Decentralized Matching Markets with Endogeneous Salaries*

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Abstract

In a Shapley-Shubik assignment problem with a supermodular output matrix, we consider games in which each firm chooses an applicant making a take-it-or-leave-it salary offer to her, and a match is made only when the offer is accepted by her. We consider both one-shot and multi-stage games. In a one-shot game, in a mixed strategy equilibrium, applicants' expected utilities are lower than the minimum competitive salaries, while firms' expected payoffs are exactly the same as the ones under the minimum competitive equilibrium. In a multi-stage model, we show that there is a stationary Markov perfect equilibrium that achieves the minimum competitive salary vector if each offer is valid until applicant accepts or rejects it. Actually, this is the best salary rquilibrium, and there are many other stationary Markov perfect equilibria with lower salaries including a zero-salary equilibrium. As a by-product of the multi-stage model, we analyze an unraveling model by Roth and Xing (1994). We show that even if the highest salary stationary Markov perfect equilibrium is played in each period, a full unraveling may occur under an additional assumption that is satisfied under the standard nxn multiplicative output matrix. In this unraveling equilibrium, all firms and all applicants are (weakly) better off and worse off by unraveling, respectively.

STILL VERY PRELIMINARY

1 Introduction

As is described in Roth (1984), the centralized matching procedure in the US medical resident market - National Residency Matching Program (NRMP) - has been a tremendous success.[1] NRMP uses the deferred-acceptance matching algorithm in Gale and Shapley (1961) with a salary vector chosen by hospitals prior to the matching procedure in order to assign seniors in medical schools

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[1] See also Roth (1991) and Roth and Peranson (1999).

to residency programs in participating hospitals. The introduction of NRMP put down early contract craze and congestion in the last minute under the decentralized system, and the participation rate to NRMP from the two sides have been very high.

In 2002, however, a lawsuit against teaching hospitals and NRMP was filed complaining that they violated the federal antitrust law (such as restraint of competition), although it has been dismissed in 2004 following the President signing a law exempting NRMP from antitrust lawsuits.[2] This lawsuit could have had a significant impact: one possible consequence was abandonment of the NRMP and other medical matching programs.

Recently, Bulow and Levin (forthcoming) set up an interesting matching model that can compare the centralized matching mechanism that has characteristics of NRMP with a decentralized market. Bulow and Levin (forthcoming) employ a simplified version of the assignment model in Shapley and Shubik (1972), and considered a two-stage game. In the first stage, hospitals simultaneously decide salaries, and in the second stage the Gale-Shapley deferred-acceptance matching algorithm takes place with preferences of hospitals and residents that are generated by the price vector determined in the first stage. So, their game mimics the matching program of the NRMP. Bulow and Levin (forthcoming) characterized a mixed strategy equilibrium of the game, and compared the expected equilibrium salary of each resident in the game with her *minimal competitive salary.* The minimal competitive salary (vector) is the lowest market equilibrium salary vector under which the surplus-maximizing assignment of hospitals and residents is stable (Shapley and Shubik, 1972). The main finding of their paper is that under the centralized system, salaries and workers' payoffs are suppressed than the ones under the decentralized system.

Although the result of Bulow and Levin (forthcoming) appears to suggest the benefits of decentralized market,[3] the notion of the minimal competitive salary does not really fit with an equilibrium outcome of a decentralized resident-hospital matching market. Since each resident is a heterogenous commodity, the minimal competitive salary can be attained as a result of a Vickrey auction in the *centralized multi-object auction* market. However, this mechanism also requires the centralized auction market. It is not clear what kind of decentralized salary vector emerges under decentralized market with usual bilateral job offers in the resident-hospital market have been often bilateral.

In this paper, we analyze equilibria of noncooperative games that describe decentralized markets by using the Shapley-Shubik assignment model. Following Bulow and Levin (forthcoming), we assume that the output matrix is supermodular (slightly more general than the one in Bulow and Levin), and that we consider games in which each firm chooses an applicant and makes a take-it-or-leave-it offer to her, and a match is made only when the offer is accepted by the applicant. We consider both one-shot and multi-stage games. Although these

[2] The Pension Funding Equity Act of 2004. This contained a rider regarding the NRMP.

