} + W(R) f_0^{\mathrm{univ}},$$

and take the displaying matrix of $\mathbf{d}^{\mathrm{univ}}$ with respect to this basis to be $\begin{pmatrix} [x_0] & 1 \\ 1 & \end{pmatrix}$. Comparing with the displaying matrix of the standard supersingular display we see that the reduction of $\mathbf{d}_0^{\mathrm{univ}}$ to $\mathbb{F}$ is isomorphic to $\mathbf{d}_0$.

For any display $\mathbf{D}_0 = (P_0, Q_0, F, V^{-1})$ over an object $R$ of **ProArt** define a new display $\mathbf{D}_0 \otimes \mathbb{Z}_{p^2} = (P, Q, F, V^{-1})$ over $R$ in the obvious way; thus

$$P = P_0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2} \qquad Q = Q_0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$$

and $F$ and $V^{-1}$ are the $\mathbb{Z}_{p^2}$-linear extensions of the corresponding operators of $\mathbf{D}_0$. In particular we define

$$\mathbf{d} = \mathbf{d}_0 \otimes \mathbb{Z}_{p^2}.$$

Using the homomorphism $\mathbb{Z}_p^\circ \to W(\mathbb{Z}_p^\circ)$ of [9, §17.6] we obtain a canonical $\mathbb{Z}_p$-algebra homomorphism

$$\mathbb{Z}_p^\circ \to W(R)$$

for any $\mathbb{Z}_p^\circ$-algebra (and in particular any object of **ProArt**) $R$, and hence a canonical homomorphism $\mathbb{Z}_{p^2} \to W(R)$. Let $\epsilon_1$ and $\epsilon_2$ be the usual idempotents in

$$W(R) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2} \cong W(R) \times W(R)$$

indexed so that

$$(1 \otimes \alpha)\epsilon_1 = (\alpha \otimes 1)\epsilon_1 \qquad (1 \otimes \alpha)\epsilon_2 = (\alpha^\sigma \otimes 1)\epsilon_2$$

for all $\alpha \in \mathbb{Z}_{p^2}$. Recalling that the underlying $\mathbb{Z}_p^\circ$-module of $\mathbf{d}_0$ has basis $\{e_0, f_0\}$, define a basis $\{e_1, e_2, f_1, f_2\}$ of the underlying $\mathbb{Z}_p^\circ$-module of $\mathbf{d}$ by

$$e_i = \epsilon_i e_0 \qquad f_i = \epsilon_i f_0.$$

With respect to this basis $\mathbf{d}$ has displaying matrix

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}. \tag{7}$$

The normalized action of $\mathcal{O}_{E_0}$ on $\mathbf{d}_0$ determines an action of $\mathcal{O}_E$ on $\mathbf{d}$. If $(D, F, V)$ denotes the Dieudonné module associated to $\mathbf{d}$ then $D$ is the free $\mathbb{Z}_p^\circ$-module on the generators $\{e_1, e_2, f_1, f_2\}$, the operators $F$ and $V$ both satisfy

$$e_1 \mapsto f_2 \qquad e_2 \mapsto f_1 \qquad f_1 \mapsto pe_2 \qquad f_2 \mapsto pe_1,$$

and the action of $\mathbb{Z}_{p^2}$ on $D$ is given by

$$\alpha * e_1 = \alpha e_1 \qquad \alpha * e_2 = \alpha^\sigma e_2 \qquad \alpha * f_1 = \alpha f_1 \qquad \alpha * f_2 = \alpha^\sigma f_2. \tag{8}$$

Now abbreviate $R = \mathbb{Z}_p^\circ[[x_1, x_2]]$ and define a display $\mathbf{d}^{\mathrm{univ}} = (P, Q, F, V^{-1})$ over $R$ as follows. The underlying $W(R)$-module $P$ of $\mathbf{d}^{\mathrm{univ}}$ is free on four generators $\{e_1^{\mathrm{univ}}, e_2^{\mathrm{univ}}, f_1^{\mathrm{univ}}, f_2^{\mathrm{univ}}\}$. The $W(R)$- submodule $Q \subset P$ is defined by

$$Q = I_R e_1^{\mathrm{univ}} + I_R e_2^{\mathrm{univ}} + W(R) f_1^{\mathrm{univ}} + W(R) f_2^{\mathrm{univ}},$$

and the displaying matrix of $\mathbf{d}^{\mathrm{univ}}$ with respect to this basis is

$$\begin{pmatrix} & [x_1] & & 1 \\ [x_2] & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}. \tag{9}$$

We endow $\mathbf{d}^{\mathrm{univ}}$ with an action of $\mathbb{Z}_{p^2}$ using the formulas (8), replacing $e_i$ with $e_i^{\mathrm{univ}}$ and $f_i$ with $f_i^{\mathrm{univ}}$. One can check that setting $x_2 = x_1 = x_0$ in (9) recovers the displaying matrix of $\mathbf{d}_0^{\mathrm{univ}} \otimes \mathbb{Z}_{p^2}$, and hence $\mathbf{d}_0^{\mathrm{univ}} \otimes \mathbb{Z}_{p^2}$ is $\mathbb{Z}_{p^2}$-linearly isomorphic to the base change of $\mathbf{d}^{\mathrm{univ}}$ through the isomorphism

$$\mathbb{Z}_p^\circ[[x_1, x_2]]/(x_1 - x_2) \xrightarrow{x_i \mapsto x_0} \mathbb{Z}_p^\circ[[x_0]].$$

Comparing (7) with (9), the base change of $\mathbf{d}^{\mathrm{univ}}$ to the residue field of $\mathbb{Z}_p[[x_1, x_2]]$ is $\mathbb{Z}_{p^2}$-linearly isomorphic to $\mathbf{d} = \mathbf{d}_0 \otimes \mathbb{Z}_{p^2}$, and hence (by Lemma 2.1.1) is $\mathbb{Z}_{p^2}$-linearly isomorphic to the display of the $p$-Barsotti-Tate group $\mathfrak{g} \cong \mathfrak{g}_0 \otimes \mathbb{Z}_{p^2}$. Thus we have a commutative diagram of functors on **Art**

$$\begin{array}{ccc} \mathrm{Spf}(\mathbb{Z}_p^\circ[[x_0]]) & \longrightarrow & \mathfrak{M}_0 \\ \downarrow & & \downarrow {\scriptstyle \otimes \mathbb{Z}_{p^2}} \\ \mathrm{Spf}(\mathbb{Z}_p^\circ[[x_1, x_2]]) & \longrightarrow & \mathfrak{M} \end{array}$$

in which the top horizontal arrow sends a morphism $\mathbb{Z}_p^\circ[[x_0]] \to R$ in **ProArt** to the $p$-divisible group associated to the base change of $\mathbf{d}_0^{\mathrm{univ}}$ to $R$, the bottom horizontal arrow sends a morphism $\mathbb{Z}_p^\circ[[x_1, x_2]] \to R$ to the $p$- divisible group associated to the base change of $\mathbf{d}^{\mathrm{univ}}$ to $R$, and the vertical arrow on the left is $x_i \mapsto x_0$.

**Proposition 2.1.3.** *The horizontal arrows in the above diagram are isomorphisms.*

*Proof.* Consider the top horizontal arrow. By [4, Theorem 1.5.3(3)] it suffices to prove that the induced map on tangent spaces

$$\mathrm{Hom}_{\mathbf{ProArt}}(\mathbb{Z}_p^\circ[[x_0]], \mathbb{F}[\epsilon]) \to \mathfrak{M}_0(\mathbb{F}[\epsilon])$$

is a bijection, and this is proved by Zink [28, §2.2] using his deformation theory of displays. The proof for the bottom horizontal arrow is similar; details can be found in [6, §6.11] and [7]. □

To paraphrase the proposition: we may identify $R_{\mathfrak{M}} \cong \mathbb{Z}_p^\circ[[x_1, x_2]]$ in such a way that the closed immersion $\mathfrak{M}_0 \to \mathfrak{M}$ is identified with the closed immersion

$$\mathrm{Spf}(\mathbb{Z}_p^\circ[[x_0]]) \to \mathrm{Spf}(\mathbb{Z}_p^\circ[[x_1, x_2]])$$

defined by $x_i \mapsto x_0$.

2.2. **The Hasse-Witt locus.** Using Proposition 2.1.3 we identify

$$R_{\mathfrak{M}} \cong \mathbb{Z}_p^\circ[[x_1, x_2]],$$

so that the ideal defining the closed formal subscheme $\mathfrak{M}_0$ is generated by $x_1 - x_2$. The display of the universal deformation of $\mathfrak{g}$ is given by the displaying matrix (9), and the Lie algebra of the universal deformation is the free $\mathbb{Z}_p^\circ[[x_1, x_2]]$-module on the generators $\{e_1^{\mathrm{univ}}, e_2^{\mathrm{univ}}\}$. Using [28, Example 23], after base change from $\mathbb{Z}_p^\circ[[x_1, x_2]]$ to $\mathbb{F}[[x_1, x_2]]$ the Verschiebung morphism on Lie algebras can be read off from the matrix (9), and is given by

$$e_1^{\mathrm{univ}} \mapsto x_2 e_2^{\mathrm{univ}} \qquad e_2^{\mathrm{univ}} \mapsto x_1 e_1^{\mathrm{univ}}.$$

Thus the Hasse-Witt locus $\mathfrak{M}_{\mathrm{HW}} \to \mathfrak{M}$ (in the sense of §1.2) is seen to be the closed formal subscheme defined by the ideal $I_{\mathrm{HW}} = (p, x_1 x_2)$. We find that $\mathfrak{M}_{\mathrm{HW}}$ has two irreducible components which we denote by

$$\begin{aligned} \mathfrak{C}_1^{\mathrm{ver}} &= \mathrm{Spf}(R_{\mathfrak{M}}/(p, x_2)) \cong \mathrm{Spf}(\mathbb{F}[[x_1]]) \\ \mathfrak{C}_2^{\mathrm{ver}} &= \mathrm{Spf}(R_{\mathfrak{M}}/(p, x_1)) \cong \mathrm{Spf}(\mathbb{F}[[x_2]]). \end{aligned}$$

Each of $\mathfrak{C}_i^{\mathrm{ver}}$ meets $\mathfrak{M}_0$ transversely in the sense that

$$\mathfrak{C}_i^{\mathrm{ver}} \times_{\mathfrak{M}} \mathfrak{M}_0 \cong \mathrm{Spf}(\mathbb{F}). \tag{10}$$

According to Proposition 1.2.12 any vertical component of $\mathfrak{Y}$ is contained in the Hasse-Witt locus, and so must be one of $\mathfrak{C}_i^{\mathrm{ver}}$.

We will examine the component $\mathfrak{C}_1^{\mathrm{ver}}$. Let $(\mathfrak{G}_1^{\mathrm{ver}}, \rho_1^{\mathrm{ver}}) \in \mathfrak{M}(\mathbb{F}[[x_1]])$ be the restriction to $\mathfrak{C}_1^{\mathrm{ver}}$ of the universal deformation of $\mathfrak{g}$ over $\mathfrak{M}$. We wish to determine which endomorphisms of $\mathfrak{g}$ lift to endomorphisms of $\mathfrak{G}_1^{\mathrm{ver}}$. Let

$$\mathbf{D}_1^{\mathrm{ver}} = (P_1^{\mathrm{ver}}, Q_1^{\mathrm{ver}}, F, V^{-1})$$

be the display over $\mathbb{F}[[x_1]]$ corresponding to $\mathfrak{G}_1^{\mathrm{ver}}$. According to §2.1 the $W(\mathbb{F}[[x_1]])$-module $P_1^{\mathrm{ver}}$ is free on the generators $\{e_1, e_2, f_1, f_2\}$, the submodule $Q_1^{\mathrm{ver}}$ is

$$I_{\mathbb{F}[[x_1]]} e_1 + I_{\mathbb{F}[[x_1]]} e_2 + W(\mathbb{F}[[x_1]]) f_1 + W(\mathbb{F}[[x_1]]) f_2,$$

and the operators $F$ and $V^{-1}$ are determined by the displaying matrix over $W(\mathbb{F}[[x_1]])$ with respect to the basis $\{e_1, e_2, f_1, f_2\}$

$$\begin{pmatrix} & [x_1] & & 1 \\ 0 & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

Let $\mathrm{Fr} : \mathbb{Z}_p^\circ[[x_1]] \to \mathbb{Z}_p^\circ[[x_1]]$ be the unique ring homomorphism which is the Frobenius on $\mathbb{Z}_p^\circ$ and satisfies $x_1 \mapsto x_1^p$. The *window* [27] associated to $\mathbf{D}_1^{\mathrm{ver}}$ with respect to the *frame* $\mathbb{Z}_p^\circ[[x_1]] \to \mathbb{F}[[x_1]]$ is the triple $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ where $M_1^{\mathrm{ver}}$ is the free $\mathbb{Z}_p^\circ[[x_1]]$-module on the generators $\{e_1, e_2, f_1, f_2\}$, the submodule $N_1^{\mathrm{ver}}$ is generated by $\{pe_1, pe_2, f_1, f_2\}$, and $\Phi : M_1^{\mathrm{ver}} \to M_1^{\mathrm{ver}}$ is the Fr-linear map satisfying

$$e_1 \mapsto f_2 \qquad e_2 \mapsto x_1 e_1 + f_1 \qquad f_1 \mapsto pe_2 \qquad f_2 \mapsto pe_1.$$

The action of $\mathbb{Z}_{p^2}$ on $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ is by the rule (8). The window of $\mathfrak{g}$ with respect to the frame $\mathbb{Z}_p^\circ \to \mathbb{F}$ is obtained as the base change $(M, N, \Phi)$ of the triple $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ via the map $\mathbb{Z}_p^\circ[[x_1]] \to \mathbb{Z}_p^\circ$ defined by $x_1 \mapsto 0$, and every $\Gamma \in \mathrm{End}_{\mathbb{Z}_{p^2}}(\mathfrak{g})$ is determined by its matrix with respect to the ordered basis $\{e_1, f_1, e_2, f_2\}$ of $M$. The condition that $\Gamma$ commutes with the $\mathbb{Z}_{p^2}$-action is equivalent to this matrix having the block diagonal form

$$\Gamma = \begin{pmatrix} Y & \\ & Z \end{pmatrix},$$

while the condition that $\Gamma$ commutes with $\Phi$ and preserves the submodule $N$ is equivalent to $Y$ and $Z$ having the form

$$Y = \begin{pmatrix} a & pb \\ c & d \end{pmatrix} \qquad Z = \begin{pmatrix} d^\sigma & pc^\sigma \\ b^\sigma & a^\sigma \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}_{p^2}$. The subring $\mathrm{End}(\mathfrak{g}_0) \subset \mathrm{End}_{\mathbb{Z}_{p^2}}(\mathfrak{g})$ consists of those $\Gamma$ for which $d = a^\sigma$ and $c = b^\sigma$.

For each $k \geq 0$ set

$$A[k] = \mathbb{Z}_p^\circ[[x_1]]/(x_1^{p^k}) \qquad R[k] = \mathbb{F}[[x_1]]/(x_1^{p^k})$$

and let $(M[k], N[k], \Phi)$ be the base change of $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ via $\mathbb{Z}_p^\circ[[x_1]] \to A[k]$, so that $(M[k], N[k], \Phi)$ is the window associated to $\mathfrak{G}_{1/R[k]}^{\mathrm{ver}}$ with respect to the frame $A[k] \to$

$R[k]$. We will lift the endomorphism $\Gamma[0] = \Gamma$ of $(M[0], N[0], \Phi) = (M, N, \Phi)$ to a quasi-endomorphism $\Gamma_1^{\mathrm{ver}}$ of $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ by successively lifting from $A[k]$ to $A[k+1]$. With respect to the ordered basis $e_1, f_1, e_2, f_2$ any $\mathbb{Z}_{p^2}$-linear quasi-endomorphism $\Gamma[k]$ of $(M[k], N[k], \Phi)$ must have the form

$$\Gamma[k] = \begin{pmatrix} Y[k] & \\ & Z[k] \end{pmatrix},$$

where $Y[k]$ and $Z[k]$ are $2 \times 2$ matrices with entries in $A[k] \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ which satisfy

$$(11) \qquad Y[k] \cdot \begin{pmatrix} x_1 & p \\ 1 & \end{pmatrix} = \begin{pmatrix} x_1 & p \\ 1 & \end{pmatrix} \cdot \mathrm{Fr}(Z[k]) \qquad Z[k] \cdot \begin{pmatrix} & p \\ 1 & \end{pmatrix} = \begin{pmatrix} & p \\ 1 & \end{pmatrix} \cdot \mathrm{Fr}(Y[k])$$

(these conditions are equivalent to $\Gamma[k]$ commuting with $\Phi$), and such a $\Gamma[k]$ is an endomorphism if and only if $Y[k]$ and $Z[k]$ have entries in $A[k]$ and each of the upper right entries is in $pA[k]$ (so that the submodule $N[k] \subset M[k]$ is preserved). The crucial observation is the following: the ring homomorphism $\mathrm{Fr} : A[k+1] \to A[k+1]$ factors through the quotient map $A[k+1] \to A[k]$, so that there is a commutative diagram

$$\begin{array}{ccc} A[k+1] & \xrightarrow{\mathrm{Fr}} & A[k+1] \\ \downarrow & \nearrow_{\mathrm{Fr}} & \\ A[k]. & & \end{array}$$

Replacing $k$ by $k+1$ in (11) and then solving for $Y[k+1]$ and $Z[k+1]$ we find that to successively lift $\Gamma[0]$ to a quasi-endomorphism of each $(M[k], N[k], \Phi)$ is equivalent to solving the recursion relations

$$(12) \qquad \begin{aligned} Y[k+1] &= \frac{1}{p} \begin{pmatrix} x_1 & p \\ 1 & \end{pmatrix} \cdot \mathrm{Fr}(Z[k]) \cdot \begin{pmatrix} & p \\ 1 & -x_1 \end{pmatrix} \\ Z[k+1] &= \frac{1}{p} \begin{pmatrix} & p \\ 1 & \end{pmatrix} \cdot \mathrm{Fr}(Y[k]) \cdot \begin{pmatrix} & p \\ 1 & \end{pmatrix} \end{aligned}$$

in $A[k] \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$. By trial and error one can explicitly solve the recursion (12). If we define matrices $Y_1^{\mathrm{ver}}$ and $Z_1^{\mathrm{ver}}$ in $\mathbb{Z}_p^\circ[[x_1]] \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ by

$$\begin{aligned} Y_1^{\mathrm{ver}} &= \begin{pmatrix} a & pb \\ c & d \end{pmatrix} + f(x_1) \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix} - g(x_1) \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \\ Z_1^{\mathrm{ver}} &= \begin{pmatrix} d^\sigma & pc^\sigma \\ b^\sigma & a^\sigma \end{pmatrix} + f(x_1^p) \begin{pmatrix} -c^\sigma & 0 \\ \frac{(d-a)^\sigma}{p} & c^\sigma \end{pmatrix} - g(x_1^p) \begin{pmatrix} 0 & 0 \\ \frac{c^\sigma}{p} & 0 \end{pmatrix} \end{aligned}$$

where

$$f(x) = x^{p^0} + x^{p^2} + x^{p^4} + x^{p^6} + x^{p^8} + \cdots$$

and

$$\begin{aligned} g(x) = {} & x^{p^0}(x^{p^0}) + x^{p^2}(2x^{p^0} + x^{p^2}) + x^{p^4}(2x^{p^0} + 2x^{p^2} + x^{p^4}) \\ & + x^{p^6}(2x^{p^0} + 2x^{p^2} + 2x^{p^4} + x^{p^6}) + x^{p^8}(2x^{p^0} + 2x^{p^2} + 2x^{p^4} + 2x^{p^6} + x^{p^8}) + \cdots \end{aligned}$$

then $Y[k]$ and $Z[k]$ are the images of $Y_1^{\mathrm{ver}}$ and $Z_1^{\mathrm{ver}}$ under the quotient map $\mathbb{Z}_p^\circ[[x_1]] \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ \to A[k] \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$. From this it is clear that $\Gamma[0] = \Gamma$ lifts from an endomorphism of $\mathfrak{g}$ to a quasi-endomorphism of $\mathfrak{G}_1^{\mathrm{ver}}$ given by

$$(13) \qquad \Gamma_1^{\mathrm{ver}} = \begin{pmatrix} Y_1^{\mathrm{ver}} & \\ & Z_1^{\mathrm{ver}} \end{pmatrix}$$

and that this quasi-endomorphism is integral if and only if $a - d \in p\mathbb{Z}_{p^2}$ and $c \in p\mathbb{Z}_{p^2}$.

**Proposition 2.2.1.** *Recall from §1.1 that $c_0$ is defined by $\mathbb{Z}_p[\gamma_0] = \mathbb{Z}_p + p^{c_0}\mathcal{O}_{E_0}$.*

(a) *If $c_0 = 0$ then $\mathfrak{Y}$ has no vertical components.*
(b) *If $c_0 > 0$ then $\mathfrak{Y}$ has exactly two vertical components, $\mathfrak{C}_1^{\mathrm{ver}}$ and $\mathfrak{C}_2^{\mathrm{ver}}$, each satisfying $I_{\mathfrak{M}}(\mathfrak{C}_i^{\mathrm{ver}}, \mathfrak{M}_0) = 1$.*

*Proof.* Consider first the closed formal subscheme $\mathfrak{C}_1^{\mathrm{ver}} = \mathrm{Spf}(\mathbb{F}[[x_1]])$ of $\mathfrak{M}$ defined by the ideal $(p, x_2) \subset R_{\mathfrak{M}}$. Let $(\mathfrak{G}_1^{\mathrm{ver}}, \rho_1^{\mathrm{ver}})$ be the restriction of the universal deformation of $\mathfrak{g}$ to $\mathfrak{C}_1$. As above, the action of the endomorphism $j(\gamma) \in \mathrm{End}_{\mathbb{Z}_{p^2}}(\mathfrak{g})$ is determined by its action on the window $(M, N, \Phi)$ associated to $\mathfrak{g}$, and is given by a matrix of the form

$$\Gamma = \begin{pmatrix} a & pb & & \\ b^\sigma & a^\sigma & & \\ & & a & pb \\ & & b^\sigma & a^\sigma \end{pmatrix}$$

with $a, b \in \mathbb{Z}_{p^2}$. By what was said above, $\mathfrak{C}_1^{\mathrm{ver}}$ is contained in $\mathfrak{Y}$ if and only if the unique lift of $j(\gamma)$ to a quasi-endomorphism of $\mathfrak{G}_1^{\mathrm{ver}}$ is an endomorphism, which is equivalent to $p$ dividing both $a - a^\sigma$ and $b$. Using the fact that $\mathbb{Z}_p[\gamma_0]$ is isomorphic to the $\mathbb{Z}_p$-subalgebra of $M_2(\mathbb{Z}_{p^2})$ generated by $\begin{pmatrix} a & pb \\ b^\sigma & a^\sigma \end{pmatrix}$ it is easy to see that

$$p \mid (a - a^\sigma) \text{ and } p \mid b \iff \mathbb{Z}_p[\gamma_0] \subset \mathbb{Z}_p + p\mathcal{O}_{E_0} \iff c_0 > 0.$$

The same argument with $\mathfrak{C}_1^{\mathrm{ver}}$ replaced by $\mathfrak{C}_2^{\mathrm{ver}}$ shows that each of the two components of the Hasse-Witt locus of $\mathfrak{M}$ is contained in $\mathfrak{Y}$ if and only if $c_0 > 0$. The final claim concerning the intersection multiplicity is just a restatement of (10). □

2.3. **Vertical multiplicities: $E_0$ unramified.** Assume that $E_0/\mathbb{Q}_p$ is unramified and fix some $\eta \in E_0$ such that $\mathcal{O}_{E_0} = \mathbb{Z}_p[\eta]$. Thus

$$\mathbb{Z}_p[\gamma_0] = \mathbb{Z}_p[p^{c_0}\eta] \qquad \mathbb{Z}_{p^2}[\gamma] = \mathbb{Z}_{p^2}[p^{c_0}\eta]$$

and

$$U \stackrel{\mathrm{def}}{=} \Psi(\eta) - \overline{\Psi}(\eta) \in (\mathbb{Z}_p^\circ)^\times.$$

Identify the window of $\mathfrak{g}$ equipped with its action induced by $j : \mathcal{O}_E \to \mathrm{End}_{\mathbb{Z}_{p^2}}(\mathfrak{g})$ with the window $(M, N, \Phi)$ of §2.2 equipped with the normalized action in the sense of §2.1. The action of $\eta$ on $(M, N, \Psi)$ with respect to the ordered basis $\{e_1, f_1, e_2, f_2\}$ is given by the matrix

$$\Gamma = \begin{pmatrix} Y & \\ & Z \end{pmatrix}$$

where

$$Y = \begin{pmatrix} \Psi(\eta) & \\ & \overline{\Psi}(\eta) \end{pmatrix} \qquad Z = \begin{pmatrix} \Psi(\eta) & \\ & \overline{\Psi}(\eta) \end{pmatrix}$$

and $\Psi, \overline{\Psi} : \mathcal{O}_{E_0} \to \mathbb{Z}_p^\circ$ are as in §2.1. As in §2.2 we let $(\mathfrak{G}_1^{\mathrm{ver}}, \rho_1^{\mathrm{ver}}) \in \mathfrak{M}(\mathbb{F}[[x_1]])$ be the restriction of the universal $p$-Barsotti-Tate group over $\mathfrak{M}$ to the closed formal subscheme

$$\mathfrak{C}_1^{\mathrm{ver}} = \mathrm{Spf}(\mathbb{F}[[x_1]])$$

and let $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ be the associated window over $\mathbb{F}[[x_1]]$ with respect to the frame $\mathbb{Z}_p^\circ[[x_1]] \to \mathbb{F}[[x_1]]$. We saw in §2.2 (especially (13)) that the endomorphism $\eta$ of $\mathfrak{g}$ lifts to the quasi-endomorphism of $\mathfrak{G}_1^{\mathrm{ver}}$ whose action on $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ is given by the matrix

$$\Gamma_1^{\mathrm{ver}} = \begin{pmatrix} Y_1^{\mathrm{ver}} & \\ & Z_1^{\mathrm{ver}} \end{pmatrix}$$

where

$$Y_1^{\mathrm{ver}} = \begin{pmatrix} \Psi(\eta) & -Uf(x_1) \\ 0 & \overline{\Psi}(\eta) \end{pmatrix} \qquad Z_1^{\mathrm{ver}} = \begin{pmatrix} \Psi(\eta) & 0 \\ \frac{U}{p} f(x_1^p) & \overline{\Psi}(\eta) \end{pmatrix}$$

and $f(x_1) = x_1^{p^0} + x_1^{p^2} + x_1^{p^4} + \cdots$. Note in particular that $p\eta$ lifts to an endomorphism of $\mathfrak{G}_1^{\mathrm{ver}}$.

We next attempt to lift the quasi-endomorphism $\Gamma_1^{\mathrm{ver}}$ to a natural family of deformations of $\mathfrak{G}_1^{\mathrm{ver}}$. For every $k \geq 0$ set

$$R[k] = \mathbb{Z}_p^{\circ}[[x_1, x_2]]/(p^{2k+1}, x_2^{p^k}) \qquad A[k] = \mathbb{Z}_p^{\circ}[[x_1, x_2]]/(x_2^{p^k}). \tag{14}$$

If we equip $A[k]$ with the unique continuous ring homomorphism $\mathrm{Fr} : A[k] \to A[k]$ which satisfies $x_i \mapsto x_i^p$ and whose restriction to $\mathbb{Z}_p^{\circ}$ is the usual Frobenius then $A[k] \to R[k]$ is a frame in Zink's sense. Denote by $\mathfrak{G}[k]$ the base change to $R[k]$ of the universal $p$-Barsotti-Tate group over $R_{\mathfrak{M}} \cong \mathbb{Z}_p^{\circ}[[x_1, x_2]]$, and note that $\mathfrak{G}[0]$ is simply the $p$-Barsotti-Tate group $\mathfrak{G}_1^{\mathrm{ver}}$ over $\mathbb{F}[[x_1]]$. There is a canonical isomorphism $\mathfrak{G}[k+1]_{/R[k]} \cong \mathfrak{G}[k]$ and so each $\mathfrak{G}[k]$ is naturally a deformation of $\mathfrak{G}_1^{\mathrm{ver}}$. Starting from the display (9) of the universal deformation of $\mathfrak{g}$ to $R_{\mathfrak{M}}$, base-changing from $R_{\mathfrak{M}}$ to $R[k]$, and applying Zink's equivalence of categories between displays over $R[k]$ and windows over $R[k]$, we find that the window associated to $\mathfrak{G}[k]$ is the triple $(M[k], N[k], \Phi)$ in which $M[k]$ is the free $A[k]$-module on generators $\{e_1, e_2, f_1, f_2\}$, the submodule $N[k] \subset M[k]$ is generated by $\{p^{2k+1}e_1, p^{2k+1}e_2, f_1, f_2\}$, and $\Phi : M[k] \to M[k]$ is the Fr-linear map determined by

$$e_1 \mapsto x_2 e_2 + f_2 \qquad e_2 \mapsto x_1 e_1 + f_1 \qquad f_1 \mapsto p e_2 \qquad f_2 \mapsto p e_1.$$

As always, the action of $\mathbb{Z}_{p^2}$ is by (8). By the previous paragraph the quasi-endomorphism of $\mathfrak{G}[0] \cong \mathfrak{G}_1^{\mathrm{ver}}$ induced by lifting $\eta$ from $\mathfrak{g}$ to $\mathfrak{G}_1^{\mathrm{ver}}$ corresponds to the quasi-endomorphism of the window $(M[0], N[0], \Phi)$ whose matrix with respect to the ordered basis $\{e_1, f_1, e_2, f_2\}$ is

$$\Gamma[0] = \begin{pmatrix} Y[0] & \\ & Z[0] \end{pmatrix}$$

where $Y[0], Z[0] \in M_2(A[0]) \otimes_{\mathbb{Z}_p^{\circ}} \mathbb{Q}_p^{\circ}$ are defined by

$$Y[0] = \begin{pmatrix} \Psi(\eta) & -Uf(x_1) \\ 0 & \overline{\Psi}(\eta) \end{pmatrix} \qquad Z[0] = \begin{pmatrix} \Psi(\eta) & 0 \\ \frac{U}{p} f(x_1^p) & \overline{\Psi}(\eta) \end{pmatrix}. \tag{15}$$

By the argument used in §2.2, the quasi-endomorphism $\Gamma[0]$ of $(M[0], N[0], \Phi)$ lifts to the quasi- endomorphism of $(M[k], N[k], \Phi)$ given by the matrix

$$\Gamma[k] = \begin{pmatrix} Y[k] & \\ & Z[k] \end{pmatrix}$$

where $Y[k], Z[k] \in M_2(A[k]) \otimes_{\mathbb{Z}_p^{\circ}} \mathbb{Q}_p^{\circ}$ satisfy the recursion relations

$$\begin{aligned} Y[k+1] &= \frac{1}{p} \begin{pmatrix} x_1 & p \\ 1 & \end{pmatrix} \cdot \mathrm{Fr}(Z[k]) \cdot \begin{pmatrix} & p \\ 1 & -x_1 \end{pmatrix} \\ Z[k+1] &= \frac{1}{p} \begin{pmatrix} x_2 & p \\ 1 & \end{pmatrix} \cdot \mathrm{Fr}(Y[k]) \cdot \begin{pmatrix} & p \\ 1 & -x_2 \end{pmatrix}. \end{aligned} \tag{16}$$

The matrices $Y[0]$ and $Z[0]$ admit obvious lifts to $M_2(\mathbb{Z}_p^\circ[[x_1,x_2]])\otimes_{\mathbb{Z}_p^\circ}\mathbb{Q}_p^\circ$, defined again by the equations (15). We then lift each $Y[k]$ and $Z[k]$ to a matrix in $M_2(\mathbb{Z}_p^\circ[[x_1,x_2]])\otimes_{\mathbb{Z}_p^\circ}\mathbb{Q}_p^\circ$ in the unique way for which the relations (16) continue to hold. For $k\geq 1$ set

$$Y_k = p^k(Y[k]-Y[k-1]) \qquad Z_k = p^k(Z[k]-Z[k-1]) \tag{17}$$

so that (by directly computing $Y[1]$ and $Z[1]$)

$$\begin{aligned} Y_1 &= 0 \\ Z_1 &= U\cdot\begin{pmatrix} x_2 f(x_1^p) & -px_2 - x_2^2 f(x_1^p) \\ & -x_2 f(x_1^p)\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned} Y_{k+1} &= \begin{pmatrix} x_1 & p \\ 1 & \end{pmatrix}\cdot \mathrm{Fr}(Z_k)\cdot\begin{pmatrix} & p \\ 1 & -x_1\end{pmatrix} \\ Z_{k+1} &= \begin{pmatrix} x_2 & p \\ 1 & \end{pmatrix}\cdot \mathrm{Fr}(Y_k)\cdot\begin{pmatrix} & p \\ 1 & -x_2\end{pmatrix}.\end{aligned}$$

For notational consistency we also define $Y_0 = Y[0]$ and $Z_0 = Z[0]$, so that for $k\geq 0$

$$\begin{aligned} p^kY[k] &= p^kY_0 + p^{k-1}Y_1 + p^{k-2}Y_2 + \cdots + pY_{k-1} + Y_k \\ p^kZ[k] &= p^kZ_0 + p^{k-1}Z_1 + p^{k-2}Z_2 + \cdots + pZ_{k-1} + Z_k.\end{aligned}$$

It is clear from the recursion relations and initial conditions that $Y_k$ and $Z_k$ have entries in $\mathbb{Z}_p^\circ[[x_1,x_2]]$ for $k\geq 1$, that $Y_k=0$ for $k$ odd, and that $Z_k = 0$ for $k$ even. Let $y_k$ and $z_k$ be the upper right entries of $Y_k$ and $Z_k$, respectively and define (for $\ell\geq 0$)

$$\epsilon_+(\ell) = \prod_{\substack{0\leq i<\ell \\ i \text{ even}}} x_2^{2p^i} \qquad \epsilon_-(\ell) = \prod_{\substack{0\leq i<\ell \\ i \text{ odd}}} x_2^{2p^i}.$$

From the fact that $Z_1$ is divisible by $x_2$ one deduces that $Z_\ell$ is divisible by $x_2^{p^{\ell-1}}$ for $\ell$ odd and positive, and that $Y_\ell$ is divisible by $x_2^{p^{\ell-1}}$ for $\ell$ even and positive. Furthermore one can easily compute the images of $y_\ell$ and $z_\ell$ in $\mathbb{F}[[x_1,x_2]]$ by solving the above recursion relations modulo $p$. One finds that

$$\begin{aligned} y_0 &= u_0\epsilon_-(0) \\ z_1 &= u_1\epsilon_+(1) + px_2g_1 \\ y_2 &= u_2\epsilon_-(2) + px_2^pg_2 \\ z_3 &= u_3\epsilon_+(3) + px_2^{p^2}g_3 \\ y_4 &= u_4\epsilon_-(4) + px_2^{p^3}g_4 \\ z_5 &= u_5\epsilon_+(5) + px_2^{p^4}g_5 \\ &\vdots \end{aligned} \tag{18}$$

for some $g_k\in\mathbb{Z}_p^\circ[[x_1,x_2]]$ and some $u_k\in\mathbb{Z}_p^\circ[[x_1]]$ with nonzero image in $\mathbb{F}[[x_1]]$. As already noted

$$\ell \text{ odd} \implies y_\ell = 0 \qquad \ell \text{ even} \implies z_\ell = 0. \tag{19}$$

Let $\alpha_k, \beta_k \in \mathbb{Z}_p^\circ[[x_1, x_2]]$ be the upper right entries of $p^k Y[k]$ and $p^k Z[k]$, respectively, so that

$$\begin{aligned} \alpha_k &= p^k y_0 + p^{k-1} y_1 + p^{k-2} y_2 + \cdots + p y_{k-1} + y_k \\ \beta_k &= p^k z_0 + p^{k-1} z_1 + p^{k-2} z_2 + \cdots + p z_{k-1} + z_k. \end{aligned} \tag{20}$$

Suppose $k \geq 1$. By what has been said above the endomorphism $p^k \eta$ of $\mathfrak{G}[0]$ lifts to a quasi-endomorphism of $\mathfrak{G}[k]$ which we denote in the same way. Our interest in the constants $\alpha_k, \beta_k \in \mathbb{Z}_p^\circ[[x_1, x_2]]$ comes from the following fundamental fact.

**Proposition 2.3.1.** *Suppose $k \geq 1$ and let $R[k] \to S$ be the maximal quotient of $R[k]$ for which the quasi-endomorphism $p^k \eta$ of $\mathfrak{G}[k]_{/S}$ is an endomorphism. Then*

$$S = R[k]/(\alpha_k, \beta_k).$$

*Proof.* As base change for displays is easier to work with than base change for windows, we first use Zink's equivalence [27] between windows and displays to construct the display associated to $(M[k], N[k], \Phi)$. As in the introduction to [27] there is a continuous ring homomorphism $A[k] \to W(R[k])$ lifting the map $A[k] \to R[k]$. This map takes $x_i \in A[k]$ to the Teichmuller lift $[x_i] \in W(R[k])$. The display $(P, Q, F, V^{-1})$ associated to $(M[k], N[k], \Phi)$ consists of the free $W(R[k])$-module

$$P = M[k] \otimes_{A[k]} W(R[k])$$

the submodule

$$Q = I_{R[k]} e_1 + W(R[k]) f_1 + I_{R[k]} e_2 + W(R[k]) f_2 \subset P,$$

and two operators $F : P \to P$ and $V^{-1} : Q \to P$ whose definition does not concern us. If we let $J = (p^{2k+1})$ denote the kernel of $A[k] \to R[k]$ then there is a canonical isomorphisms of $R[k]$-modules

$$P/I_{R[k]}P \cong M[k]/JM[k]$$

which identifies the submodule $Q/I_{R[k]}P$ on the left with the submodule $N[k]/JM[k]$ on the right. The quasi-endomorphism $p^k \eta$ of $(M[k], N[k], \Phi)$ is represented by the matrix $p^k \Gamma[k]$ described above, which has entries in $A[k]$. In particular $p^k \Gamma[k]$ is an endomorphism of the $A[k]$-module $M[k]$ and induces an endomorphism of the $W(R[k])$-module $P$. The induced $R[k]$-module map $Q/IP \to P/Q$ (which, speaking intuitively, measures the extent to which the quasi- endomorphism $p^k \eta$ of $\mathfrak{G}[k]$ fails to be an endomorphism) is determined by

$$f_1 \mapsto \alpha_k e_1 \qquad f_2 \mapsto \beta_k e_2.$$

First suppose that $R[k] \to S$ is any quotient over which the quasi-endomorphism $p^k \eta$ of $\mathfrak{G}[k]_{/S}$ is an endomorphism. The base change of $(P, Q, F, V^{-1})$ has $P_S = P \otimes_{W(R[k])} W(S)$ as its underlying $W(S)$-module, with submodule $Q_S \subset P_S$ equal to the image of $Q$ under $P \to P_S$. By hypothesis the endomorphism $p^k \Gamma[k]$ of $P_S$ preserves $Q_S$ and hence (letting $I_S$ denote the kernel of $W(S) \to S$) the induced map

$$(Q/IP) \otimes_{R[k]} S \cong Q_S/I_S P_S \xrightarrow{p^k \Gamma[k]} P_S/Q_S \cong (P/Q) \otimes_{R[k]} S \tag{21}$$

is trivial. This implies that $\alpha_k$ and $\beta_k$ lie in the kernel of $R[k] \to S$.