[3] Bulow and Levin (forthcoming) carefully caution that their result does not directly indicate that NRMP suppresses the wages of medical residents, since NRMP can use enormous amount of information nationally while decentralized system tend to match agents locally.

games are way too simpler than the real-world market institutions, this is the first systematic attempt to analyze equilibrium salary vectors of the resident-hospital decentralized matching problem in an noncooperative manner.

The results of the paper are as follows. In a one-shot simultaneous move game, there is only a mixed strategy equilibrium, and the highest possible realization of a salary vector is the minimal competitive equilibrium salary vector: i.e., applicants' expected utilities are lower than the minimal competitive salaries. On the other hand, firms get exactly the same expected payoffs as under the minimal competitive equilibrium. In a multi-stage model under an open offer assumption,[4] we show that there is a stationary Markov perfect equilibrium that achieves the minimal competitive salary vector under some assumptions. However, this equilibrium actually attains the highest salary vector among all stationary Markov perfect equilibria in pure strategies.[5] These results seem to indicate that the minimal competitive salary vector is the best case scenario in the decentralized market for student applicants. Thus, we may be able to say that the reference salary vector adopted by Bulow and Levin (forthcoming) for decentralized market outcome might be too optimistic.

As a by-product of the multi-stage model, we analyze an unraveling model by Roth and Xing (1994) with a (semi-)open offer assumption. Although there are many recent interesting papers on unraveling such as Niederle and Roth (2003) and Damiano, Li and Suen (forthcoming), no paper but Roth and Xing (1994) has endogeneized salaries. In contrast Roth and Xing (1994) which showed that unraveling occurs in every subgame perfect equilibrium, we show that there is a stationary Markov perfect equilibrium in which the minimal competitive salary vector is attained in each period subgame (if the period has been reached), and a full unraveling occurs over periods under an additional assumption that is satisfied under a standard $n \times n$ multiplicative output matrix. In this unraveling equilibrium, all firms and all applicants are weakly better off and worse off by unraveling, respectively.

Our games are definitely too simplistic in comparison with the real world market institutions. Roth (1984) and Roth and Sotomayor (1990) describe how the NRMP was adopted in the medical resident market in the history. The medical-residency was first introduced around the turn of the century as an optional form of postgraduate medical education. For students, residencies offered concentrated exposure to clinical medicine, and for hospitals they offered a supply of relatively cheap labor. Due to shortage of interns, there was considerable competition among hospitals for residents. Because of this, hospitals attempted to finalize binding agreements with residents a little earlier than competitors. As a result, the date by which most residents had been finalized crept forward (*unraveling*), and in 1944 the date of appointment reached the beginning of the

[4] An offer is called **open** if the offer is valid until it has been accepted or rejected by the applicant who received it. In contrast, an offer is called **exploding** if an applicant who receives it needs to either accept or reject it immediately, and she will not be able to receive another offer in future if she rejects it.

[5] There are many other stationary Markov perfect equilibria with lower salaries including a zero-salary vector.

junior year. In order to fix this coordination problem, the Association of American Medical Colleges (AAMC) prohibited releasing students' transcripts and letters of reference prior to a certain dates. Although this practice proved to be effective in resolving unravelling, a new problem arose by this. The problem was that a student who got offers from a hospital waited as long as possible before accepting the offered position in the hope of getting an offer from his/her more preferable hospital. In order to speed up the matching process, deadlines of offers became shorter and shorter (ten days in 1945, eight days in 1946, and immediate (*exploding offers*) from 1949 to 51). Finally, in 1952, NRMP was introduced to the market in order to avoid such congestion in the market (so many exploding offers and students' decisions go back and forth in a very short period). Thus, in order to evaluate the performance of NRMP in comparison with decentralized market, ideally we should use equilibrium salary vectors of the games that mimic the real-world market institutions as reference salary vectors of decentralized matching markets. However, exploding offers are very difficult to analyze especially with endogenized salaries.[6] We plan to extend our analysis by adopting more realistic assumptions in our future research.

The rest of the paper is organized as follows. In Section 2, the Shapley-Shubik model with a spermodular and increasing output matrix is presented. In Section 3, a simultaneous move game is analyzed. In Section 4, a multistage game with open offers is analyzed. In Section 5, a multi-stage game is presented and the possibility of a full unravelling equilibrium is investigated. Section 6 concludes.