Now set $S = R[k]/(\alpha_k, \beta_k)$ so that (21) is trivial. It follows that $p^k \Gamma[k]$ preserves the submodule $Q_S \subset P_S$. As we already know that $p^k \Gamma[k]$ is a quasi-endomorphism of $(P, Q, F, V^{-1})_{/S}$ some $\mathbb{Z}_p$- multiple of $p^k \Gamma[k]$ commutes with both $F$ and $V^{-1}$. But $W(S)$ is $\mathbb{Z}_p$-torsion-free and $P_S$ is free over $W(S)$; thus $P_S$ has no $\mathbb{Z}_p$-torsion and we deduce that

$p^k\Gamma[k]$ itself commutes with $F$ and $V^{-1}$. In other words $p^k\Gamma[k]$ is an endomorphism of $(P, Q, F, V^{-1})_{/S}$. □

Explicitly computing $\alpha_k$ and $\beta_k$ by solving the recursion (12) is prohibitively difficult; luckily everything we need to know about $\alpha_k$ and $\beta_k$ can be deduced from (18), (19), and (20) without knowing the actual values of the $u_\ell$'s and $g_\ell$'s. Let $\mathcal{A}$ denote the completed local ring of $\mathbb{Z}_p^\circ[[x_1, x_2]]$ at the prime ideal $(p, x_2)$, and note that the constants $u_\ell$ appearing in (18) satisfy

$$u_\ell \in \mathcal{A}^\times$$

as they have nonzero image in the residue field $\mathcal{A}/\mathfrak{m}_\mathcal{A} \cong \mathbb{F}((x_1))$.

**Lemma 2.3.2.** *Suppose that $p$ is odd and $k \geq 1$. The $\mathcal{A}$-module $Q_k = \mathcal{A}/(\alpha_k, \beta_k)$ is Artinian of length*

$$\text{length}_\mathcal{A}(Q_k) = 2p^{k-1} + 4p^{k-2} + 6p^{k-3} + 8p^{k-4} + \cdots + (2k)p^0$$

*and is annihilated by $x_2^{2+2p+2p^2+\cdots+2p^{k-1}}$.*

*Proof.* As we assume that $p > 2$ we have the easy inequality

$$\sum_{0\leq i<\ell} 2p^i < p^\ell \tag{22}$$

for all $\ell \geq 0$. Hence $x_2^{p^\ell}$ is a multiple of $x_2 \cdot \epsilon_\pm(\ell)$ and we may define a nonunit $C_\ell \in A$ by the relations

$$\begin{aligned} C_\ell \cdot \epsilon_-(\ell) &= g_{\ell+1} x_2^{p^\ell} \qquad \text{if } \ell \text{ even} \\ C_\ell \cdot \epsilon_+(\ell) &= g_{\ell+1} x_2^{p^\ell} \qquad \text{if } \ell \text{ odd.} \end{aligned}$$

With this notation (18), (19), and (20) can be rewritten as

$$\begin{aligned} \alpha_k &= \sum_{\substack{0\leq\ell\leq k \\ \ell \text{ even}}} u_\ell p^{k-\ell} \epsilon_-(\ell) + \sum_{\substack{0\leq\ell<k \\ \ell \text{ odd}}} C_\ell p^{k-\ell} \epsilon_+(\ell) \\ \beta_k &= \sum_{\substack{0\leq\ell\leq k \\ \ell \text{ odd}}} u_\ell p^{k-\ell} \epsilon_+(\ell) + \sum_{\substack{0\leq\ell<k \\ \ell \text{ even}}} C_\ell p^{k-\ell} \epsilon_-(\ell). \end{aligned} \tag{23}$$

For $0 \leq j \leq k$ abbreviate

$$Q_k^{(j)} = x_2^{2+2p+2p^2+\cdots+2p^{j-1}} \cdot Q_k.$$

We will construct $\alpha_k^{(j)}, \beta_k^{(j)} \in \mathcal{A}$ such that

$$\mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j)}) \cong Q_k^{(j)}$$

as $\mathcal{A}$-modules and such that

$$\begin{aligned} \alpha_k^{(j)} &= \sum_{\substack{j\leq\ell\leq k \\ \ell \text{ even}}} u_\ell^{(j)} p^{k-\ell} \frac{\epsilon_-(\ell)}{\epsilon_-(j)} + \sum_{\substack{j\leq\ell\leq k \\ \ell \text{ odd}}} C_\ell^{(j)} p^{k-\ell} \frac{\epsilon_+(\ell)}{\epsilon_-(j)} \\ \beta_k^{(j)} &= \sum_{\substack{j\leq\ell\leq k \\ \ell \text{ odd}}} u_\ell^{(j)} p^{k-\ell} \frac{\epsilon_+(\ell)}{\epsilon_+(j)} + \sum_{\substack{j\leq\ell\leq k \\ \ell \text{ even}}} C_\ell^{(j)} p^{k-\ell} \frac{\epsilon_-(\ell)}{\epsilon_+(j)} \end{aligned}$$

for some units $u_\ell^{(j)} \in \mathcal{A}$ and some nonunits $C_\ell^{(j)} \in \mathcal{A}$. The construction is recursive, beginning with $\alpha_k^{(0)} = \alpha_k$ and $\beta_k^{(0)} = \beta_k$. We now assume that $\alpha_k^{(j)}$ and $\beta_k^{(j)}$ have already been constructed and proceed to construct $\alpha_k^{(j+1)}$ and $\beta_k^{(j+1)}$.

Suppose first that $j$ is even. The $\ell = j$ term in $\alpha_k^{(j)}$ is a unit multiple of $p^{k-j}$ while the $\ell = j$ term in $\beta_k^{(j)}$ is a multiple of $p^{k-j}$. Therefore we may eliminate the $\ell = j$ term from $\beta_k^{(j)}$ by adding a suitable multiple of $\alpha_k^{(j)}$. After collecting terms we find that there are units $u_\ell^{(j+1)}$ and nonunits $C_\ell^{(j+1)}$ (for $\ell$ odd and even, respectively) such that

$$\begin{aligned}\beta_k^{(j)} &- \alpha_k^{(j)} \cdot \frac{C_j^{(j)}\epsilon_-(j)}{u_j^{(j)}\epsilon_+(j)} \\ &= \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ odd}}} u_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_+(\ell)}{\epsilon_+(j)} + \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ even}}} C_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_-(\ell)}{\epsilon_+(j)}.\end{aligned}$$

Each term on the right hand side is divisible by $x^{2p^j}$, and dividing out $x^{2p^j}$ leaves

$$\beta_k^{(j+1)} \stackrel{\text{def}}{=} \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ odd}}} u_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_+(\ell)}{\epsilon_+(j+1)} + \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ even}}} C_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_-(\ell)}{\epsilon_+(j+1)}.$$

Note that by construction of $\beta_k^{(j+1)}$ we have

$$\left(\alpha_k^{(j)}, \beta_k^{(j)}\right) = \left(\alpha_k^{(j)}, x_2^{2p^j}\beta_k^{(j+1)}\right). \tag{24}$$

The $\ell = j+1$ term in $\beta_k^{(j+1)}$ is a unit multiple of $p^{k-j-1}$, while the $\ell = j$ term in $\alpha_k^{(j)}$ is a unit multiple of $p^{k-j}$. Therefore we may eliminate the $\ell = j$ term from $\alpha_k^{(j)}$ by adding a suitable multiple of $\beta_k^{(j+1)}$. Collecting common terms we find that

$$\begin{aligned}\alpha_k^{(j)} &- \beta_k^{(j+1)} \cdot \frac{u_j^{(j)} p}{u_{j+1}^{(j+1)}} \\ &= \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ even}}} u_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_-(\ell)}{\epsilon_-(j)} + \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ odd}}} C_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_+(\ell)}{\epsilon_-(j)}\end{aligned}$$

for some units $u_\ell^{(j+1)}$ and nonunits $C_\ell^{(j+1)}$ (for $\ell$ even and odd, respectively). As we are assuming that $j$ is even $\epsilon_-(j) = \epsilon_-(j+1)$, and the above expression is equal to

$$\alpha_k^{(j+1)} \stackrel{\text{def}}{=} \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ even}}} u_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_-(\ell)}{\epsilon_-(j+1)} + \sum_{\substack{j+1\le\ell\le k\\ \ell \text{ odd}}} C_\ell^{(j+1)} p^{k-\ell}\frac{\epsilon_+(\ell)}{\epsilon_-(j+1)}.$$

By construction of $\alpha_k^{(j+1)}$

$$\left(\alpha_k^{(j+1)}, \beta_k^{(j+1)}\right) = \left(\alpha_k^{(j)}, \beta_k^{(j+1)}\right). \tag{25}$$

As we are assuming $Q_k^{(j)} \cong \mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j)})$ the exact sequence

$$0 \to Q_k^{(j+1)} \to Q_k^{(j)} \to Q_k^{(j)}/Q_k^{(j+1)} \to 0$$

can be identified with

$$0 \to (x_2^{2p^j})/(x_2^{2p^j}) \cap (\alpha_k^{(j)}, \beta_k^{(j)}) \to \mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j)}) \to \mathcal{A}/(x_2^{2p^j}, \alpha_k^{(j)}, \beta_k^{(j)}) \to 0.$$

Using (24) and (25) and the fact that $\alpha_k^j$ is not divisible by $x_2$ we obtain isomorphisms

$$\mathcal{A}/(\alpha_k^{(j+1)}, \beta_k^{(j+1)}) \cong \mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j+1)}) \xrightarrow{x_2^{2p^j}} (x_2^{2p^j})/(x_2^{2p^j}) \cap (\alpha_k^{(j)}, \beta_k^{(j)}) \cong Q_k^{(j+1)}$$

as desired.

Now suppose that $j$ is odd. The method is the same as in the even case: we first eliminate the $\ell = j$ term of $\alpha_k^{(j)}$ by adding a multiple of $\beta_k^{(j)}$ to $\alpha_k^{(j)}$, resulting in

$$\begin{aligned}
&\alpha_k^{(j)} - \beta_k^{(j)} \frac{C_j^{(j)}\epsilon_+(j)}{u_j^{(j)}\epsilon_-(j)} \\
&= \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ even}}} u_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_-(\ell)}{\epsilon_-(j)} + \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ odd}}} C_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_+(\ell)}{\epsilon_-(j)}
\end{aligned}$$

for some units $u_\ell^{(j+1)}$ and nonunits $C_\ell^{(j+1)}$ (with $\ell$ even and odd, respectively). Each term on the right is divisible by $x_2^{2p^j}$, and dividing gives

$$\alpha_k^{(j+1)} \stackrel{\text{def}}{=} \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ even}}} u_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_-(\ell)}{\epsilon_-(j+1)} + \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ odd}}} C_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_+(\ell)}{\epsilon_-(j+1)}.$$

By construction

$$(\alpha_k^{(j)}, \beta_k^{(j)}) = (x_2^{2p^j} \alpha_k^{(j+1)}, \beta_k^{(j)}).$$

One now checks that

$$\begin{aligned}
&\beta_k^{(j)} - \alpha_k^{(j+1)} \frac{u_j^{(j)} p}{u_{j+1}^{(j+1)}} \\
&= \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ odd}}} u_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_+(\ell)}{\epsilon_+(j)} + \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ even}}} C_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_-(\ell)}{\epsilon_+(j)}
\end{aligned}$$

for some units $u_\ell^{(j+1)}$ and nonunits $C_\ell^{(j+1)}$ (with $\ell$ odd and even, respectively). As $j$ is odd we have $\epsilon_+(j) = \epsilon_+(j+1)$, and so the above quantity is equal to

$$\beta_k^{(j+1)} \stackrel{\text{def}}{=} \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ odd}}} u_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_+(\ell)}{\epsilon_+(j+1)} + \sum_{\substack{j+1 \le \ell \le k \\ \ell \text{ even}}} C_\ell^{(j+1)} p^{k-\ell} \frac{\epsilon_-(\ell)}{\epsilon_+(j+1)}.$$

By construction

$$(\alpha_k^{(j+1)}, \beta_k^{(j+1)}) = (\alpha_k^{(j+1)}, \beta_k^{(j)})$$

and exactly in the even case we find isomorphisms

$$\mathcal{A}/(\alpha_k^{(j+1)}, \beta_k^{(j+1)}) \cong \mathcal{A}/(\alpha_k^{(j+1)}, \beta_k^{(j)}) \xrightarrow{x_2^{2p^j}} (x_2^{2p^j})/(x_2^{2p^j}) \cap (\alpha_k^{(j)}, \beta_k^{(j)}) \cong Q_k^{(j+1)}.$$

Both parts of the lemma now follow easily. As one of $\alpha_k^{(k)}$ and $\beta_k^{(k)}$ is a unit we have

$$x_2^{2+2p+2p^2+\cdots+2p^{k-1}} \cdot Q_k = Q_k^{(k)} \cong \mathcal{A}/(\alpha_k^{(k)}, \beta_k^{(k)}) = 0.$$

The length of $Q_k$ may be computed as

$$\begin{aligned}\mathrm{length}_{\mathcal{A}}(Q_k) &= \sum_{0\le j<k} \mathrm{length}_{\mathcal{A}}(Q_k^{(j)}/Q_k^{(j+1)}) \\ &= \sum_{0\le j<k} \mathrm{length}_{\mathcal{A}}(\mathcal{A}/(x_2^{2p^j}, \alpha_k^{(j)}, \beta_k^{(j)})).\end{aligned}$$

If $j$ is even then by (24)

$$(x_2^{2p^j}, \alpha_k^{(j)}, \beta_k^{(j)}) = (x_2^{2p^j}, \alpha_k^{(j)}, x_2^{2p^j}\beta_k^{(j+1)}) = (x_2^{2p^j}, \alpha_k^{(j)}) = (x_2^{2p^j}, p^{k-j}).$$

For the final equality we have used the fact that for $\ell > j$

$$x_2^{2p^j} \text{ divides} \begin{cases} \epsilon_-(\ell)/\epsilon_-(j) & \text{if } \ell \text{ is even} \\ \epsilon_+(\ell)/\epsilon_-(j) & \text{if } \ell \text{ is odd.} \end{cases}$$

Similarly if $j$ is odd then

$$(x_2^{2p^j}, \alpha_k^{(j)}, \beta_k^{(j)}) = (x_2^{2p^j}, x_2^{2p^j}\alpha_k^{(j+1)}, \beta_k^{(j)}) = (x_2^{2p^j}, \beta_k^{(j)}) = (x_2^{2p^j}, p^{k-j}).$$

In either case

$$\mathrm{length}_{\mathcal{A}}(\mathcal{A}/(x_2^{2p^j}, \alpha_k^{(j)}, \beta_k^{(j)})) = 2p^j(k-j)$$

completing the proof of

$$\mathrm{length}_{\mathcal{A}}(Q_k) = \sum_{0\le j<k} 2p^j(k-j).$$

□

We continue to let $\mathcal{A}$ denote the completed local ring of $\mathbb{Z}_p^\circ[[x_1, x_2]]$ at $(p, x_2)$, with maximal ideal $\mathfrak{m}_{\mathcal{A}} = (p, x_2)$.

**Lemma 2.3.3.** *If $p$ is odd and $k \ge 1$ then there is an inclusion of ideals in $\mathcal{A}$*

$$(p^{2k+1}, x_2^{p^k}) \subset \mathfrak{m}_{\mathcal{A}} \cdot (\alpha_k, \beta_k).$$

*Proof.* Let $Q_k = \mathcal{A}/(\alpha_k, \beta_k)$ and for $0 \le j \le k$ set

$$Q_k^{(j)} = p^{2k-2j} Q_k.$$

We will recursively define $\alpha_k^{(j)}$ and $\beta_k^{(j)}$ in $\mathcal{A}$ with the property

$$\mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j)}) \cong Q_k^{(j)}$$

as $\mathcal{A}$-modules and such that

$$\begin{aligned}\alpha_k^{(j)} &= \sum_{\substack{0\le\ell\le j \\ \ell \text{ even}}} u_\ell^{(j)} p^{j-\ell}\epsilon_-(\ell) + \sum_{\substack{0\le\ell\le j \\ \ell \text{ odd}}} C_\ell^{(j)} p^{j-\ell}\epsilon_+(\ell) \\ \beta_k^{(j)} &= \sum_{\substack{0\le\ell\le j \\ \ell \text{ odd}}} u_\ell^{(j)} p^{j-\ell}\epsilon_+(\ell) + \sum_{\substack{0\le\ell\le j \\ \ell \text{ even}}} C_\ell^{(j)} p^{j-\ell}\epsilon_-(\ell)\end{aligned}$$

for some units $u_\ell^{(j)} \in \mathcal{A}$ and some nonunits $C_\ell^{(j)} \in \mathcal{A}$. After (23) we may begin by defining

$$\alpha_k^{(k)} = \alpha_k \qquad \beta_k^{(k)} = \beta_k.$$

Assuming that $\alpha_k^{(j)}$ and $\beta_k^{(j)}$ have been constructed we now construct $\alpha_k^{(j-1)}$ and $\beta_k^{(j-1)}$.

Assume first that $j$ is even. The $\ell = j$ term of $\alpha_k^{(j)}$ is a unit multiple of $\epsilon_-(j)$ while the $\ell = j$ term of $\beta_k^{(j)}$ is a multiple of $\epsilon_-(j)$. Subtracting a suitable multiple of $\alpha_k^{(j)}$ from $\beta_k^{(j)}$ removes the $\ell = j$ term from $\beta_k^{(j)}$ resulting in

$$\begin{aligned}&\beta_k^{(j)} - \alpha_k^{(j)} \frac{C_j^{(j)}}{u_j^{(j)}} \\ &\quad = \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ odd}}} u_\ell^{(j-1)} p^{j-\ell} \epsilon_+(\ell) + \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ even}}} C_\ell^{(j-1)} p^{j-\ell} \epsilon_-(\ell)\end{aligned}$$

for some units $u_\ell^{(j-1)}$ and nonunits $C_\ell^{(j-1)}$ (with $\ell$ odd and even, respectively). Dividing this quantity by $p$ results in

$$\beta_k^{(j-1)} \stackrel{\text{def}}{=} \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ odd}}} u_\ell^{(j-1)} p^{j-1-\ell} \epsilon_+(\ell) + \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ even}}} C_\ell^{(j-1)} p^{j-1-\ell} \epsilon_-(\ell),$$

and by construction

$$(\alpha_k^{(j)}, \beta_k^{(j)}) = (\alpha_k^{(j)}, p\beta_k^{(j-1)}). \tag{26}$$

Subtracting a suitable multiple of $\beta_k^{(j-1)}$ removes the $\ell = j$ term from $\alpha_k^{(j)}$, resulting in

$$\begin{aligned}&\alpha_k^{(j)} - \beta_k^{(j-1)} \frac{u_j^{(j)} \epsilon_-(j)}{u_{j-1}^{(j-1)} \epsilon_+(j-1)} \\ &\quad = \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ even}}} u_\ell^{(j-1)} p^{j-\ell} \epsilon_-(\ell) + \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ odd}}} C_\ell^{(j-1)} p^{j-\ell} \epsilon_+(\ell)\end{aligned}$$

for some units $u_\ell^{(j-1)}$ and nonunits $C_\ell^{(j-1)}$ (with $\ell$ even and odd, respectively). Dividing this last expression by $p$ we obtain

$$\alpha_k^{(j-1)} \stackrel{\text{def}}{=} \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ even}}} u_\ell^{(j-1)} p^{j-1-\ell} \epsilon_-(\ell) + \sum_{\substack{0 \le \ell \le j-1 \\ \ell \text{ odd}}} C_\ell^{(j-1)} p^{j-1-\ell} \epsilon_+(\ell).$$

By construction

$$(\alpha_k^{(j)}, \beta_k^{(j-1)}) = (p\alpha_k^{(j-1)}, \beta_k^{(j-1)}). \tag{27}$$

The exact sequence

$$0 \to pQ_k^{(j)} \to Q_k^{(j)} \to Q_k^{(j)}/pQ_k^{(j)} \to 0$$

can be identified with

$$0 \to (p)/(p) \cap (\alpha_k^{(j)}, \beta_k^{(j)}) \to \mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j)}) \to \mathcal{A}/(p, \alpha_k^{(j)}, \beta_k^{(j)}) \to 0.$$

As $\alpha_k^{(j)}$ is not divisible by $p$ we find, using (26), isomorphisms

$$\mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j-1)}) \xrightarrow{p} (p)/(p) \cap (\alpha_k^{(j)}, p\beta_k^{(j-1)}) \cong pQ_k^{(j)}.$$

The exact sequence

$$0 \to Q_k^{(j-1)} \to pQ_k^{(j)} \to pQ_k^{(j)}/p^2 Q_k^{(j)} \to 0$$

is then identified with

$$0 \to (p)/(p) \cap (\alpha_k^{(j)}, \beta_k^{(j-1)}) \to \mathcal{A}/(\alpha_k^{(j)}, \beta_k^{(j-1)}) \to \mathcal{A}/(p, \alpha_k^{(j)}, \beta_k^{(j-1)}) \to 0$$

and we find, using (27) and the observation that $\beta_k^{(j-1)}$ is not divisible by $p$, isomorphisms

$$\mathcal{A}/(\alpha_k^{(j-1)},\beta_k^{(j-1)}) \xrightarrow{p} (p)/(p)\cap(p\alpha_k^{(j-1)},\beta_k^{(j-1)}) \cong Q_k^{(j-1)}.$$

The construction of $\alpha_k^{(j-1)}$ and $\beta_k^{(j-1)}$ for $j$ odd is similar. We first write

$$\begin{aligned}\alpha_k^{(j)} - \beta_k^{(j)}\frac{C_j^{(j)}}{u_j^{(j)}} \\ = \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ even}}} u_\ell^{(j-1)}p^{j-\ell}\epsilon_-(\ell) + \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ odd}}} C_\ell^{(j-1)}p^{j-\ell}\epsilon_+(\ell)\end{aligned}$$

for some units $u_\ell^{(j-1)}$ and nonunits $C_\ell^{(j-1)}$ (with $\ell$ even and odd, respectively). Dividing this quantity by $p$ results in

$$\alpha_k^{(j-1)} \stackrel{\text{def}}{=} \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ even}}} u_\ell^{(j-1)}p^{j-1-\ell}\epsilon_-(\ell) + \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ odd}}} C_\ell^{(j-1)}p^{j-1-\ell}\epsilon_+(\ell).$$

We next write

$$\begin{aligned}\beta_k^{(j)} - \alpha_k^{(j-1)}\frac{u_j^{(j)}\epsilon_+(j)}{u_{j-1}^{(j-1)}\epsilon_-(j-1)} \\ = \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ odd}}} u_\ell^{(j-1)}p^{j-\ell}\epsilon_+(\ell) + \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ even}}} C_\ell^{(j-1)}p^{j-\ell}\epsilon_-(\ell)\end{aligned}$$

for some units $u_\ell^{(j-1)}$ and nonunits $C_\ell^{(j-1)}$ (with $\ell$ even and odd, respectively). Dividing this last expression by $p$ we obtain

$$\beta_k^{(j-1)} \stackrel{\text{def}}{=} \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ odd}}} u_\ell^{(j-1)}p^{j-1-\ell}\epsilon_+(\ell) + \sum_{\substack{0\le\ell\le j-1\\ \ell \text{ even}}} C_\ell^{(j-1)}p^{j-1-\ell}\epsilon_-(\ell).$$

Using

$$(\alpha_k^{(j)},\beta_k^{(j)}) = (p\alpha_k^{(j-1)},\beta_k^{(j)}) \qquad (\alpha_k^{(j-1)},\beta_k^{(j)}) = (\alpha_k^{(j-1)},p\beta_k^{(j-1)})$$

one proves that

$$\mathcal{A}/(\alpha_k^{(j-1)},\beta_k^{(j-1)}) \cong p^2Q_k^{(j)} = Q_k^{(j-1)}.$$

As $\alpha_k^{(0)}$ is a unit we find that

$$p^{2k}Q_k \cong \mathcal{A}/(\alpha_k^{(0)},\beta_k^{(0)}) = 0$$

and so $p^{2k}\in(\alpha_k,\beta_k)$. This proves that $p^{2k+1}\in\mathfrak{m}_\mathcal{A}\cdot(\alpha_k,\beta_k)$. On the other hand Lemma 2.3.2 shows that

$$x_2^{2+2p+2p^2+\cdots+2p^{k-1}} \in (\alpha_k,\beta_k),$$

and so the inequality (22) shows that

$$x_2^{p^k}\in\mathfrak{m}_\mathcal{A}\cdot(\alpha_k,\beta_k).$$

□

**Proposition 2.3.4.** *Assume that* $c_0 > 0$ *and that* $p > 2$. *If* $\mathfrak{C}_i^{\mathrm{ver}}$ *is either of the vertical components of* $\mathfrak{Y}$ *found in Proposition 2.2.1 then (in the notation of Definition 1.1.1)*

$$\begin{aligned} \mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C}_i^{\mathrm{ver}}) &= 2p^{c_0-1} + 4p^{c_0-2} + 6p^{c_0-3} + 8p^{c_0-4} + \cdots + 2c_0p^0 \\ &= \frac{2p(p^{c_0}-1) - 2c_0(p-1)}{(p-1)^2}. \end{aligned}$$

*Proof.* We give the proof for $i = 1$, the proof for $i = 2$ being entirely similar. Set $k = c_0$ and let $J \subset R_{\mathfrak{M}}$ be the ideal of definition of the closed formal subscheme $\mathfrak{Y} \to \mathfrak{M}$. As

$$\mathbb{Z}_{p^2}[p^k\eta] = \mathbb{Z}_{p^2} + p^{c_0}\mathcal{O}_E = \mathbb{Z}_{p^2}[\tau]$$

the quotient $R_{\mathfrak{M}}/J \cong R_{\mathfrak{Y}}$ is the maximal quotient over which the endomorphism $j(p^k\eta)$ of $\mathfrak{g}$ lifts to an endomorphism of the universal deformation of $\mathfrak{g}$. Let $\mathfrak{q} \subset R_{\mathfrak{Y}}$ be the ideal of definition of the closed formal subscheme $\mathfrak{C}_1^{\mathrm{ver}} \to \mathfrak{Y}$, and denote again by $\mathfrak{q}$ the kernel of

$$R_{\mathfrak{M}} \to R_{\mathfrak{Y}} \to R_{\mathfrak{Y}}/\mathfrak{q} \cong \mathbb{F}[[x_1]]$$

so that $\mathfrak{q} = (p, x_2)$. Proposition 2.3.1 tells us that

$$J + (p^{2k+1}, x_2^{p^k}) = (\alpha_k, \beta_k) + (p^{2k+1}, x_2^{p^k})$$

as ideals of $R_{\mathfrak{M}} \cong \mathbb{Z}_p^{\circ}[[x_1, x_2]]$. If we define $J_{\mathcal{A}} = J \cdot \mathcal{A}$, where $\mathcal{A}$ is the completed local ring of $R_{\mathfrak{M}}$ at $\mathfrak{q}$, then we obtain the equality of ideals of $\mathcal{A}$

$$J_{\mathcal{A}} + (p^{2k+1}, x_2^{p^k}) = (\alpha_k, \beta_k) + (p^{2k+1}, x_2^{p^k}).$$

Lemma 2.3.3 gives the inclusion of ideals $(p^{2k+1}, x_2^{p^k}) \subset (\alpha_k, \beta_k)$ in $\mathcal{A}$ and so

$$J_{\mathcal{A}} + (p^{2k+1}, x_2^{p^k}) = (\alpha_k, \beta_k).$$

From this we obviously have $J_{\mathcal{A}} \subset (\alpha_k, \beta_k)$. To prove the reverse inclusion we apply the inclusion of Lemma 2.3.3 to obtain

$$(\alpha_k, \beta_k) \subset J_{\mathcal{A}} + (p^{2k+1}, x_2^{p^k}) \subset J_{\mathcal{A}} + \mathfrak{m}_{\mathcal{A}}(\alpha_k, \beta_k),$$

and an induction argument then proves that

$$(\alpha_k, \beta_k) \subset J_{\mathcal{A}} + \mathfrak{m}_{\mathcal{A}}^{\ell}(\alpha_k, \beta_k)$$

for every $\ell > 0$. In particular $(\alpha_k, \beta_k) \subset J_{\mathcal{A}} + \mathfrak{m}_{\mathcal{A}}^{\ell+1}$ for every $\ell$, and topological considerations prove that $(\alpha_k, \beta_k) \subset J_{\mathcal{A}}$. Having proved $J_{\mathcal{A}} = (\alpha_k, \beta_k)$, Lemma 2.3.2 now shows that $\mathcal{A}/J_{\mathcal{A}}$ is Artinian of length

$$\mathrm{length}_{\mathcal{A}}(\mathcal{A}/J_{\mathcal{A}}) = 2p^{k-1} + 4p^{k-2} + 6p^{k-3} + \cdots + (2k)p^0.$$

Using the isomorphisms

$$\mathcal{A}/J_{\mathcal{A}} \cong R_{\mathfrak{M},\mathfrak{q}}/JR_{\mathfrak{M},\mathfrak{q}} \cong R_{\mathfrak{Y},\mathfrak{q}}$$

we find that $R_{\mathfrak{Y},\mathfrak{q}}$ has length $2p^{k-1} + 4p^{k-2} + 6p^{k-3} + \cdots + (2k)p^0$, completing the proof of the first equality in the statement of the proposition. The proof of the second is then an elementary exercise. □

2.4. **Vertical multiplicities: $E_0$ ramified.** Assume that $E_0/\mathbb{Q}_p$ is ramified. We assume throughout all of §2.4 that $p > 2$. This allows us to choose a trace-free uniformizing parameter $\varpi_{E_0} \in \mathcal{O}_{E_0}$. In the notation of §2.1 $\varpi_{E_0}$ is a root of $x^2 - bb^\sigma p$ for some $b \in \mathbb{Z}_{p^2}^\times$, and the normalized action of $\varpi_{E_0}$ on the window $(M, N, \Phi)$ of the $p$-Barsotti-Tate group $\mathfrak{g}$ (with respect to the frame $\mathbb{Z}_p^\circ \to \mathbb{F}$, as in §2.2) is through the matrix

$$\Gamma = \begin{pmatrix} & pb & & \\ b^\sigma & & & \\ & & & pb \\ & & b^\sigma & \end{pmatrix}.$$

Let $(\mathfrak{G}_1^{\mathrm{ver}}, \rho_1^{\mathrm{ver}})$ be the restriction of the universal deformation of $\mathfrak{g}$ over $\mathfrak{M} \cong \mathrm{Spf}(\mathbb{Z}_p^\circ[[x_1, x_2]])$ to the closed formal subscheme

$$\mathfrak{C}_1^{\mathrm{ver}} = \mathrm{Spf}(\mathbb{F}[[x_1]])$$

and let $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ be the window of $\mathfrak{G}_1^{\mathrm{ver}}$ with respect to the frame $\mathbb{Z}_p^\circ[[x_1]] \to \mathbb{F}[[x_1]]$ so that, by the calculations of §2.2, the endomorphism $\Gamma$ of $(M, N, \Phi)$ lifts to the quasi-endomorphism of $(M_1^{\mathrm{ver}}, N_1^{\mathrm{ver}}, \Phi)$ given by

$$\Gamma_1^{\mathrm{ver}} = \begin{pmatrix} Y_1^{\mathrm{ver}} & \\ & Z_1^{\mathrm{ver}} \end{pmatrix}$$

with

$$\begin{aligned} Y_1^{\mathrm{ver}} &= \begin{pmatrix} & pb \\ b^\sigma & \end{pmatrix} + b^\sigma \begin{pmatrix} f(x_1) & -g(x_1) \\ & -f(x_1) \end{pmatrix} \\ Z_1^{\mathrm{ver}} &= \begin{pmatrix} & pb \\ b^\sigma & \end{pmatrix} + b \begin{pmatrix} -f(x_1^p) & \\ -\frac{g(x_1^p)}{p} & f(x_1^p) \end{pmatrix}. \end{aligned}$$

We now imitate the method of §2.3. Define $R[k]$ and $A[k]$ by (14), let $\mathfrak{G}[k]$ be the base change of the universal $p$-Barsotti-Tate group through $\mathbb{Z}_p^\circ[[x_1, x_2]] \to R[k]$, and let $(M[k], N[k], \Phi)$ be the window of $\mathfrak{G}[k]$ with respect to the frame $A[k] \to R[k]$. Viewing each $\mathfrak{G}[k]$ as a deformation of $\mathfrak{G}[0] = \mathfrak{G}_1^{\mathrm{ver}}$ the quasi-endormorphism $\Gamma[0] = \Gamma$ of $(M[0], N[0], \Phi) = (M, N, \Phi)$ lifts to the quasi-endomorphism of $(M[k], N[k], \Phi)$ given by the matrix

$$\Gamma[k] = \begin{pmatrix} Y[k] & \\ & Z[k] \end{pmatrix}$$

in which $Y[k], Z[k] \in M_2(A[k]) \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ satisfy the recursion (16) and the initial conditions

$$Y[0] = Y_1^{\mathrm{ver}} \qquad Z[0] = Z_1^{\mathrm{ver}}.$$

As in §2.3 we lift each $Y[k]$ and $Z[k]$ to a matrix in $M_2(\mathbb{Z}_p^\circ[[x_1, x_2]]) \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ in such a way that (16) continues to hold. For $k \geq 0$ define $Y_k, Z_k \in M_2(\mathbb{Z}_p^\circ[[x_1, x_2]]) \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ by (17) and set $Y_0 = Y[0]$ and $Z_0 = Z[0]$. Explicitly computing $Y[1]$ and $Z[1]$ we find that

$$\begin{aligned} Y_1 &= 0 \\ Z_1 &= x_2 \begin{pmatrix} -b^\sigma g(x_1^p) & b^\sigma x_2 g(x_1^p) \\ & b^\sigma g(x_1^p) \end{pmatrix} + p x_2 \begin{pmatrix} b^\sigma & 2bf(x_1^p) - b^\sigma x_2 \\ & -b^\sigma \end{pmatrix}. \end{aligned}$$

For $k \geq 1$ we again let $y_k, z_k \in M_2(\mathbb{Z}_p^\circ[[x_1, x_2]])$ denote the upper right entries of $Y_k$ and $Z_k$, and let $\alpha_k, \beta_k \in M_2(\mathbb{Z}_p^\circ[[x_1, x_2]])$ denote the upper right entries of $p^k Y[k]$ and $p^k Z[k]$. By the same reasoning as in §2.3 (that is, by computing the images of $Y_k$ and $Z_k$ in $\mathbb{F}[[x_1, x_2]]$ and using the fact that $Z_1$ is divisible by $x_2$) we find that $y_k$ and $z_k$ satisfy (18) for some

$g_k \in \mathbb{Z}_p^\circ[[x_1, x_2]]$ and some $u_k \in \mathbb{Z}_p^\circ[[x_1]]$ with nonzero image in $\mathbb{F}[[x_1]]$, and that (19) and (20) hold.

**Proposition 2.4.1.** *Assume that $c_0 > 0$ and that $p > 2$. If $\mathfrak{C}_i^{\mathrm{ver}}$ is either of the vertical components of $\mathfrak{Y}$ found in Proposition 2.2.1 then (in the notation of Definition 1.1.1)*

$$\begin{aligned} \mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C}_i^{\mathrm{ver}}) &= 2p^{c_0-1} + 4p^{c_0-2} + 6p^{c_0-3} + 8p^{c_0-4} + \cdots + 2c_0 p^0 \\ &= \frac{2p(p^{c_0}-1) - 2c_0(p-1)}{(p-1)^2}. \end{aligned}$$

*Proof.* The statement is exactly that of Proposition 2.3.4, which dealt with the case of $E_0/\mathbb{Q}_p$ unramified. In the ramified situation the statements of Lemmas 2.3.2 and 2.3.3 continue to hold, as the proofs only require that $\alpha_k$ and $\beta_k$ satisfy (18), (19), and (20). Similarly the statement and proof of Proposition 2.3.1 hold verbatim in the ramified case (taking $\eta = \varpi_{E_0}$). The proof of the proposition is now the same, word-for-word, as that of Proposition 2.3.4 (again taking $\eta = \varpi_{E_0}$). □

## 3. Horizontal components

In this subsection we will compute all horizontal components of $\mathfrak{Y}$ in the sense of Definition 1.1.1. The strategy is to start from the components of $\mathfrak{Y}_0$, all of which are known to us by the Gross-Keating theory of quasi- canonical lifts described in §3.1, and apply the action of $\mathcal{O}_E^\times$ on $\mathfrak{Y}$ as described in §1.1 to produce more components. In the case in which $E_0/\mathbb{Q}_p$ is unramified we will show that this construction accounts for all horizontal components of $\mathfrak{Y}$. More precisely, Proposition 3.2.5 shows that every $\mathcal{O}_E^\times$-orbit of horizontal components of $\mathfrak{Y}$ contains a unique horizontal component of $\mathfrak{Y}_0$. When $E_0/\mathbb{Q}_p$ is ramified the situation is slightly more complicated. Taking $\mathcal{O}_E^\times$-orbits of components of $\mathfrak{Y}_0$ yields exactly half of the horizontal components of $\mathfrak{Y}$. The components constructed in this way we call *standard* components. To construct the remaining *nonstandard* components we first construct a closed formal subscheme $\mathfrak{Y}_0' \to \mathfrak{M}_0$ different from $\mathfrak{Y}_0$, but whose image in $\mathfrak{M}$ is still contained in $\mathfrak{Y}$. Taking $\mathcal{O}_E^\times$-orbits of components of $\mathfrak{Y}_0'$ yields the nonstandard components of $\mathfrak{Y}$ (Proposition 3.3.6). Having thus constructed all horizontal components $\mathfrak{C}$ of $\mathfrak{Y}$, we use results of Keating on the endomorphism rings of reductions of quasi-canonical lifts to compute the intersection number $I_{\mathfrak{M}}(\mathfrak{C}, \mathfrak{M}_0)$ for those components which meet $\mathfrak{M}_0$ properly (the *proper* components of Definition 1.1.1). We know from Corollary 1.2.6 that every horizontal component $\mathfrak{C}$ satisfies $\mathrm{mult}_{\mathfrak{Y}}(\mathfrak{C}) = 1$, and so our results give a complete picture of the horizontal part of $\mathfrak{Y}$ (at least for $p > 2$, a hypothesis which will first appear in Proposition 3.3.8).

3.1. **Quasi-canonical lifts.** Recall from §2.1 that $M_0$ is the fraction field of $W_0$. Viewing $E_0$ as a subfield of $M_0$ via $\Psi$, the reciprocity map of class field theory provides an isomorphism

$$\mathrm{rec} : \mathcal{O}_{E_0}^\times \to \mathrm{Gal}(\mathcal{E}_0^{\mathrm{ab}}/M_0)$$

where $\mathcal{E}_0^{\mathrm{ab}}$ is the completion of the maximal abelian extension of $E_0$ (inside the completion of an algebraic closure of $\mathbb{Q}_p$ containing $M_0$), and for each nonnegative integer $k$ we let $M_0 \subset M_k \subset \mathcal{E}_0^{\mathrm{ab}}$ be the intermediate extension satisfying

$$\mathcal{O}_{E_0}^\times / (\mathbb{Z}_p + p^k \mathcal{O}_{E_0})^\times \cong \mathrm{Gal}(M_k/M_0).$$

Note that $M_k/\mathbb{Q}_p^\circ$ is Galois, as $(\mathbb{Z}_p + p^k\mathcal{O}_{E_0})^\times$ is stable under the action of $\mathrm{Gal}(E_0/\mathbb{Q}_p)$. Let $W_k$ denote the ring of integers of $M_k$.

Recall that $\mathbb{Z}_p[\gamma_0] = \mathbb{Z}_p + p^{c_0}\mathcal{O}_{E_0}$. For each $0 \le k \le c_0$ Gross [8] (see also [26]) has constructed $[M_k : \mathbb{Q}_p^\circ]$ distinct surjective $\mathbb{Z}_p^\circ$-algebra homomorphisms $R_{\mathfrak{Y}_0} \to W_k$. A deformation

$$(\mathfrak{G}_0, \rho_0) \in \mathfrak{Y}_0(W_k) \cong \mathrm{Hom}_{\mathbf{ProArt}}(R_{\mathfrak{Y}_0}, W_k)$$

corresponding to such a surjection is called a *quasi-canonical* lift of level $k$. A quasi-canonical lift of level 0 is called a *canonical* lift. The action of $\mathbb{Z}_p[\gamma_0]$ on a quasi-canonical lift $\mathfrak{G}_0$ of level $k$ can be extended to an action of the larger order $\mathbb{Z}_p + p^k\mathcal{O}_{E_0}$, and this enlarged action makes the $p$-adic Tate module $\mathrm{Ta}_p(\mathfrak{G}_0)$ into a free $\mathbb{Z}_p + p^k\mathcal{O}_{E_0}$-module of rank one. Furthermore, if $\phi_0 : R_{\mathfrak{Y}_0} \to W_k$ is the homomorphism defining a quasi-canonical lift of level $k$ then for any $\xi \in \mathcal{O}_{E_0}^\times$ there is a commutative diagram

$$\begin{array}{ccc} R_{\mathfrak{Y}_0} & \xrightarrow{\phi_0} & W_k \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \mathrm{rec}(\xi^{-1})} \\ R_{\mathfrak{Y}_0} & \xrightarrow[\phi_0]{} & W_k \end{array}$$

where the vertical arrow on the left is the automorphism of (3).