2 The Model

2.1 The Shapley-Shubik Assignment Problem

There are two disjoint finite sets of agents: applicants and firms are denoted by $A = \{a_1, ..., a_m\}$ and $F = \{f_1, ..., f_n\}$, respectively. We assume that each firm has exactly one position each.

The firm-applicant matching problem is described as an assignment model in Shapley and Shubik (1972). Each firm hires at most one applicant and output that each pair of firms and workers can produce is described in the following $n \times m$ **output matrix**:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nm} \end{pmatrix},$$

where Y_{ij} denotes the amount of output a firm-worker pair (f_i, w_j) can produce together. We assume that every argument of the matrix is positive. An

[6] For exploding offers, see Roth and Xing (1994) and Niederle and Roth (2003) for example.

assignment matrix is an $m \times n$ matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix},$$

where (i) for each applicant $a_j \in A$ and each firm $i \in F$, we have $x_{ji} \in \{0,1\}$, and (ii) for each applicant $a_j \in A$, $\sum_{i=1}^{n} x_{ji} = 1$ and for each firm $f_i \in F$, $\sum_{j=1}^{m} x_{ji} = 1$. This matrix describes matching between applicants and firms. An **optimal assignment matrix** X^* is an assignment matrix that maximizes the total production of this industry:

$$X^*Y = \sum_{i=1}^{n}\sum_{j=1}^{m} x^*_{ji} Y_{ij} = \max_X \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ji} Y_{ij}.$$

An **outcome** of the assignment problem is a list of assignment and profit and salary vectors $(X, \boldsymbol{\pi}, \mathbf{s})$ such that $\boldsymbol{\pi} = (\pi_1, ..., \pi_m) \in \mathbb{R}^m_+$, $\mathbf{s} = (s_1, ..., s_n) \in \mathbb{R}^n_+$, and $x_{ij} = 1$ iff $\pi_i + s_j = Y_{ij}$ and $s_j = 0$ for $a_j \in A$ with $\sum_{j=1}^{n} x_{ij} = 0$ (zero salaries for applicants who are not assigned to any firm). An outcome of an assignment problem is **stable** if for any $f_i \in F$ and for any $a_j \in A$ $\pi_i + s_j \geq Y_{ij}$. If an outcome $(X, \boldsymbol{\pi}, \mathbf{s})$ is stable then $\mathbf{s}$ is said a **competitive salary vector**, and if $\mathbf{s}^*$ is said the **minimal competitive salary vector** if $\mathbf{s}^*$ is competitive salary vector, and $\mathbf{s}^* \leq \mathbf{s}'$ holds for any competitive salary vector $\mathbf{s}'$. The minimal competitive salary vector can be found by a multi-object ascending price aution algorithm in Demange, Gale and Sotomayor (1986). Strict supermodularity and strict increasingness give us an explicit formula for the minimal competitive salary.

In this paper, we additionally assume that the output matrix is **strictly supermodular** if for any $f_i \in F$ and for any $a_j \in A$

$$Y_{ij} - Y_{ij+1} > Y_{i+1j} - Y_{i+1j+1},$$

and **strictly increasing** if for any $f_i \in F$ and for any $a_j \in A$

$$Y_{ij} - Y_{ij+1} > 0 \text{ and } Y_{ij} - Y_{i+1j} > 0.$$

We assume that Y is strictly supermodular and strictly increasing throughout the paper.[7] These assumptions guarantee that there is a unique optimal assignment matrix X^* that is **assortative** i.e. $x_{ii} = 1$ for any $i = \{1, ..., m\}$.

Lemma 1. If the output matrix Y is strictly supermodular and strictly increasing then the minimal competitive salary vector is $\mathbf{s}^* = (s^*_1, ..., s^*_m)$ where (i) $s^*_j = \sum_{j'=j}^{n-1}(Y_{j'+1j'} - Y_{j'+1j'+1})$ for any $j \leq n-1$ and $s^*_j = 0$ for $j \geq n$ when

[7]Frequently used multiplicative output matrix and semi-multiplicative output matrix in Bulow and Levin (forthcoming) satisfy both supermodularity and increasingness. However, note that for simplicity, we do not allow indifference (by assuming 'strictness').