**Proposition 3.1.1** (Gross, Keating)**.** *The closed immersion of formal schemes $\mathfrak{Y}_0 \to \mathfrak{M}_0$ can be identified with*

$$\mathrm{Spf}(\mathbb{Z}_p^\circ[[x]]/I_0) \to \mathrm{Spf}(\mathbb{Z}_p^\circ[[x]])$$

*where the ideal $I_0$ is generated by a power series of the form*

$$g_{c_0}(x) = \prod_{k=0}^{c_0} \varphi_k(x)$$

*with each $\varphi_k(x)$ an Eisenstein polynomial satisfying $\mathbb{Z}_p^\circ[[x]]/(\varphi_k) \cong W_k$.*

*Proof.* By [22, Theorem 3.8] we may fix an isomorphism $R_{\mathfrak{M}_0} \cong \mathbb{Z}_p^\circ[[x]]$. By [26, Theorem 5.1] the quotient $R_{\mathfrak{Y}_0}$ then has the form $\mathbb{Z}_p^\circ[[x]]/(g)$ for some power series $g(x)$. By the Weierstrass preparation theorem we may choose $g(x)$ to be a polynomial of degree $f$ satisfying $g(x) \equiv x^f \pmod p$. The quasi-canonical lifts described above give, for each $0 \le k \le c_0$, a surjection $R_{\mathfrak{Y}_0} \to W_k$ whose kernel is generated by an Eisenstein polynomial $\varphi_k(x)$ which must divide $g(x)$. This implies that we may factor $g(x) = \varphi_0(x) \cdots \varphi_{c_0}(x)h(x)u(x)$ with $u(x)$ a unit in $\mathbb{Z}_p^\circ[[x]]$ and $h(x)$ a polynomial of degree $e$ satisfying $h(x) \equiv x^e \pmod p$. Thus

$$\mathrm{length}_{\mathbb{F}}(R_{\mathfrak{Y}_0}/(p)) = f = e + \sum_{k=0}^{c_0} [M_k : \mathbb{Q}_p^\circ].$$

But calculations of Keating give an explicit formula for $\mathrm{length}_{\mathbb{F}}(R_{\mathfrak{Y}_0}/(p))$. Using the remarks following [12, Theorem 1.1] one can show that

$$\mathrm{length}_{\mathbb{F}}(R_{\mathfrak{Y}_0}/(p)) = \begin{cases} 2(1 + p + \cdots + p^{c_0-1}) + p^{c_0} & \text{if } E_0/\mathbb{Q}_p \text{ is unramified} \\ 2(1 + p + \ldots + p^{c_0}) & \text{if } E_0/\mathbb{Q}_p \text{ is ramified.} \end{cases}$$

Comparing with the values (for $k > 0$)

$$[M_k : \mathbb{Q}_p^\circ] = \begin{cases} (p+1)p^{k-1} & \text{if } E_0/\mathbb{Q}_p \text{ is unramified} \\ 2p^k & \text{if } E_0/\mathbb{Q}_p \text{ is ramified} \end{cases}$$

we deduce that $e = 0$, and so $h(x)$ is a unit. ∎

3.2. $E_0$ **unramified.** Throughout all of §3.2 we assume that $E_0/\mathbb{Q}_p$ is unramified. The calculation of the horizontal components of $\mathfrak{Y}$ will be an easy consequence of Proposition 1.2.10 and Proposition 3.2.2 below. First we need a simple lemma.

**Lemma 3.2.1.** *If $\mathbb{Z}_{p^2}[\gamma] = \mathcal{O}_E$ then $\mathfrak{Y} \cong \mathfrak{Y}_0 \cong \mathrm{Spf}(\mathbb{Z}_p^\circ)$.*

*Proof.* As we assume that $E_0$ is unramified $\mathcal{O}_{E_0} \cong \mathbb{Z}_{p^2}$ as $\mathbb{Z}_p$-algebras, and $\mathcal{O}_E \cong \mathcal{O}_{E_0} \times \mathcal{O}_{E_0}$. The two idempotents on the right determine a splitting

$$\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2} \cong \mathfrak{g}_0 \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E \cong \mathfrak{g}_0 \times \mathfrak{g}_0$$

and similarly for any deformation of $\mathfrak{g}$ for which the $\mathcal{O}_E$-action lifts. This splitting of deformations of $\mathfrak{g}$ induces an isomorphism of formal schemes over $\mathbb{Z}_p^\circ$

$$\mathfrak{Y} \cong \mathfrak{Y}_0 \times_{\mathrm{Spf}(\mathbb{Z}_p^\circ)} \mathfrak{Y}_0.$$

But according to Proposition 3.1.1, applied with $\mathbb{Z}_p[\gamma_0] = \mathcal{O}_{E_0}$ and $c_0 = 0$, we have $\mathfrak{Y}_0 \cong \mathrm{Spf}(\mathbb{Z}_p^\circ)$. The claim follows. □

**Proposition 3.2.2.** *Let $R$ be the ring of integers of a finite extension of $\mathbb{Q}_p^\circ$. In the terminology of Definition 1.2.8, if two elements of $\mathfrak{Y}(R)$ have the same geometric CM order, then they have the same reflex type.*

*Proof.* Fix a deformation $(\mathfrak{G}, \rho) \in \mathfrak{Y}(R)$ and abbreviate $\mathcal{O} = \mathcal{O}(\mathfrak{G})$ for the geometric CM order of $\mathfrak{G}$, so that $\mathrm{Ta}_p(\mathfrak{G})$ is free of rank one over $\mathcal{O}$. Let $s \in \mathbb{Z}$ be defined by

$$\mathcal{O} = \mathbb{Z}_{p^2} + p^s \mathcal{O}_E.$$

As in Remark 1.2.9 the reflex type of $\mathfrak{G}$ is determined by the action of $\mathbb{Z}_p[\gamma_0]$ on $\mathrm{Lie}(\mathfrak{G})$. We will prove that this action is through $\Psi : \mathcal{O}_{E_0} \to \mathbb{Z}_p^\circ$ if $s$ is even and through $\overline{\Psi} : \mathcal{O}_{E_0} \to \mathbb{Z}_p^\circ$ if $s$ is odd, where $\Psi$ is the homomorphism defined in §2.1 and $\overline{\Psi}(x) = \Psi(x^\sigma)$.

Using [25, Theorem 1.3] we see that there is $p$-Barsotti-Tate group $\mathfrak{G}^*$ with $\mathcal{O}_E$-action satisfying

$$\mathrm{Ta}_p(\mathfrak{G}^*) \cong \mathrm{Ta}_p(\mathfrak{G}) \otimes_{\mathcal{O}} \mathcal{O}_E$$

and that the $\mathcal{O}$-linear inclusion of $\mathrm{Ta}_p(\mathfrak{G})$ into $\mathrm{Ta}_p(\mathfrak{G}^*)$ arises from an $\mathcal{O}$-linear isogeny of $p$-Barsotti-Tate groups $f : \mathfrak{G} \to \mathfrak{G}^*$ of degree $p^{2s}$. Let $\mathfrak{g}^*$ denote the reduction of $\mathfrak{G}^*$ to $\mathbb{F}$ with its action of $\mathcal{O}_E$. The claim is that one can find a $p$-Barsotti-Tate group $\mathfrak{g}_0^*$ over $\mathbb{F}$ equipped with an action of $\mathcal{O}_{E_0}$ in such a way that $\mathfrak{g}^*$ is $\mathcal{O}_E$-linearly isomorphic to $\mathfrak{g}_0^* \otimes \mathbb{Z}_{p^2}$. Let $(D^*, F, V)$ be the covariant Dieudonné module of $\mathfrak{g}^*$ and identify (Lemma 2.1.1) the covariant Dieudonné module of $\mathfrak{g}_0$ with its $\mathcal{O}_{E_0}$-action with the standard superspecial Dieudonné module $(D_0, F, V)$ with its normalized action of $\mathcal{O}_{E_0}$, as defined in §2.1. The Dieudonné module of $\mathfrak{g}$ is then $D = D_0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$ with its $\mathbb{Z}_{p^2}$-linear extensions of $F$ and $V$. Let $\{\epsilon_1, \epsilon_2\}$ be the idempotents in $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^\circ \cong \mathbb{Z}_p^\circ \times \mathbb{Z}_p^\circ$ indexed so that

$$(\alpha \otimes 1)\epsilon_1 = (1 \otimes \alpha)\epsilon_1 \qquad (\alpha \otimes 1)\epsilon_2 = (1 \otimes \alpha^\sigma)\epsilon_2.$$

The $\mathbb{Z}_p^\circ$-module $D$ then has

$$e_1 = \epsilon_1 e_0 \qquad e_2 = \epsilon_2 e_0 \qquad f_1 = \epsilon_1 f_0 \qquad f_2 = \epsilon_2 f_0$$

as a basis, the action of $\alpha \in \mathbb{Z}_{p^2}$ is via (8) and the action of $F$ (and $V$) is via

$$e_1 \mapsto f_2 \qquad e_2 \mapsto f_1 \qquad f_1 \mapsto pe_2 \qquad f_2 \mapsto pe_1.$$

Using the reduction of the isogeny $f$ to identify $D^*$ with an $\mathcal{O}_E$-stable superlattice of $D$ in $D \otimes_{\mathbb{Z}_p^\circ} \mathbb{Q}_p^\circ$ we find that the action of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^\circ \cong (\mathbb{Z}_p^\circ)^4$ decomposes $D^*$ into a direct sum of four rank one $\mathbb{Z}_p^\circ$- submodules, each spanned by some $\mathbb{Q}_p^\circ$-multiple of one of the four basis elements $\{e_1, e_2, f_1, f_2\}$. The condition that $D^*$ be stable under $F$ and $V$ then forces $D^*$ to have the form

$$D^* = \frac{1}{p^a}\mathbb{Z}_p^\circ e_1 \oplus \frac{1}{p^b}\mathbb{Z}_p^\circ e_2 \oplus \frac{1}{p^{b+\delta}}\mathbb{Z}_p^\circ f_1 \oplus \frac{1}{p^{a+\delta}}\mathbb{Z}_p^\circ f_2$$

with $\delta \in \{0,1\}$, $a, b \geq 0$, and $a+b+\delta = s$. If we define $D_0^* \subset D^*$ to be the $\mathbb{Z}_p^\circ$-span of

$$e_0^* = \frac{1}{p^a}e_1 + \frac{1}{p^b}e_2 \qquad f_0^* = \frac{1}{p^{b+\delta}}f_1 + \frac{1}{p^{a+\delta}}f_2$$

then $D_0^*$ is stable under the actions of $\mathcal{O}_{E_0}$, $F$, and $V$, and satisfies $D_0^* \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2} \cong D^*$. Hence we may take $\mathfrak{g}_0^*$ to be the $p$-Barsotti-Tate group associated to the Dieudonné module $(D_0^*, F, V)$.

Fix an isomorphism $\mathfrak{g}^* \cong \mathfrak{g}_0^* \otimes \mathbb{Z}_{p^2}$ as in the previous paragraph and let $\mathfrak{Y}_0^*$ and $\mathfrak{Y}^*$ be the functors on **Art** classifying, respectively, deformations of $\mathfrak{g}_0^*$ with its $\mathcal{O}_{E_0}$- action and $\mathfrak{g}^*$ with its $\mathcal{O}_E$-action. By Lemma 3.2.1 we have

$$\mathfrak{Y}^* \cong \mathfrak{Y}_0^* \cong \mathrm{Spf}(\mathbb{Z}_p^\circ)$$

and so there is an $\mathcal{O}_E$-linear isomorphism $\mathfrak{G}^* \cong \mathfrak{G}_0^* \otimes \mathbb{Z}_{p^2}$ where $\mathfrak{G}_0^*$ is the unique deformation to $R$ of $\mathfrak{g}_0^*$ with its $\mathcal{O}_{E_0}$-action. Fix an isomorphism of $\mathcal{O}_{E_0}$-modules

$$\mathrm{Ta}_p(\mathfrak{G}_0^*) \cong \mathcal{O}_{E_0}$$

and let $\mathfrak{G}_0'$ be the $p$-Barsotti-Tate group over $R$ whose $p$-adic Tate module is the sublattice

$$\mathrm{Ta}_p(\mathfrak{G}_0') \cong \mathcal{O} \subset \mathcal{O}_{E_0} \cong \mathrm{Ta}_p(\mathfrak{G}_0^*).$$

Let $f_0' : \mathfrak{G}_0' \to \mathfrak{G}_0^*$ be the associated isogeny of degree $p^s$, set $\mathfrak{G}' = \mathfrak{G}_0' \otimes \mathbb{Z}_{p^2}$, and let $f' : \mathfrak{G}' \to \mathfrak{G}^*$ be the isogeny $f' = f_0' \otimes \mathrm{id}$. Then the $p$-adic Tate modules of $\mathfrak{G}$ and $\mathfrak{G}'$ are each free rank one $\mathcal{O}$- submodules of $\mathrm{Ta}_p(\mathfrak{G}^*) \cong \mathcal{O}_E$ of index $p^{2s}$, and such submodules are permuted simply transitively by $\mathcal{O}_E^\times/\mathcal{O}^\times$. Thus there is a $\xi \in \mathcal{O}_E^\times$ such that the automorphism of $\mathrm{Ta}_p(\mathfrak{G}^*)$ induced by $\xi$ carries the $p$-adic Tate module of $\mathfrak{G}$ isomorphically onto the $p$-adic Tate module of $\mathfrak{G}'$. This isomorphism of $p$-adic Tate modules arises from an isomorphism $\xi : \mathfrak{G} \to \mathfrak{G}'$ making the diagram

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\ \xi\ } & \mathfrak{G}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathfrak{G}^* & \xrightarrow[\ \xi\ ]{} & \mathfrak{G}^* \end{array}$$

commute. Reducing these isogenies to $\mathbb{F}$ we obtain a commutative diagram of $\mathcal{O}_E$-linear isogenies

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\ \xi\circ\rho\ } & \mathfrak{g}' & \xrightarrow{\ \cong\ } & \mathfrak{g}_0' \otimes \mathbb{Z}_{p^2} \\ {\scriptstyle f\circ\rho}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_0'\otimes\mathrm{id}} \\ \mathfrak{g}^* & \xrightarrow[\ \xi\ ]{} & \mathfrak{g}^* & \xrightarrow{\ \cong\ } & \mathfrak{g}_0^* \otimes \mathbb{Z}_{p^2}. \end{array}$$

As the horizontal arrows are isomorphisms there are $\mathcal{O}_E$-linear isomorphisms

$$\mathrm{Lie}(\mathfrak{g}) \cong \mathrm{Lie}(\mathfrak{g}') \cong \mathrm{Lie}(\mathfrak{g}_0') \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}.$$

The action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{g})$ is through $\psi$, and hence so is the action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{g}_0')$. Applying Lemma 2.1.2 to the reduction of the isogeny $f_0'$, the action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{g}_0')$ is therefore through

$$\phi = \begin{cases} \psi & \text{if } s \text{ even} \\ \overline{\psi} & \text{if } s \text{ odd.} \end{cases}$$

Let $\Phi : \mathcal{O}_{E_0} \to \mathbb{Z}_p^\circ$ be the unique (recall that $E_0/\mathbb{Q}_p$ is unramified) $\mathbb{Z}_p$-algebra homomorphism lifting $\phi$, so that

$$\Phi = \begin{cases} \Psi & \text{if } s \text{ even} \\ \overline{\Psi} & \text{if } s \text{ odd} \end{cases}$$

By what we have shown, the action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{G}_0^*)$ is through $\Phi$, and hence so is the action of $\mathcal{O}_{E_0}$ on

$$\mathrm{Lie}(\mathfrak{G}^*) \cong \mathrm{Lie}(\mathfrak{G}_0^*) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}.$$

Finally, the isogeny $f$ induces an $\mathbb{Z}_p[\gamma_0]$-linear injection

$$\mathrm{Lie}(\mathfrak{G}) \to \mathrm{Lie}(\mathfrak{G}^*)$$

and we at last deduce that $\mathbb{Z}_p[\gamma_0]$ acts on $\mathrm{Lie}(\mathfrak{G})$ through $\Phi$. □

**Corollary 3.2.3.** *Let $R$ be the ring of integers of a finite extension of $\mathbb{Q}_p^\circ$. Every $\mathcal{O}_E^\times$-orbit in $\mathfrak{Y}(R)$ contains an element of $\mathfrak{Y}_0(R)$.*

*Proof.* Fix a deformation $(\mathfrak{G}, \rho) \in \mathfrak{Y}(R)$ and let

$$\mathcal{O} = \mathbb{Z}_{p^2} + p^s\mathcal{O}_E$$

be the geometric CM-order of $\mathfrak{G}$. As $\mathcal{O} \supset \mathbb{Z}_{p^2}[\gamma] = \mathbb{Z}_p[\gamma_0] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$ we must have $0 \le s \le c_0$. Let $R'$ be the ring of integers in a finite extension of $\mathrm{Frac}(R)$ large enough to contain a subfield isomorphic to $M_s$, and fix a quasi-canonical lift $(\mathfrak{G}_0', \rho_0') \in \mathfrak{Y}_0(W_s)$ of $\mathfrak{g}_0$ of level $s$. The deformations $(\mathfrak{G}, \rho)_{/R'}$ and $(\mathfrak{G}_0', \rho_0')_{/R'} \otimes \mathbb{Z}_{p^2}$ have the same geometric CM-order $\mathcal{O}$, and hence have the same reflex type by Proposition 3.2.2. By Proposition 1.2.10 there is a $\xi \in \mathcal{O}_E^\times$ such that

$$\xi * (\mathfrak{G}, \rho)_{/R'} \cong (\mathfrak{G}_0', \rho_0')_{/R'} \otimes \mathbb{Z}_{p^2}.$$

The pro-representability of $\mathfrak{Y}_0$ and $\mathfrak{Y}$ imply that

$$\mathfrak{Y}(R) \cap \mathfrak{Y}_0(R') = \mathfrak{Y}_0(R)$$

and so $\xi * (\mathfrak{G}, \rho) \in \mathfrak{Y}_0(R)$. □

The homomorphism $\mathcal{O}_E^\times \to \mathrm{Aut}_{\mathbf{ProArt}}(R_{\mathfrak{Y}})$ constructed in §1.1 determines an action of $\mathcal{O}_E^\times$ on the set of components of $\mathfrak{Y}$. To be explicit, if $\mathfrak{C}$ is the component corresponding to a minimal prime $\mathfrak{p}$ of $R_{\mathfrak{Y}}$ then $\xi * \mathfrak{C}$ corresponds to the kernel $\xi(\mathfrak{p})$ of

$$R_{\mathfrak{Y}} \xrightarrow{\xi^{-1}} R_{\mathfrak{Y}} \to R_{\mathfrak{Y}}/\mathfrak{p}.$$

Recalling Proposition 3.1.1, for each $0 \le s \le c_0$ let $\mathfrak{p}_{0,s} \subset R_{\mathfrak{Y}_0}$ be the unique minimal prime for which $R_{\mathfrak{Y}_0}/\mathfrak{p}_{0,s} \cong W_s$ and define a component of $\mathfrak{Y}_0$ by

$$\mathfrak{C}_0(s) = \mathrm{Spf}(R_{\mathfrak{Y}_0}/\mathfrak{p}_{0,s}).$$

Let $q : R_{\mathfrak{Y}} \to R_{\mathfrak{Y}_0}$ be the homomorphism inducing the closed immersion $\mathfrak{Y}_0 \to \mathfrak{Y}$ and write $\mathfrak{C}(s)$ for $\mathfrak{C}_0(s)$ viewed as a closed subscheme of $\mathfrak{Y}$, so that

$$\mathfrak{C}(s) \cong \mathrm{Spf}(R_{\mathfrak{Y}}/\mathfrak{p}_s)$$

where $\mathfrak{p}_s = q^{-1}(\mathfrak{p}_{0,s})$. Define a subgroup $H_s \subset \mathcal{O}_E^\times$ by

$$H_s = \mathcal{O}_{E_0}^\times \cdot (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times.$$

*Remark* 3.2.4. As Proposition 3.1.1 gives a complete list of the minimal primes of $R_{\mathfrak{Y}_0}$, it follows that $\mathfrak{C}(0), \dots, \mathfrak{C}(c_0)$ is a complete list of the improper components of $\mathfrak{Y}$.