$n \leq m$, and (ii) $s_j^* = Y_{mm} + \sum_{j'=j}^{m-1}(Y_{j'+1j'} - Y_{j'+1j'+1})$ for any $j \leq m-1$ and $s_m^* = Y_{mm}$ when $n > m$.

Proof : Shapley and Shubik (1972) showed that if an outcome of an assignment problem is stable then the assignment matrix associated with it is an optimal assignment. Under strict supermodularity and strict increasingness, the unique optimal assignment of the output matrix Y is an assortative matrix X^*. Thus, what is left to show is that the minimal salary vector that supports this assignment is $\mathbf{s}^*$ where (i) $s_j^* = \sum_{j'=j}^{n-1}(Y_{j'+1j'} - Y_{j'+1j'+1})$ for any $j \leq n-1$ and $s_j^* = 0$ for any $j \geq n$ when $m \geq n$, and (ii) $s_j^* = Y_{m+1m} + \sum_{j'=j}^{m-1}(Y_{j'+1j'} - Y_{j'+1j'+1})$ for any $j \leq m-1$ and $s_m^* = Y_{m+1m}$ when $n > m$.

Suppose not. Then there is a competitive salary vector $\mathbf{s}'$ with $s'_j < s_j^*$ for some j. First assume $n \leq m$. Obviously, such j must belong to $\{1, ...n-1\}$, suppose that $s'_{n-1} < s^*_{n-1}$, thus $s'_{n-1} < Y_{nn-1} - Y_{nn}$. Consider a deviation by a pair (f_n, a_{n-1}). Since $s'_n \geq s^*_n = 0$, $\pi'_n \leq Y_{nn}$. Now, $s'_{n-1} + \pi'_n < Y_{nn-1} - Y_{nn} + Y_{nn} = Y_{nn-1}$. This violates the stability, and contradicts with $\mathbf{s}'$ being a competitive salary. Thus $s'_{n-1} \geq s^*_{n-1}$. Suppose that $s'_{n-2} < s^*_{n-2}$, thus $s'_{n-2} < Y_{n-1n-2} - (Y_{n-1n-1} - (Y_{nn-1} - Y_{nn}))$. From the previous step, we know $s'_{n-1} \geq s^*_{n-1}$, and thus $\pi'_{n-1} \leq Y_{n-1n-1} - s^*_{n-1} = Y_{n-1n-1} - (Y_{nn-1} - Y_{nn})$. Thus, we have

$$\begin{aligned} s'_{n-2} + \pi'_{n-1} &< Y_{n-1n-2} - (Y_{n-1n-1} - (Y_{nn-1} - Y_{nn})) + Y_{n-1n-1} - (Y_{nn-1} - Y_{nn}) \\ &= Y_{n-1n-2}. \end{aligned}$$

This violates the stability, and contradicts with $\mathbf{s}'$ being a competitive salary. Thus $s'_{n-2} \geq s^*_{n-2}$. Repeated applications of the same logic conclude that any competitive salary vector $\mathbf{s}'$ satisfies $\mathbf{s}' \geq \mathbf{s}^*$.

Second assume $n > m$. Then such j with $s'_j < s^*_j$ must satisfy $j \leq m$ obviously. Suppose $s'_m < s^*_m = Y_{m+1m}$. Then, a deviation by a pair (f_{m+1}, a_m) can block the allocation, since in the allocation $\pi_{m+1} = 0$ and $s'_m < Y_{m+1m}$. This is a contradiction. Thus $s'_m \geq s^*_m$. Since in the allocation firm f_m gets $\pi_m = Y_{mm} - s'_m \leq Y_{mm} - s^*_m = 0$, we conclude $s'_m = s^*_m$. Now suppose $s'_{m-1} < s^*_{m-1}$. In this case, a deviation by a pair (f_m, a_{m-1}) can improve upon the allocation since $\pi_m = 0$ and $s'_{m-1} < Y_{mm} + (Y_{mm-1} - Y_{mm}) = Y_{mm-1}$. This is a contradiction, and we conclude $s'_{m-1} \geq s^*_{m-1}$. Now suppose that $s'_{m-2} < s^*_{m-2}$, thus $s'_{m-2} < Y_{m-1m-2} - Y_{m-1m-1} + Y_{mm-1}$. From the previous step, we know $s'_{m-1} \geq s^*_{m-1} = Y_{mm-1}$, and thus $\pi'_{m-1} \leq Y_{m-1m-1} - s^*_{m-1} = Y_{m-1m-1} - Y_{mm-1}$. Thus, we have

$$\begin{aligned} s'_{m-2} + \pi'_{m-1} &< Y_{m-1m-2} - Y_{m-1m-1} + Y_{mm-1} + Y_{m-1m-1} - Y_{mm-1} \\ &= Y_{m-1m-2}. \end{aligned}$$