**Proposition 3.2.5.** *Every horizontal component of $\mathfrak{Y}$ can be expressed uniquely in the form*

$$\mathfrak{C}(s,\xi) \stackrel{\text{def}}{=} \xi * \mathfrak{C}(s)$$

*with $0 \le s \le c_0$ and $\xi \in \mathcal{O}_E^\times / H_s$.*

*Proof.* Fix a horizontal component $\mathfrak{C} = \mathrm{Spf}(R_{\mathfrak{Y}}/\mathfrak{p})$ of $\mathfrak{Y}$ and let $R$ be the ring of integers in the fraction field of $R_{\mathfrak{Y}}/\mathfrak{p}$, a finite extension of $\mathbb{Q}_p^\circ$ by Corollary 1.2.6. Applying Corollary 3.2.3 to the deformation $(\mathfrak{G}, \rho) \in \mathfrak{Y}(R)$ determined by the composition map $R_{\mathfrak{Y}} \to R_{\mathfrak{Y}}/\mathfrak{p} \to R$ we find a $\xi \in \mathcal{O}_E^\times$ such that

$$\xi * (\mathfrak{G}, \rho) \cong (\mathfrak{G}_0, \rho_0) \otimes \mathbb{Z}_{p^2}$$

where $(\mathfrak{G}_0, \rho_0)$ is obtained as the base change of a quasi-canonical lift of level $s$ through some embedding $W_s \to R$. The integer $s$ is determined by the condition that $\mathbb{Z}_{p^2} + p^s\mathcal{O}_E$ is the geometric CM-order of $(\mathfrak{G}, \rho)$. This can be restated as saying that the composition

$$R_{\mathfrak{Y}} \xrightarrow{\xi^{-1}} R_{\mathfrak{Y}} \to R$$

factors through $R_{\mathfrak{Y}_0}$, and the kernel of this composition is $\mathfrak{p}_s$. Therefore $\xi(\mathfrak{p}) = \mathfrak{p}_s$ and so $\xi * \mathfrak{C} = \mathfrak{C}(s)$.

The only thing left to prove is that $H_s \subset \mathcal{O}_E^\times$ is the stabilizer of $\mathfrak{C}(s)$. Fix a quasi-canonical lift $(\mathfrak{G}_0, \rho_0) \in \mathfrak{Y}_0(W_s)$ corresponding to some surjection of $\mathbb{Z}_p^\circ$-algebras $f_0 : R_{\mathfrak{Y}_0} \to W_s$ and let

$$(\mathfrak{G}, \rho) = (\mathfrak{G}_0, \rho_0) \otimes \mathbb{Z}_{p^2} \in \mathfrak{Y}(W_s)$$

be the deformation corresponding to $f = f_0 \circ q$. The action of $\mathcal{O}_{E_0}^\times$ on $R_{\mathfrak{Y}_0}$ must stabilize the prime $\mathfrak{p}_{0,s}$, as $\mathfrak{p}_{0,s}$ is characterized as the unique minimal prime of $R_{\mathfrak{Y}_0}$ with the property $R_{\mathfrak{Y}_0}/\mathfrak{p}_{0,s} \cong W_s$. Thus the action of $\mathcal{O}_{E_0}^\times$ on $R_{\mathfrak{Y}}$ leaves $\mathfrak{p}_s$ fixed, proving that $\mathcal{O}_{E_0}^\times$ stabilizes $\mathfrak{C}(s)$. Now suppose we are given some $\xi \in (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times$. By the comments preceding Proposition 3.1.1 the action of $\mathbb{Z}_p[\gamma_0]$ on $\mathfrak{G}_0$ can be extended to an action of $\mathbb{Z}_p + p^s\mathcal{O}_K$, and hence the action of $\mathbb{Z}_{p^2}[\gamma]$ on $\mathfrak{G}$ can be extended to an action of $\mathbb{Z}_{p^2} + p^s\mathcal{O}_E$. The existence of a lifting of $\xi \in \mathrm{Aut}(\mathfrak{g})$ to an automorphism of $\mathfrak{G}$ is equivalent to the existence of an isomorphism of deformations

$$(\mathfrak{G}, \rho) \cong (\mathfrak{G}, \rho \circ \xi^{-1}),$$

which proves that $f = f \circ \xi^{-1}$. As $\mathfrak{p}_s = \ker(f)$ we conclude that $\mathfrak{p}_s = \xi(\mathfrak{p}_s)$. We have now shown that both $\mathcal{O}_{E_0}^\times$ and $(\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times$ are contained in the stabilizer of $\mathfrak{C}(s)$.

Now start with some $\xi \in \mathcal{O}_E^\times$ which satisfies $\mathfrak{p}_s = \xi(\mathfrak{p}_s)$. This implies that the two homomorphisms $f, f \circ \xi^{-1} : R_{\mathfrak{Y}} \to W_s$ have the same kernel, and as $f = f_0 \circ q$ factors through the quotient map $q : R_{\mathfrak{Y}} \to R_{\mathfrak{Y}_0}$ we may also factor $f \circ \xi^{-1} = f_0' \circ q$ for some $f_0' : R_{\mathfrak{M}_0} \to W_s$. As $f_0, f_0' : R_{\mathfrak{Y}_0} \to W_s$ are surjective homomorphisms of $\mathbb{Z}_p^\circ$- algebras with the same kernel $\mathfrak{p}_{0,s}$, there is a

$$\tau \in \mathrm{Aut}_{\mathbb{Z}_p^\circ}(W_s) \cong \mathrm{Gal}(M_s/M_0)$$

such that $f_0 = \tau \circ f_0'$. Recalling the discussion after Proposition 3.1.1 we may write $\tau = \mathrm{rec}(\xi_0^{-1})$ for some $\xi_0 \in \mathcal{O}_{E_0}^\times$ to obtain

$$f = f_0 \circ q = f_0' \circ \xi_0 \circ q = f_0' \circ q \circ \xi_0 = f \circ \xi^{-1} \circ \xi_0.$$

In other words there is an isomorphism of deformations

$$(\mathfrak{G}, \rho) \cong (\mathfrak{G}, \rho \circ \xi^{-1}\xi_0).$$

This implies that the automorphism $\xi\xi_0^{-1} \in \mathcal{O}_E^\times$ of $\mathfrak{g}$ lifts to an automorphism of $\mathfrak{G}$, and so acts on the $p$-adic Tate module of $\mathfrak{G}$. But $\mathfrak{G}$ has geometric CM order

$$\mathcal{O}(\mathfrak{G}) = \mathbb{Z}_{p^2} + p^s\mathcal{O}_E$$

and we conclude that $\xi\xi_0^{-1} \in (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times$. Hence $\xi \in H_s$. □

Note that for each $0 \le s \le c_0$ the group $\mathcal{O}_E^\times/H_s$ has a decreasing filtration

$$\{1\} = H_s/H_s \subset H_{s-1}/H_s \subset \cdots H_0/H_s = \mathcal{O}_E^\times/H_s.$$

**Proposition 3.2.6.** *Suppose $0 \le t < s \le c_0$, $\xi \in H_t/H_s$, and $\xi \notin H_{t+1}/H_s$. The horizontal component $\mathfrak{C}(s,\xi)$ of $\mathfrak{Y}$ defined in Proposition 3.2.5 satisfies*

$$I_{\mathfrak{M}}(\mathfrak{C}(s,\xi), \mathfrak{M}_0) = 1 + \frac{(p+1)(p^t-1)}{p-1}.$$

*Proof.* Let $(\mathfrak{G}_0, \rho_0) \in \mathfrak{Y}_0(W_s)$ be a quasi-canonical lift of level $s$ and let $(\mathfrak{G}, \rho) = (\mathfrak{G}_0, \rho_0) \otimes \mathbb{Z}_{p^2}$ be its image in $\mathfrak{Y}(W_s)$. If we write $\mathfrak{m}$ for the maximal ideal of $W_s$ and $R_k = W_s/\mathfrak{m}^{k+1}$ then

$$I_{\mathfrak{M}}(\mathfrak{C}(s,\xi), \mathfrak{M}_0) = k+1$$

where $k$ is the largest nonnegative integer for which

$$(\mathfrak{G}, \rho \circ \xi^{-1})_{/R_k} \in \mathrm{Image}\big(\mathfrak{M}_0(R_k) \to \mathfrak{M}(R_k)\big).$$

The coset $\xi \in H_t/H_s$ admits a representative $\xi \in (\mathbb{Z}_{p^2} + p^t\mathcal{O}_E)^\times$ not contained in $(\mathbb{Z}_{p^2} + p^{t+1}\mathcal{O}_E)^\times$, which we now fix.

For any $k \ge 0$ abbreviate

$$a(k) = \frac{(p+1)(p^k-1)}{p-1}.$$

Keating [12, Theorem 5.2] has computed the endomorphism ring of $\mathfrak{G}_{0/R_k}$, and found that the largest order of $\mathcal{O}_{E_0}$ for which the action $j_0 : \mathcal{O}_{E_0} \to \mathrm{End}(\mathfrak{g}_0)$ lifts to the deformation $(\mathfrak{G}_0, \rho_0)_{/W_k}$ is

$$\begin{array}{rll} \mathcal{O}_{E_0} & \text{if} & 0 \le k < a(0)+1 \\ \mathbb{Z}_p + p\mathcal{O}_{E_0} & \text{if} & a(0)+1 \le k < a(1)+1 \\ \mathbb{Z}_p + p^2\mathcal{O}_{E_0} & \text{if} & a(1)+1 \le k < a(2)+1 \\ & \vdots & \\ \mathbb{Z}_p + p^{s-1}\mathcal{O}_{E_0} & \text{if} & a(s-2)+1 \le k < a(s-1)+1 \\ \mathbb{Z}_p + p^s\mathcal{O}_{E_0} & \text{if} & a(s-1)+1 \le k. \end{array}$$

We remark that Keating refers the reader to his unpublished thesis for the proof of [12, Theorem 5.2], but a complete proof may be found in [24]. Using $\mathrm{End}(\mathfrak{G}_{/R_k}) \cong \mathrm{End}(\mathfrak{G}_{0/R_k}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$

we find that the largest order of $\mathcal{O}_E$ for which the action $j : \mathcal{O}_E \to \mathrm{End}(\mathfrak{g})$ lifts to the deformation $(\mathfrak{G}, \rho)_{/R_k}$ is

$$\begin{array}{rl} \mathcal{O}_E & \text{if} \quad 0 \le k < a(0)+1 \\ \mathbb{Z}_{p^2} + p\mathcal{O}_E & \text{if} \quad a(0)+1 \le k < a(1)+1 \\ \mathbb{Z}_{p^2} + p^2\mathcal{O}_E & \text{if} \quad a(1)+1 \le k < a(2)+1 \\ & \vdots \\ \mathbb{Z}_{p^2} + p^{s-1}\mathcal{O}_E & \text{if} \quad a(s-2)+1 \le k < a(s-1)+1 \\ \mathbb{Z}_{p^2} + p^{s}\mathcal{O}_E & \text{if} \quad a(s-1)+1 \le k. \end{array}$$

Set $k = a(t)$ so that the automorphism $\xi \in (\mathbb{Z}_{p^2} + p^t\mathcal{O}_E)^\times$ of $\mathfrak{g}$ lifts to the deformation $(\mathfrak{G}, \rho)_{/R_k}$ but not to $(\mathfrak{G}, \rho)_{/R_{k+1}}$. The existence of a lift of $\xi$ over $R_k$ is equivalent to the existence of an isomorphism of deformations

$$(\mathfrak{G}, \rho)_{/R_k} \cong (\mathfrak{G}, \rho \circ \xi^{-1})_{/R_k},$$

and as $(\mathfrak{G}, \rho)_{/R_k}$ lies in the image of $\mathfrak{M}_0(R_k) \to \mathfrak{M}(R_k)$ so does $(\mathfrak{G}, \rho \circ \xi^{-1})_{/R_k}$.

Now set $k = a(t)+1$ and $\Delta_0 = \mathrm{End}(\mathfrak{g}_0)$ so that the largest order $\mathcal{O} \subset \mathcal{O}_{E_0}$ for which the restriction to $\mathcal{O}$ of the action $j_0 : \mathcal{O}_{E_0} \to \Delta_0$ lifts to $(\mathfrak{G}_0, \rho_0)_{/R_k}$ is

$$\mathcal{O} = \mathbb{Z}_p + p^{t+1}\mathcal{O}_{E_0}.$$

As in §1.1 we fix an embedding $\mathbb{Z}_{p^2} \to M_2(\mathbb{Z}_p)$ and identify the functor $\mathfrak{G}'_0 \mapsto \mathfrak{G}'_0 \otimes \mathbb{Z}_{p^2}$ with the functor $\mathfrak{G}'_0 \mapsto \mathfrak{G}'_0 \times \mathfrak{G}'_0$. This identification determines an isomorphism

$$\mathrm{End}(\mathfrak{G}'_0 \otimes \mathbb{Z}_{p^2}) \cong M_2(\mathrm{End}(\mathfrak{G}'_0))$$

and in particular $\mathrm{End}(\mathfrak{g}) \cong M_2(\Delta_0)$. Moreover, one easily checks that the closed formal subscheme $\mathfrak{M}_0 \to \mathfrak{M}$ is the locus of deformations for which the action of $M_2(\mathbb{Z}_p)$ on $\mathfrak{g}$ lifts. To obtain a contradiction let us suppose that

$$(\mathfrak{G}, \rho \circ \xi^{-1})_{/R_k} \in \mathrm{Image}\big(\mathfrak{M}_0(R_k) \to \mathfrak{M}(R_k)\big)$$

so that the action of $M_2(\mathbb{Z}_p)$ on $\mathfrak{g}$ lifts to the deformation $(\mathfrak{G}, \rho \circ \xi^{-1})_{/R_k}$, and, equivalently, the endomorphism $\xi \circ \mu \circ \xi^{-1}$ of $\mathfrak{g}$ lifts to the deformation $(\mathfrak{G}, \rho)$ for every $\mu \in M_2(\mathbb{Z}_p)$. Using $j_0 : \mathcal{O}_{E_0} \to \Delta_0$ we view $\mathcal{O}_{E_0}$ as a subring of $\Delta_0$ and $\mathcal{O}_E$ as a subring of $M_2(\mathcal{O}_{E_0})$. We also view

$$\mathrm{End}(\mathfrak{G}_{/R_k}) \cong M_2(\mathrm{End}(\mathfrak{G}_{0/R_k}))$$

as a subring of $\mathrm{End}(\Delta_0)$ using the reduction map $R_k \to \mathbb{F}$. Under these identifications

$$\mathrm{End}(\mathfrak{G}_{/R_k}) \cap M_2(\mathcal{O}_{E_0}) = M_2(\mathcal{O})$$

and we deduce that $\xi \circ \mu \circ \xi^{-1} \in M_2(\mathcal{O})$ for every $\mu \in M_2(\mathbb{Z}_p)$. As the subalgebra of $M_2(\Delta_0)$ generated by $M_2(\mathbb{Z}_p)$ and $\mathcal{O}$ is $M_2(\mathcal{O})$, we further deduce that $\xi \circ \mu \circ \xi^{-1} \in M_2(\mathcal{O})$ for every $\mu \in M_2(\mathcal{O})$. Some elementary linear algebra then shows that $\xi \in \mathcal{O}_{E_0}^\times \cdot \mathrm{GL}_2(\mathcal{O})$. Thus there is a $\xi_0 \in \mathcal{O}_{E_0}$ for which $\xi\xi_0^{-1} \in M_2(\mathcal{O})$, and so

$$\xi\xi_0^{-1} \in \mathrm{End}(\mathfrak{G}_{/R_k}) \cap \mathcal{O}_E^\times = (\mathbb{Z}_{p^2} + p^{t+1}\mathcal{O}_E)^\times.$$

But this means that $\xi \in H_{t+1}$, a contradiction.

We have shown that $(\mathfrak{G}, \rho \circ \xi^{-1})_{/R_k}$ lies in the image of $\mathfrak{M}_0(R_k)$ for $k = a(t)$ but not for $k = a(t)+1$, completing the proof. □

**Corollary 3.2.7.** *We have*

$$\sum_{\mathfrak{C}} I_{\mathfrak{M}}(\mathfrak{C}, \mathfrak{M}_0) = \frac{-4p(p^{c_0}-1)+2c_0(p-1)+c_0p^{c_0}(p^2-1)}{(p-1)^2}$$

*where the sum is over all proper horizontal components* $\mathfrak{C} \to \mathfrak{Y}$.

*Proof.* If $0 \le t < s \le c_0$ then combining

$$|H_t/H_s| = \begin{cases} p^{s-t} & \text{if } t \neq 0 \\ p^{s-1}(p-1) & \text{if } t = 0 \end{cases}$$

with Proposition 3.2.6 shows that

$$\sum_{\substack{\xi \in \mathcal{O}_E^\times/H_s \\ \xi \neq 1}} I_{\mathfrak{M}}(\mathfrak{C}(s,\xi), \mathfrak{M}_0) = p^{s-1}(p-2) + p^{s-1}(p+1)(s-1) - \frac{2(p^{s-1}-1)}{p-1}.$$

Summing over $0 \le s \le c_0$ and using Proposition 3.2.5 yields the desired result. □

3.3. $E_0$ **ramified.** Throughout all of §3.3 we assume that $E_0/\mathbb{Q}_p$ is ramified. Suppose $R$ is the ring of integers of a finite extension of $\mathbb{Q}_p^\circ$ and $(\mathfrak{G}, \rho) \in \mathfrak{Y}(R)$. As in Remark 1.2.9 there is a decomposition of the Lie algebra

$$\mathrm{Lie}(\mathfrak{G}) \cong \Lambda_1 \oplus \Lambda_2$$

in which each $\Lambda_i$ is free of rank one over $R$, $\mathbb{Z}_{p^2}$ acts through $\mathbb{Z}_{p^2} \to \mathbb{Z}_p^\circ \to R$ on $\Lambda_1$ and through the conjugate embedding on $\Lambda_2$. The action of $\mathbb{Z}_{p^2}[\gamma]$ on $\Lambda_i$ is through some homomorphism $\Phi_i : \mathcal{O}_E \to R$. We have $\Phi_1 \neq \Phi_2$ after restriction to $\mathbb{Z}_{p^2}$, and the restriction of the pair $(\Phi_1, \Phi_2)$ to $\mathcal{O}_{E_0}$ determines the reflex type of $(\mathfrak{G}, \rho)$.

**Definition 3.3.1.** If $\Phi_1 = \Phi_2$ when restricted to $\mathcal{O}_{E_0}$ then we say that the deformation $(\mathfrak{G}, \rho)$ has a *standard* reflex type. If $\Phi_1 \neq \Phi_2$ when restricted to $\mathcal{O}_{E_0}$ then we say that the reflex type is *nonstandard*.

There is a disjoint union

$$\mathfrak{Y}(R) = \mathfrak{Y}^+(R) \bigcup \mathfrak{Y}^-(R)$$

where $\mathfrak{Y}^+(R)$ is the subset of deformations having standard reflex type and $\mathfrak{Y}^-(R)$ is the subset of deformations with nonstandard reflex type. Each subset is $\mathcal{O}_E^\times$-stable, as the reflex type is constant on $\mathcal{O}_E^\times$- orbits. Note that the above decomposition makes sense for any objects $R$ of **ProArt**, but is not functorial: after base change through a homomorphism $R \to R'$ one may have elements of $\mathfrak{Y}^-(R)$ whose image lies in $\mathfrak{Y}^+(R')$. In this subsection we will only consider those $R$ which are integer rings of finite extensions of $\mathbb{Q}_p^\circ$, and for such rings any (necessarily injective) map $R \to R'$ induces an inclusion $\mathfrak{Y}(R) \to \mathfrak{Y}(R')$ which respects the decompositions into standard and nonstandard reflex types.