This violates the stability, and contradicts with $\mathbf{s}'$ being a competitive salary. Thus $s'_{m-2} \geq s^*_{m-2}$. Repeated applications of the same logic conclude that any competitive salary vector $\mathbf{s}'$ satisfies $\mathbf{s}' \geq \mathbf{s}^*$. ■

3 One-Shot Game

Consider a one shot game. Each firm f_i decides to which applicant it makes an offer, and how much salary to pay her. We assume that *an offer is a take-it-or-leave-it offer.* All firms make simultaneous offers, and applicants choose the best offer if they receive multiple offers. We assume the following **tie-breaking rule**: if an applicant a_j receives offers from f_i and $f_{i'}$ with the same salaries $(i < i')$, then a_j prefers f_i to $f_{i'}$. If an applicant accepts an offer a match is made. Since each firm has an incentive to reduce a salary as long as a positive salary is paid, there is no pure strategy equilibrium in this game.

Let $G = (G_{ij})_{f_i \in F, a_j \in A}$ be a mixed strategy profile, where $G_{ij}(s) \in [0, 1]$ is the probability that f_i offers a salary $s' \leq s$ to applicant a_j ($G_i = (G_{ij})_{a_j \in A}$ is a distribution function $G_i : \mathbb{R}^n_+ \to [0, 1]$ be such that $G_i(s_1, ..., s_j, ..., s_n) = \sum_{j=1}^{n} G_{ij}(s_j)$). Let $\bar{s}_{ij}$ be the smallest upper bound for the set $\{s_{ij} \in \mathbb{R}_+ : G_{ij}(\bar{s}_{ij}) = G_{ij}(\infty)\}$, and let $\bar{s}_j = \max_i \bar{s}_{ij}$. Given a strategy profile $(G_{ij})_{f_i \in F, a_j \in A}$, let $w_{ij}(s) = \Pi_{i' \neq i} (1 - (G_{i'j}(\bar{s}_j) - G_{i'j}(s)))$. This function denotes f_i's winning probability of a_j by offering s to a_j, since $G_{i'j}(\bar{s}_j) - G_{i'j}(s)$ is the probability that $f_{i'}$ makes a better offer to a_j. Let $u_i(G)$ and $v_j(G)$ be the equilibrium payoff of f_i and a_j, respectively ($i = 1, ..., n$ and $j = 1, ..., m$). The main result of this section is as follows:

Proposition 1. Suppose that Y is strictly supermodular and strictly increasing. In any mixed strategy equilibrium G, it follows that

1. When $n \leq m$ (applicants are abundant), $\bar{s}_j = s^*_j$ holds for all $j = 1, ..., m$, $u_i(G) = Y_{ii} - s^*_i$ for all $i = 1, ..., n$, $v_j(G) < s^*_j$ for all $j = 1, ..., n - 1$, and $v_j(G) = s^*_j = 0$ for all $j = n, ..., m$.
2. When $n > m$ (applicants are scarce), $\bar{s}_j = s^*_j$ holds for all $j = 1, ..., m$, $u_i(G) = Y_{ii} - s^*_i$ for all $i = 1, ..., m$, $v_j(G) < s^*_j$ for all $j = 1, ..., m$.

This proposition has an important implication. As long as we consider a natural simultaneous move game, applicants cannot expect to earn the same salaries under the minimal competitive salary vector in the assignment problem (the very best outcome is the minimal competitive salary, but it happens with a zero measure), while firms earn exactly the same expected profits under the minimal competitive salary vector. We will prove Proposition 1 by a sequence of following claims.

Claim 1. (No spikes) For all a_j, no firm f_i plays (a_j, s) for any $s \in (0, \bar{s}_j]$ with a positive probability when $\bar{s}_j > 0$.

Proof. Suppose that a firm f_i makes an offer of salary $s > 0$ to a_j with a positive probability. Then for all other firms winning probability function $w_{i'j}$

jumps up at s. Thus, no other firm plays (a_j, s') with positive density for $s' = s - \epsilon$ for $\epsilon > 0$ small enough. This gives f_i an incentive to shift the spike at s slightly lower. Thus, in equilibrium, f_i would not play (a_j, s) with a positive probability for any $s \in (0, \bar{s}_j]$.■

Claim 2. (No gap for at least a pair of firms) For all a_j, and all interval $\left(s'_j, s''_j\right) \subset (0, \bar{s}_j)$, there are at least two firms with $G_{ij}(s'_j) < G_{ij}(s''_j)$.

Proof. First note that there is a firm f_i that makes an offer to a_j with salary $\bar{s}_j$. This firm obtains expected payoff of $Y_{ij} - \bar{s}_j$ by this offer since $w_{ij}(\bar{s}_j) = 1$. Thus, there is at least one firm that plays (a_j, s) for some $s \in (s''_j, \bar{s}_j]$. Note that $s''_j < \bar{s}_j$ (by the definition of $\bar{s}_j$). Without loss of generality, we can let $G_{i'j}(\bar{s}_j) - G_{i'j}(s''_j) > 0$ for some $f_{i'}$ (by adjusting s''_j). Focus on this firm. Suppose that there is no firm that offers any of $s_j \in (s'_j, s''_j)$ to a_j. Then, $w_{i'j}(s'_j) = w_{i'j}(s''_j)$, which is a contradiction. Thus, there is at least one such a firm. Moreover, if firm $f_{i'}$ is the only such a firm, then again $w_{i'j}(s'_j) = w_{i'j}(s''_j)$ follows. This is again a contradiction, and we have shown that at least two firms makes salary offers within the interval (s'_j, s''_j) to a_j with positive probabilities.■

Note that

$$\begin{aligned} w_{ij}(s_j) &= \Pi_{i' \neq i} \left(1 - (G_{i'j}(\bar{s}_j) - G_{i'j}(s_j))\right) \\ &= \Pi_{i'=1}^{n} \left(1 - (G_{i'j}(\bar{s}_j) - G_{i'j}(s_j))\right) \times \frac{1}{(1 - (G_{ij}(\bar{s}_j) - G_{ij}(s_j)))} \end{aligned} \tag{1}$$

This implies that $w_{ij}(s_j) > w_{i'j}(s_j)$ if and only if $1 - (G_{ij}(\bar{s}_j) - G_{ij}(s_j)) > 1 - (G_{i'j}(\bar{s}_j) - G_{i'j}(s_j))$. Thus, we have the following:

Claim 3. For all $i, i' = 1, ..., n$ and all $j = 1, ..., m$, $0 < \bar{s}_{ij} < \bar{s}_{i'j} = \bar{s}_j$ implies $i' < i$.

Proof. By definition, we have $w_{ij}(\bar{s}_{ij})\,(Y_{ij} - \bar{s}_{ij}) = u_i(G) \geq Y_{ij} - \bar{s}_j$. Thus.

$$w_{ij}(\bar{s}_{ij}) \geq \frac{Y_{ij} - \bar{s}_j}{Y_{ij} - \bar{s}_{ij}}.$$

Similarly, since $\bar{s}_{i'j} = \bar{s}_j$, we have $w_{i'j}(\bar{s}_{ij})\,(Y_{i'j} - \bar{s}_{ij}) \leq Y_{i'j} - \bar{s}_j = u_{i'}(G)$. Thus,

$$w_{i'j}(\bar{s}_{ij}) \leq \frac{Y_{i'j} - \bar{s}_j}{Y_{i'j} - \bar{s}_{ij}}.$$

Since $G_{i'j}(\bar{s}_{ij}) < G_{i'j}(\bar{s}_j)$ (from $\bar{s}_{i'j} = \bar{s}_j$), (1) implies $w_{ij}(\bar{s}_{ij}) < w_{i'j}(\bar{s}_{ij})$. Thus, we have

$$\frac{Y_{ij} - \bar{s}_j}{Y_{ij} - \bar{s}_{ij}} < \frac{Y_{i'j} - \bar{s}_j}{Y_{i'j} - \bar{s}_{ij}},$$

or $Y_{ij} < Y_{i'j}$. This completes the proof.■

Claim 4. For any $j = 1, ..., n-1$, we have (i) $\bar{s}_j = \bar{s}_{jj} = \bar{s}_{j+1j}$ and (ii) $\bar{s}_{kj} < \bar{s}_j$ for $k \neq j, j+1$.