**Lemma 3.3.2.** *Let $R$ be the ring of integers of a finite extension of $\mathbb{Q}_p^\circ$. Then $\mathfrak{Y}_0(R) \subset \mathfrak{Y}^+(R)$ and every $\mathcal{O}_E^\times$-orbit in $\mathfrak{Y}^+(R)$ contains an element of $\mathfrak{Y}_0(R)$.*

*Proof.* First start with some $(\mathfrak{G}_0, \rho_0) \in \mathfrak{Y}_0(R)$. The action of $\mathbb{Z}_p[\gamma_0]$ on $\mathrm{Lie}(\mathfrak{G}_0)$ is through some $\mathbb{Z}_p$-algebra map $\Phi_0 : \mathcal{O}_{E_0} \to R$, and so the action of $\mathbb{Z}_p[\gamma_0]$ on the Lie algebra $\mathrm{Lie}(\mathfrak{G}_0) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^2}$ of

$$(\mathfrak{G}_0, \rho_0) \otimes \mathbb{Z}_{p^2} \in \mathfrak{Y}(R)$$

is again through $\Phi_0$. After restriction to $\mathcal{O}_{E_0}$ we now have $\Phi_1 = \Phi_0 = \Phi_2$, and hence $(\mathfrak{G}_0, \rho_0) \otimes \mathbb{Z}_{p^2}$ has a standard reflex type, proving $\mathfrak{Y}_0(R) \subset \mathfrak{Y}^+(R)$.

Now suppose we start with some $(\mathfrak{G},\rho)\in\mathfrak{Y}^+(R)$, so that $\mathbb{Z}_p[\gamma_0]$ acts on $\mathrm{Lie}(\mathfrak{G})$ through some embedding $\Phi_0:\mathcal{O}_{E_0}\to R$. Let $\mathbb{Z}_{p^2}+p^s\mathcal{O}_E$ be the geometric CM-order of $(\mathfrak{G},\rho)$ and consider a quasi-canonical lift $(\mathfrak{G}_0,\rho_0)\in\mathfrak{Y}_0(W_s)$ of level $s$. The action of $\mathbb{Z}_p[\gamma_0]$ on $\mathrm{Lie}(\mathfrak{G}_0)$ is through some embedding $\Phi_{00}:\mathbb{Z}_p[\gamma_0]\to W_s$, and after enlarging $R$ as in the proof of Corollary 3.2.3 we assume that there is an embedding of $\mathbb{Z}_p^\circ$-algebras $i:W_s\to R$. As $M_s/\mathbb{Q}_p^\circ$ is Galois we may choose a $\tau\in\mathrm{Gal}(M_s/\mathbb{Q}_p^\circ)$ whose restriction to $\mathrm{Gal}(M_0/\mathbb{Q}_p^\circ)$ is nontrivial, and then $i\circ\Phi_{00}$ and $i\circ\tau\circ\Phi_{00}$ give the two distinct embeddings of $\mathbb{Z}_p[\gamma_0]$ into $R$. Thus after possibly replacing $i$ by $i\circ\tau$ we may assume that $i\circ\Phi_{00}=\Phi_0$. The action of $\mathbb{Z}_p[\gamma_0]$ on the Lie algebra of $(\mathfrak{G}_0,\rho_0)_{/R}\in\mathfrak{Y}_0(R)$ is now through $\Phi_0$, and so the deformation

$$(\mathfrak{G}_0,\rho_0)_{/R}\otimes\mathbb{Z}_{p^2}\in\mathfrak{Y}(R)$$

has the same reflex type and same geometric CM-order as $(\mathfrak{G},\rho)$. Applying Proposition 1.2.10 completes the proof. □

Again let $R$ be the ring of integers in a finite extension of $\mathbb{Q}_p^\circ$. We now explain how to construct elements in $\mathfrak{Y}^-(R)$. As we assume that $E_0/\mathbb{Q}_p$ is a ramified field extension, $E$ is a biquadratic field extension of $\mathbb{Q}_p$ containing a unique quadratic subfield $E_0'$ which is ramified over $\mathbb{Q}_p$ and not equal to $E_0$. Choose an action $j_0':\mathcal{O}_{E_0'}\to\mathrm{End}(\mathfrak{g}_0)$ of $\mathcal{O}_{E_0'}$ on $\mathfrak{g}_0$ and let $j'$ be the induced $\mathbb{Z}_{p^2}$-linear action

$$j':\mathcal{O}_E\to\mathrm{End}(\mathfrak{g})$$

of $\mathcal{O}_E=\mathcal{O}_{E_0'}\otimes\mathbb{Z}_{p^2}$ on $\mathfrak{g}$.

**Lemma 3.3.3.** *There is a $\mathbb{Z}_{p^2}$-linear automorphism $w:\mathfrak{g}\to\mathfrak{g}$ satisfying*

$$w\circ j'(x)=j(x)\circ w$$

*for every $x\in\mathcal{O}_E$.*

*Proof.* Abbreviate $\Delta_0=\mathrm{End}(\mathfrak{g}_0)$ and $\Delta=\mathrm{End}(\mathfrak{g})$ so that $\Delta=\Delta_0\otimes_{\mathbb{Z}_p}\mathbb{Z}_{p^2}$. Thus $\Delta_0$ is the maximal order in a quaternion division algebra over $\mathbb{Q}_p$ and $\Delta$ is isomorphic to an Eichler order of level one in $M_2(\mathbb{Z}_{p^2})$. Fix such an isomorphism, so that $\Delta$ acts on a two-dimension $\mathbb{Q}_{p^2}$-vector space in a way which stabilizes a $\mathbb{Z}_{p^2}$-lattice $\Lambda_1$ and sublattice $\Lambda_2\subset\Lambda_1$ with $\Lambda_1/\Lambda_2\cong\mathbb{F}_{p^2}$. Let $L_i$ denote $\Lambda_i$ viewed as an $\mathcal{O}_E$-module via $j$, and let $L_i'$ denote $\Lambda_i$ viewed as an $\mathcal{O}_E$-module via $j'$. If $\mathfrak{q}\subset\mathcal{O}_E$ is the maximal ideal then $L_2=\mathfrak{q}L_1$, $L_2'=\mathfrak{q}L_1'$, and there is an isomorphism of torsion free rank one $\mathcal{O}_E$-modules $w:L_1'\to L_1$. As $w$ takes $L_2'$ to $L_2$, we have found the desired $w\in\mathrm{End}(\Lambda_2)\cap\mathrm{End}(\Lambda_1)=\Delta$. □

Choose any $\gamma_0'\in\mathcal{O}_{E_0'}$ for which $\mathbb{Z}_p[\gamma_0']=\mathbb{Z}_p+p^{c_0}\mathcal{O}_{E_0'}$ and set

$$\gamma'=\gamma_0'\otimes 1\in\mathcal{O}_E\cong\mathcal{O}_{E_0'}\otimes_{\mathbb{Z}_p}\mathbb{Z}_{p^2}$$

so that

$$\mathbb{Z}_{p^2}[\gamma']=\mathbb{Z}_{p^2}+p^{c_0}\mathcal{O}_E=\mathbb{Z}_{p^2}[\gamma].$$

Let $\mathfrak{Y}_0'\to\mathfrak{M}_0$ be the closed formal subscheme classifying deformations of $\mathfrak{g}_0$ for which the action $j_0':\mathbb{Z}_p[\gamma_0']\to\mathrm{End}(\mathfrak{g}_0)$ lifts, and let $\mathfrak{Y}'\to\mathfrak{M}$ be the closed formal subscheme classifying deformations of $\mathfrak{g}$ for which the action $j':\mathbb{Z}_{p^2}[\gamma']\to\mathrm{End}(\mathfrak{g})$ lifts. There is then an isomorphism of functors

$$\mathfrak{Y}'\xrightarrow{w}\mathfrak{Y}$$

defined by

$$(\mathfrak{G},\rho)\mapsto(\mathfrak{G},\rho\circ w^{-1})$$

which respects with the action of $\mathcal{O}_E^\times$ on each side (the actions on the left and right being defined using the actions $j'$ and $j$ of $\mathcal{O}_E^\times$ on $\mathfrak{g}$, respectively). Define $R_{\mathfrak{Y}'_0}$ and $R_{\mathfrak{Y}'}$ by

$$\mathfrak{Y}'_0 = \mathrm{Spf}(R_{\mathfrak{Y}'_0}) \qquad \mathfrak{Y}' = \mathrm{Spf}(R_{\mathfrak{Y}'}).$$

We point out that Proposition 3.1.1 applies equally well with $\mathfrak{Y}_0$ replaced by $\mathfrak{Y}'_0$, and the theory of quasi-canonical lifts applies equally well to the deformation problem $\mathfrak{Y}'_0$ as it does to $\mathfrak{Y}_0$. We also point out that the isomorphism $w : \mathfrak{Y}' \to \mathfrak{Y}$ does not restrict to an isomorphism $\mathfrak{Y}'_0 \to \mathfrak{Y}_0$, as the automorphism $w$ of $\mathfrak{g}$ does not arise from an automorphism of $\mathfrak{g}_0$. Instead we have the following pretty situation.

**Lemma 3.3.4.** *Let $R$ be the ring of integers of a finite extension of $\mathbb{Q}_p^\circ$. The bijection $w : \mathfrak{Y}'(R) \to \mathfrak{Y}(R)$ restricts to an injection $w : \mathfrak{Y}'_0(R) \to \mathfrak{Y}^-(R)$, and every $\mathcal{O}_E^\times$-orbit in $\mathfrak{Y}^-(R)$ contains an element in the image of $\mathfrak{Y}'_0(R)$.*

*Proof.* We will repeatedly use the fact that for any two distinct embeddings of $E$ into a field there is a unique quadratic subfield of $E$ on which those embeddings agree. Fix $(\mathfrak{G}'_0, \rho'_0) \in \mathfrak{Y}'_0(R)$ and define

$$(\mathfrak{G}', \rho') = (\mathfrak{G}'_0, \rho'_0) \otimes \mathbb{Z}_{p^2} \in \mathfrak{Y}'(R).$$

Let $\Phi'_0 : \mathcal{O}_{E'_0} \to R$ be the $\mathbb{Z}_p^\circ$-algebra homomorphism giving the action of $\mathbb{Z}_p[\gamma'_0]$ on $\mathrm{Lie}(\mathfrak{G}'_0)$. Decomposing the Lie algebra

$$\mathrm{Lie}(\mathfrak{G}') \cong \Lambda_1 \oplus \Lambda_2$$

as before, the order $\mathbb{Z}_{p^2}[\gamma']$ acts on $\Lambda_i$ through an embedding $\Phi_i : \mathcal{O}_E \to R$ which restricts to $\Phi'_0$ on $\mathcal{O}_{E'_0}$. The restrictions of $\Phi_1$ and $\Phi_2$ to $\mathbb{Z}_{p^2}$ are distinct, and it follows that the restrictions of $\Phi_1$ and $\Phi_2$ to $\mathcal{O}_{E_0}$ are also distinct.

If we now define

$$(\mathfrak{G}, \rho) = (\mathfrak{G}', \rho' \circ w^{-1}) \in \mathfrak{Y}(R)$$

and let $\mathrm{id} : \mathfrak{G} \to \mathfrak{G}'$ be the identity map then id is a $\mathbb{Z}_{p^2} + p^{c_0}\mathcal{O}_E$-linear isomorphism of $p$-Barsotti-Tate groups $\mathfrak{G} \cong \mathfrak{G}'$ (where the action of $\mathbb{Z}_{p^2}[\gamma]$ on $\mathfrak{G}$ lifts the action $j$ on $\mathfrak{g}$, and the action of $\mathbb{Z}_{p^2}[\gamma']$ on $\mathfrak{G}'$ lifts the action $j'$ on $\mathfrak{g}$), and so $\mathrm{Lie}(\mathfrak{G}) \cong \mathrm{Lie}(\mathfrak{G}')$ as modules over

$$\mathbb{Z}_{p^2}[\gamma] \otimes_{\mathbb{Z}_p} R \cong \mathbb{Z}_{p^2}[\gamma'] \otimes_{\mathbb{Z}_p} R.$$

By the previous paragraph the reflex type of $(\mathfrak{G}, \rho)$ is nonstandard, proving $(\mathfrak{G}, \rho) \in \mathfrak{Y}^-(R)$.

Now start with any $(\mathfrak{G}, \rho) \in \mathfrak{Y}^-(R)$, and let $\mathbb{Z}_{p^2} + p^s\mathcal{O}_E$ be the geometric CM-order of $(\mathfrak{G}, \rho)$. Decompose

$$\mathrm{Lie}(\mathfrak{G}) \cong \Lambda_1 \oplus \Lambda_2$$

as above, so that $\mathbb{Z}_{p^2}[\gamma]$ acts on $\Lambda_i$ through some $\Phi_i : \mathcal{O}_E \to R$. As $\Phi_1$ and $\Phi_2$ are distinct when restricted to $\mathbb{Z}_{p^2}$ and to $\mathcal{O}_{E_0}$ (by definition of nonstandard reflex type), they must agree when restricted to $\mathcal{O}_{E'_0}$. Call the common restriction $\Phi'_0 : \mathcal{O}_{E'_0} \to R$. Choose a quasi-canonical lift $(\mathfrak{G}'_0, \rho'_0) \in \mathfrak{Y}'_0(W_s)$ of level $s$ of $\mathfrak{g}_0$ with its action of $j'_0$, and (after enlarging $R$ as in the proof of Corollary 3.2.3) an embedding $W_s \to R$ of $\mathbb{Z}_p^\circ$-algebras. As in the proof of Lemma 3.3.2 this can be done in such a way that the action of $\mathbb{Z}_p[\gamma'_0]$ on the Lie algebra of $\mathfrak{G}'_{0/R}$ is through $\Phi'_0$. The image of the deformation $(\mathfrak{G}'_0, \rho'_0)$ under

$$\mathfrak{Y}'_0(W_s) \to \mathfrak{Y}'_0(R) \xrightarrow{w} \mathfrak{Y}^-(R)$$

then has the same geometric CM-order and reflex type as $(\mathfrak{G}, \rho)$, and so lies in the same $\mathcal{O}_E^\times$-orbit by Proposition 1.2.10. $\square$

The isomorphism of functors $w : \mathfrak{Y}' \to \mathfrak{Y}$ induces an isomorphism $w : R_{\mathfrak{Y}'} \to R_{\mathfrak{Y}}$ in such a way that the composition

$$\mathrm{Hom}_{\mathbf{ProArt}}(R_{\mathfrak{Y}'}, -) \cong \mathfrak{Y}'(-) \xrightarrow{w} \mathfrak{Y}(-) \cong \mathrm{Hom}_{\mathbf{ProArt}}(R_{\mathfrak{Y}}, -)$$

has the form $f \mapsto f \circ w^{-1}$. Let

$$q : R_{\mathfrak{Y}} \to R_{\mathfrak{Y}_0} \qquad q' : R_{\mathfrak{Y}'} \to R_{\mathfrak{Y}'_0}$$

be the natural surjections. As in §3.2, for each $0 \le s \le c_0$ let $\mathfrak{p}_{0,s}$ be the unique minimal prime of $R_{\mathfrak{Y}_0}$ for which $R_{\mathfrak{Y}_0}/\mathfrak{p}_{0,s} \cong W_s$ and let $\mathfrak{p}_s = q^{-1}(\mathfrak{p}_{0,s})$ be the corresponding prime of $R_{\mathfrak{Y}}$. Similarly let $\mathfrak{p}'_{0,s}$ be the minimal prime of $R_{\mathfrak{Y}'_0}$ corresponding to a level $s$ quasi-canonical lift of the action $j'_0 : \mathcal{O}_{E_0} \to \mathrm{End}(\mathfrak{g}_0)$, so that $R_{\mathfrak{Y}'_0}/\mathfrak{p}'_{0,s} \cong W'_s$ with $W'_s$ constructed exactly as in §3.1 but with $E_0$ and $j_0$ replaced everywhere by $E'_0$ and $j'_0$. Let $\mathfrak{p}'_s = q'^{-1}(\mathfrak{p}'_{0,s})$ be the corresponding prime of $R_{\mathfrak{Y}'}$. For each $0 \le s \le c_0$ we now define two distinguished components of $\mathfrak{Y}$,

$$\mathfrak{C}(s) = \mathrm{Spf}(R_{\mathfrak{Y}}/\mathfrak{p}_s) = \mathrm{Spf}(R_{\mathfrak{Y}_0}/\mathfrak{p}_{0,s})$$

and

$$\mathfrak{C}'(s) = \mathrm{Spf}(R_{\mathfrak{Y}}/w(\mathfrak{p}'_s)) \cong \mathrm{Spf}(R_{\mathfrak{Y}'}/\mathfrak{p}'_{0,s}) = \mathrm{Spf}(R_{\mathfrak{Y}'_0}/\mathfrak{p}'_{0,s}),$$

and two subgroups of $\mathcal{O}_E^\times$

$$H_s = \mathcal{O}_{E_0}^\times \cdot (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times \qquad H'_s = \mathcal{O}_{E'_0}^\times \cdot (\mathbb{Z}_{p^2} + p^s\mathcal{O}_E)^\times.$$

*Remark* 3.3.5. Exactly as in Remark 3.2.4, $\mathfrak{C}(0), \ldots, \mathfrak{C}(c_0)$ is a complete list of the improper components of $\mathfrak{Y}$.

We will say that a horizontal component $\mathfrak{C} = \mathrm{Spf}(R_{\mathfrak{Y}}/\mathfrak{p})$ of $\mathfrak{Y}$ is *standard* if the deformation corresponding to the quotient map $R_{\mathfrak{Y}} \to R_{\mathfrak{Y}}/\mathfrak{p}$ has standard reflex type, and say that $\mathfrak{C}$ is *nonstandard* otherwise.

**Proposition 3.3.6.** *Every standard horizontal component of $\mathfrak{Y}$ can be written uniquely in the form*

$$\mathfrak{C}(s,\xi) \stackrel{\mathrm{def}}{=} \xi * \mathfrak{C}(s)$$

*for $0 \le s \le c_0$ and $\xi \in \mathcal{O}_E^\times/H_s$, and every nonstandard horizontal component of $\mathfrak{Y}$ can be written uniquely in the form*

$$\mathfrak{C}'(s,\xi) \stackrel{\mathrm{def}}{=} \xi * \mathfrak{C}'(s)$$

*for $0 \le s \le c_0$ and $\xi \in \mathcal{O}_E^\times/H'_s$. To be clear: in both instances the action of $\xi*$ on $\mathfrak{Y}$ is that of §1.1 defined using the action $j : \mathcal{O}_E^\times \to \mathrm{End}(\mathfrak{g})$.*

*Proof.* The proof is the same as the proof of Proposition 3.2.5, almost verbatim. In place of Corollary 3.2.3 one uses Lemma 3.3.2 in the standard case and Lemma 3.3.4 in the nonstandard case. □

**Proposition 3.3.7.** *Suppose $0 \le t < s \le c_0$, $\xi \in H_t/H_s$, and $\xi \notin H_{t+1}/H_s$. The horizontal component $\mathfrak{C}(s,\xi) \to \mathfrak{Y}$ defined in Proposition 3.3.6 satisifes*

$$I_{\mathfrak{M}}(\mathfrak{C}(s,\xi), \mathfrak{M}_0) = 1 + p^t + \frac{(p+1)(p^t-1)}{p-1}.$$

*Proof.* Let $(\mathfrak{G}_0, \rho_0) \in \mathfrak{Y}_0(W_s)$ be a quasi-canonical lift of level $s$. For any $k \geq 0$ set $R_k = W_s/\mathfrak{m}^{k+1}$ and abbreviate

$$b(k) = p^k + \frac{(p+1)(p^k-1)}{p-1}.$$

Keating [12, Theorem 5.2] has computed the endomorphism ring of $\mathfrak{G}_{0/R_k}$, and found that the largest order of $\mathcal{O}_{E_0}$ for which the action $j_0 : \mathcal{O}_{E_0} \to \mathrm{End}(\mathfrak{g}_0)$ lifts to the deformation $(\mathfrak{G}_0, \rho_0)_{/R_k}$ is

$$\begin{array}{rll} \mathcal{O}_{E_0} & \text{if} & 0 \leq k < b(0)+1 \\ \mathbb{Z}_p + p\mathcal{O}_{E_0} & \text{if} & b(0)+1 \leq k < b(1)+1 \\ \mathbb{Z}_p + p^2\mathcal{O}_{E_0} & \text{if} & b(1)+1 \leq k < b(2)+1 \\ & \vdots & \\ \mathbb{Z}_p + p^{s-1}\mathcal{O}_{E_0} & \text{if} & b(s-2)+1 \leq k < b(s-1)+1 \\ \mathbb{Z}_p + p^{s}\mathcal{O}_{E_0} & \text{if} & b(s-1)+1 \leq k. \end{array}$$

Using these formulas, the proof is a direct imitation of that of Proposition 3.2.6. □

**Proposition 3.3.8.** *Suppose* $0 \leq s \leq c_0$ *and* $\xi \in \mathcal{O}_E^\times/H_s$. *If* $\mathrm{ord}_p(\mathrm{disc}(E_0)) = 1$ *then the horizontal component* $\mathfrak{C}'(s,\xi) \to \mathfrak{Y}$ *defined in Proposition 3.3.6 satisifes*

$$I_{\mathfrak{M}}(\mathfrak{C}'(s,\xi), \mathfrak{M}_0) = 1.$$

*Note that* $\mathrm{ord}_p(\mathrm{disc}(E_0)) = 1$ *is equivalent to* $p > 2$.

*Proof.* Fix an isomorphism $\mathfrak{C}'(s,\xi) \cong \mathrm{Spf}(W_s')$, let $\mathfrak{m}$ be the maximal ideal of $W_s'$, and set $R = W_s'/\mathfrak{m}^2$ so that $R$ is isomorphic to the ring of dual numbers $\mathbb{F}[\epsilon]$ of $\mathbb{F}$. The resulting closed immersion of formal schemes $\mathrm{Spf}(W_s') \to \mathfrak{Y}$ is given by the composition

$$R_{\mathfrak{Y}} \xrightarrow{\xi^{-1}} R_{\mathfrak{Y}} \xrightarrow{w^{-1}} R_{\mathfrak{Y}'} \xrightarrow{q'} R_{\mathfrak{Y}_0'} \xrightarrow{f_s} W_s' \tag{28}$$

where the final arrow $f_s$ defines a level $s$ quasi-canonical lift $(\mathfrak{G}_0', \rho_0') \in \mathfrak{Y}_0(W_s')$ of $\mathfrak{g}_0$ with respect to the embedding $j_0' : \mathcal{O}_{E_0'} \to \mathrm{End}(\mathfrak{g}_0)$. Setting $(\mathfrak{G}', \rho') = (\mathfrak{G}_0', \rho_0') \otimes \mathbb{Z}_{p^2}$, the composition (28) corresponds to the deformation

$$(\mathfrak{G}'', \rho'') = (\mathfrak{G}', \rho' \circ w^{-1} \circ \xi^{-1}) \in \mathfrak{Y}(W_s').$$

It follows from the formulas of Keating cited in the proof of Proposition 3.3.7 that the action of $\mathbb{Z}_{p^2}[\tau]$ on the deformation $\mathfrak{G}'_{/R}$ extends (uniquely) to an action of $\mathcal{O}_E$, and hence the same is true of the deformation $\mathfrak{G}''_{/R}$.We will show first that the resulting actions of $\mathcal{O}_{E_0}$ on the two summands

$$\mathrm{Lie}(\mathfrak{G}''_{/R}) \cong \Lambda_1 \oplus \Lambda_2$$

are through distinct homomorphisms $\mathcal{O}_{E_0} \to R$.

Consider the diagram

$$\begin{array}{ccc} R_{\mathfrak{Y}_0'} & \xrightarrow{f_s} & W_s' \\ \downarrow & & \downarrow \\ W_0' & \overset{?}{\dashrightarrow} & R \end{array}$$

in which the vertical arrow on the left corresponds to a canonical lift $(\mathfrak{G}_0^\dagger, \rho_0^\dagger) \in \mathfrak{Y}_0'(W_0')$ of $\mathfrak{g}_0$ with respect to the action $j_0' : \mathcal{O}_{E_0'} \to \mathrm{End}(\mathfrak{g}_0)$. Choose an isomorphism $R_{\mathfrak{M}_0} \cong \mathbb{Z}_p^\circ[[x]]$ and let $\alpha$ be the image of $x$ under

$$R_{\mathfrak{M}_0} \to R_{\mathfrak{Y}_0'} \xrightarrow{f_s} W_s' \to R.$$

There is an isomorphism $W_0' \cong \mathbb{Z}_p^\circ[[x]]/(\varphi_0')$ with $\varphi_0'$ Eisenstein of degree two, and as $\varphi_0'(\alpha) = 0$ we find that there is a surjection $W_0' \to R$ making the above diagram commute. In other words, if we set $(\mathfrak{G}^\dagger, \rho^\dagger) = (\mathfrak{G}_0^\dagger, \rho_0^\dagger) \otimes \mathbb{Z}_{p^2}$ then there is an isomorphism of deformations in $\mathfrak{Y}(R)$

$$(\mathfrak{G}'', \rho'')_{/R} \cong (\mathfrak{G}^\dagger, \rho^\dagger \circ w^{-1} \circ \xi^{-1})_{/R}$$

and so $(\mathfrak{G}'', \rho'')_{/R}$ admits a lift

$$(\mathfrak{G}^\dagger, \rho^\dagger \circ w^{-1} \circ \xi^{-1}) \in \mathfrak{Y}^-(W_0')$$

to $W_0'$ of nonstandard reflex type with the property that the action $j : \mathcal{O}_E \to \mathrm{End}(\mathfrak{g})$ also lifts. The action of $\mathcal{O}_{E_0}$ on each $\Lambda_i$ is therefore through some $\mathcal{O}_{E_0} \xrightarrow{\phi_i} W_0' \to R$, and $\phi_1 \neq \phi_2$. The hypothesis that $\mathrm{ord}_p(\mathrm{disc}(E_0)) = 1$ implies that the nontrivial Galois automorphism of $W_0'/\mathbb{Z}_p^\circ$ (which interchanges $\phi_1$ and $\phi_2$) remains nontrivial modulo the square of the maximal ideal of $W_0'$, and so the actions of $\mathcal{O}_{E_0}$ on $\Lambda_1$ and $\Lambda_2$ are through distinct homomorphisms $\mathcal{O}_{E_0} \to R$.

Is it possible that $(\mathfrak{G}, \rho)_{/R}$ lies in the image of $\mathfrak{M}_0(R) \to \mathfrak{M}(R)$? If so, then $(\mathfrak{G}, \rho)_{/R} \cong (\mathfrak{G}_0^\sim, \rho_0^\sim) \otimes \mathbb{Z}_{p^2}$ for some $(\mathfrak{G}_0^\sim, \rho_0^\sim) \in \mathfrak{Y}_0(R)$ corresponding to a surjective $\mathbb{Z}_p^\circ$-algebra map $R_{\mathfrak{Y}_0} \to R$. As above, the fact that $W_0/\mathbb{Z}_p^\circ$ is ramified implies that this map can be factored as $R_{\mathfrak{Y}_0} \to W_0 \to R$ where the first arrow corresponds to a canonical lift in $\mathfrak{Y}_0(W_0)$. This implies that $(\mathfrak{G}'', \rho'')_{/R}$ admits a lift to $\mathfrak{Y}^+(W_0)$ with the property that the action $j : \mathcal{O}_E \to \mathrm{End}(\mathfrak{g})$ also lifts. We deduce that the action of $\mathcal{O}_{E_0}$ on $\mathrm{Lie}(\mathfrak{G}''_{/R})$ is through a *single* homomorphism $\mathcal{O}_{E_0} \to R$. This contradicts what was said in the previous paragraph. Thus the reduction of $(\mathfrak{G}'', \rho'')$ to $R$ is not in the image of $\mathfrak{M}_0(R) \to \mathfrak{M}(R)$, and so $I_{\mathfrak{M}}(\mathfrak{C}'(s,\xi), \mathfrak{M}_0) = 1$. ☐

**Corollary 3.3.9.** *If $p \neq 2$ then*

$$\sum_{\mathfrak{C}} I_{\mathfrak{M}}(\mathfrak{C}, \mathfrak{M}_0) = \frac{-4p^{c_0+1} + 2p + 2}{(p-1)^2} - \frac{(2c_0+1)p^{c_0+1} + 2c_0 + 1}{p-1}$$

*where the sum is over all proper horizontal components $\mathfrak{C} \to \mathfrak{Y}$.*

*Proof.* If $0 \le t < s \le c_0$ and then combining $|H_t/H_s| = p^{s-t}$ with Proposition 3.3.7 shows that

$$\sum_{\substack{\xi \in \mathcal{O}_E^\times/H_s \\ \xi \neq 1}} I_{\mathfrak{M}}(\mathfrak{C}(s,\xi), \mathfrak{M}_0) = 2sp^s - \frac{2p^s - 2}{p-1}$$

while Proposition 3.3.8 shows that

$$\sum_{\xi \in \mathcal{O}_E^\times/H_s} I_{\mathfrak{M}}(\mathfrak{C}'(s,\xi), \mathfrak{M}_0) = p^s.$$

Summing over $0 \le s \le c_0$ and using Proposition 3.3.6 yields the desired result. ☐

## Appendix A

A.1. **A ramified variant.** In this appendix we consider a slightly modified version of the deformation problem of the text. We let $\mathfrak{g}_0$, $E_0$,

$$\mathbb{Z}_p[\gamma_0] = \mathbb{Z}_p + p^{c_0}\mathcal{O}_{E_0},$$

$j_0 : \mathcal{O}_{E_0} \to \mathrm{End}(\mathfrak{g}_0)$, and $\mathfrak{Y}_0 \to \mathfrak{M}_0$ be as in §1.1, but now choose a ramified quadratic extension $F/\mathbb{Q}_p$ whose ring of integers $\mathcal{O}_F$ will assume the role played earlier by $\mathbb{Z}_{p^2}$. Thus we define a $p$-Barsotti-Tate group $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathcal{O}_F$ over $\mathbb{F}$ of height 4 and dimension 2 and let $\mathfrak{M}$ be the formal $\mathbb{Z}_p^\circ$-scheme classifying deformations of $\mathfrak{g}$, with its $\mathcal{O}_F$ action, to objects of **Art**. Set $E = E_0 \otimes_{\mathbb{Q}_p} F$, let $\gamma = \gamma_0 \otimes 1 \in E$, and let $\mathfrak{Y}$ be the closed formal subscheme of $\mathfrak{M}$ classifying those deformations for which the action of $\gamma$ lifts. We again have the cartesian diagram of closed immersions (2) in which the horizontal arrows are now defined by the functor $\otimes\mathcal{O}_F$.

Many of the methods used in the main body of the article can be applied to study this new formal scheme $\mathfrak{Y}$. Virtually everything said in §1 holds simply by replacing $\mathbb{Z}_{p^2}$ everywhere by $\mathcal{O}_F$ (except that the action of $\mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ on $\mathfrak{g}$ deduced from $j_0$ may not extend to all of $\mathcal{O}_E$.) The methods used in §2 to study vertical components should apply in this new setting (although we have not checked this in any detail). The more difficult problem seems to be extending the methods of §3 to study the horizontal components of $\mathfrak{Y}$. The issue is that there may be deformations of $\mathfrak{g}$ with its $\mathcal{O}_F[\gamma]$-action to characteristic 0 whose full endomorphism ring is an $\mathcal{O}_F$-order in $\mathcal{O}_E$ which is not of the form $\mathcal{O} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ for any $\mathbb{Z}_p$-order $\mathcal{O} \subset \mathcal{O}_{E_0}$. One should not expect the component of $\mathfrak{Y}$ containing such a deformation to come from $\mathfrak{Y}_0$ in the way described in Propositions 3.2.5 and 3.3.6.

We will content ourselves for the moment with the following simple case, which is needed for the global intersection theory of [10].

**Proposition A.1.** *If $E_0/\mathbb{Q}_p$ is unramified and $c_0 = 0$ then each of $\mathfrak{Y}_0$ and $\mathfrak{Y}$ are isomorphic to* $\mathrm{Spf}(\mathbb{Z}_p^\circ)$*. In particular the closed immersion $\mathfrak{Y}_0 \to \mathfrak{Y}$ is an isomorphism and $\mathfrak{Y}$ is contained in $\mathfrak{M}_0$.*

*Proof.* That $\mathfrak{Y}_0 \cong \mathrm{Spf}(\mathbb{Z}_p^\circ)$ is a special case of Proposition 3.1.1, and so we turn to $\mathfrak{Y}$. Our hypotheses imply that

$$\mathcal{O}_E = \mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_F = \mathcal{O}_F[\gamma].$$

Let $\Psi, \overline{\Psi} : \mathcal{O}_{E_0} \to \mathbb{Z}_p^\circ$ be as in 2.1. For any object $R$ of **Art** we denote again by $\Psi$ and $\overline{\Psi}$ the ring homomorphisms obtain by composition with the canonical map $\mathbb{Z}_p^\circ \to W(R)$. Define a display $\mathbf{D}_R = (P_R, Q_R, F, V^{-1})$ over $R$ as follows. The $W(R)$-module $P_R$ is free on the generators $\{e_1, e_2, f_1, f_2\}$, the submodule $Q_R \subset P_R$ is

$$Q_R = I_R e_1 + I_R e_2 + W(R) f_1 + W(R) f_2,$$

and $F$ and $V^{-1}$ are determined by the stipulation that the displaying matrix [28, (9)] of $\mathbf{D}_R$ with respect to the basis $\{e_1, e_2, f_1, f_2\}$ is

$$\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}.$$

Define an action of $\mathcal{O}_E$ on $\mathbf{D}_R$ as follows. An element $r \in \mathcal{O}_{E_0}$ acts as

$$\begin{pmatrix} \Psi(r) & 0 & & \\ 0 & \Psi(r) & & \\ & & \overline{\Psi}(r) & 0 \\ & & 0 & \overline{\Psi}(r) \end{pmatrix}.$$

Choose a uniformizer $\varpi_F \in \mathcal{O}_F$ with minimal polynomial $x^2 - ax - b$ and let $\varpi_F$ act as

$$\begin{pmatrix} 0 & b & & \\ 1 & a & & \\ & & 0 & b \\ & & 1 & a \end{pmatrix}.$$

These rules define commuting actions of $\mathcal{O}_{E_0}$ and $\mathcal{O}_F$, and so determine an action of $\mathcal{O}_E$.

One easily checks that $\mathbf{D}_{\mathbb{F}}$ is $\mathcal{O}_E$-linearly isomorphic to the display of $\mathfrak{g}$. Indeed, in the notation of §2.1 there is an isomorphism $\mathbf{D}_{\mathbb{F}} \cong \mathbf{d}_0 \otimes \mathcal{O}_F$ defined by

$$e_1 \mapsto e_0 \otimes 1 \qquad e_2 \mapsto e_0 \otimes \varpi_F \qquad f_1 \mapsto f_0 \otimes 1 \qquad f_2 \mapsto f_0 \otimes \varpi_F.$$

Thus each $\mathbf{D}_R$ is a deformation to $R$ of the display of $\mathfrak{g}$ with its $\mathcal{O}_E$-action. Suppose we have a morphism $R' \to R$ in **Art** whose kernel $J$ satisfies $J^2 = 0$. By Zink's deformation theory [28, Theorem 48] the set of all deformations to $R'$ of $\mathbf{D}_R$ with its $\mathcal{O}_E$-action is in bijection with the set of $\mathcal{O}_E$-stable direct summands $T \subset P_{R'}/I_{R'}P_{R'}$ which lift the Hodge filtration

$$Q_R/I_RP_R \subset P_R/I_RP_R.$$

The condition that $T$ be stable under the action of $\mathcal{O}_{E_0} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^\circ \cong \mathbb{Z}_p^\circ \times \mathbb{Z}_p^\circ$ implies that $T = T(\Psi) \oplus T(\overline{\Psi})$ with $T(\Psi) \subset R'e_1 \oplus R'e_2$ and $T(\overline{\Psi}) \subset R'f_1 \oplus R'f_2$. As $\mathcal{O}_{E_0}$ acts on $Q_R/I_RP_R$ through $\overline{\Psi}$ we must have $T(\Psi) = 0$. Thus

$$T = T(\overline{\Psi}) \subset R'f_1 \oplus R'f_2 = Q_{R'}/I_{R'}Q_{R'}$$

from which $T = Q_{R'}/I_{R'}Q_{R'}$ follows. We deduce that $\mathbf{D}_{R'}$ is the unique deformation of $\mathbf{D}_R$ with its $\mathcal{O}_E$-action. By induction on the length of the local ring $R$ we conclude that $\mathbf{D}_R$ is the unique deformation to $R$ of $\mathbf{D}_{\mathbb{F}}$ with its $\mathcal{O}_E$-action, and it follows that $\mathfrak{Y} \cong \mathrm{Spf}(\mathbb{Z}_p^\circ)$. $\square$

## References

[1] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, 82:5–96 (1996), 1995.
[2] P. Deligne and G. Pappas. Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compositio Math.*, 90(1):59–79, 1994.
[3] M. Demazure. *Lectures on p-divisible groups*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 302.
[4] T. Dokchitser. *Deformations of p-divisible groups and p-descent on elliptic curves*. PhD thesis, University of Utrecht, 2000. http://igitur-archive.library.uu.nl/dissertations/1922434/inhoud.htm.
[5] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, and A. Vistoli. *Fundamental algebraic geometry: Grothendieck's FGA explained*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.
[6] E. Z. Goren. *Lectures on Hilbert modular varieties and modular forms*, volume 14 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2002.
[7] E. Z. Goren and F. Oort. Stratifications of Hilbert modular varieties. *J. Algebraic Geom.*, 9(1):111–154, 2000.
[8] B. Gross. On canonical and quasicanonical liftings. *Invent. Math.*, 84(2):321–326, 1986.
[9] M. Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.
[10] B. Howard. Intersection theory on Shimura surfaces II. *Preprint*.

[11] B. Howard. Intersection theory on Shimura surfaces. *Compositio Math.*, 145(2):423–475, 2009.
[12] K. Keating. Lifting endomorphisms of formal $A$-modules. *Compositio Math.*, 67(2):211–239, 1988.
[13] S. Kudla. Central derivatives of Eisenstein series and height pairings. *Ann. of Math. (2)*, 146(3):545–646, 1997.
[14] S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin L-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004.
[15] S. Kudla and M. Rapoport. Special cycles on unitary Shimura varieties I. Unramified local theory. 2008.
[16] S. Kudla, M. Rapoport, and T. Yang. *Modular forms and special cycles on Shimura curves*, volume 161 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2006.
[17] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.
[18] W. Messing. Travaux de Zink. *Astérisque*, (311):Exp. No. 964, ix, 341–364, 2007. Séminaire Bourbaki. Vol. 2005/2006.
[19] D. Mumford. *Geometric invariant theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34. Springer-Verlag, Berlin, 1965.
[20] M. Rapoport. Compactifications de l'espace de modules de Hilbert-Blumenthal. *Compositio Math.*, 36(3):255–335, 1978.
[21] M. Rapoport and Th. Zink. *Period spaces for p-divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.
[22] E. Viehmann and K. Ziegler. Formal moduli of formal $\mathcal{O}_{\mathrm{K}}$-modules. *Astérisque*, 312:57–66, 2007.
[23] I. Vollaard. On the Hilbert-Blumenthal moduli problem. *J. Inst. Math. Jussieu*, 4(4):653–683, 2005.
[24] I. Vollaard. Endomorphisms of quasi-canonical lifts. *Astérisque*, 312:105–112, 2007.
[25] W. Waterhouse. On $p$-divisible groups over complete valuation rings. *Ann. of Math. (2)*, 95:55–65, 1972.
[26] S. Wewers. Canonical and quasi-canonical liftings. *Astérisque*, 312:67–86, 2007.
[27] T. Zink. Windows for displays of $p$-divisible groups. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.*, pages 491–518. Birkhäuser, Basel, 2001.
[28] T. Zink. The display of a formal $p$-divisible group. *Astérisque*, 278:127–248, 2002.

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