Proof. By induction. Let $j = 1$. Then, by Claim 1, at least two firms make the highest salary offers to a_1. By Lemma 3, f_1 and f_2 must satisfy $\bar{s}_{11} = \bar{s}_{21} = \bar{s}_1$. Suppose that f_3 also satisfies the same condition: $\bar{s}_{31} = \bar{s}_1$. These imply

$$\begin{aligned} Y_{11} - \bar{s}_1 &\geq Y_{1j} - \bar{s}_j \\ Y_{21} - \bar{s}_1 &\geq Y_{2j} - \bar{s}_j \\ Y_{31} - \bar{s}_1 &\geq Y_{3j} - \bar{s}_j \end{aligned}$$

for all $j = 2, ..., m$. However, by strict supermodularity, $Y_{\hat{\imath}\hat{\jmath}} - Y_{\hat{\imath}\tilde{\jmath}} > Y_{\tilde{\imath}\hat{\jmath}} - Y_{\tilde{\imath}\tilde{\jmath}}$ for all $\hat{\imath} < \tilde{\imath}$ and all $\hat{\jmath} < \tilde{\jmath}$. This implies

$$\begin{aligned} Y_{11} - \bar{s}_1 &> Y_{1j} - \bar{s}_j \\ Y_{21} - \bar{s}_1 &> Y_{2j} - \bar{s}_j \\ Y_{31} - \bar{s}_1 &\geq Y_{3j} - \bar{s}_j \end{aligned}$$

for all $j = 2, ..., m$. By the above inequalities, firms f_1 and f_2 make offers only to a_1. (If f_2 makes an offer to $a_j \neq a_1$, then $\bar{s}_{2j} < \bar{s}_j$. By Claim 3, this implies $\bar{s}_{1j} = \bar{s}_1$, which is a contradiction.) Let $G_{i1}(\underline{s}_{i1}) = 0$ for $i = 1, 2$. Obviously, $\underline{s}_{21} \geq 0$. By Claim 1 and our tie-breaking rule, there is no spike in the distribution of G_{21}. This implies that $w_{11}(\underline{s}_{21}) = 0$. This implies $\underline{s}_{11} > \underline{s}_{21}$, which cannot happen. This is a contradiction. Thus, we conclude that only f_1 and f_2 make offers to a_1 with the highest salary $\bar{s}_1$. Hence, we have

$$\begin{aligned} Y_{11} - \bar{s}_1 &> Y_{1j} - \bar{s}_j \\ Y_{21} - \bar{s}_1 &\geq Y_{2j} - \bar{s}_j \end{aligned}$$

for all $j = 2, ..., m$. By supermodularity, an equality holds in the second inequality only when $j = 2$.

Now, we move on to a_2. As we have seen above, we need $w_{11}(0) > 0$ and $G_{21}(\bar{s}_1) < 1$. This shows that firm f_2 makes some salary offers to a_2 with a positive probability and the maximum offer must be $\bar{s}_2$ by Claim 3 again. By the same argument above, firms f_2 and f_3 make offers to a_2 with $\bar{s}_2$.

By repeating the same argument, we complete the proof of this Claim.■

Now, we can complete the proof of Proposition 1.

Proof of Proposition 1. We start with the case 1 ($n \leq m$). By Claim 4, we know a_n gets an offer only from f_n with a positive probability. This means that $\bar{s}_j = 0 = s^*_j$ for all $j \geq n$. Since firm f_{j+1} is indifferent between making an offer to a_j with salary $\bar{s}_j$ and making an offer to a_{j+1} with a salary $\bar{s}_{j+1}$, and in either case the winning probability is 1 ($w_{j+1j}(\bar{s}_j) = w_{j+1j+1}(\bar{s}_{j+1}) = 1$). Thus, $Y_{j+1j} - \bar{s}_j = Y_{j+1j+1} - \bar{s}_j$ for all $j = 1, ..., n-1$. By Claim 1, we conclude $\bar{s} = s^*$. Thus, applicants' payoffs are lower since firms play mixed strategies, and the best case scenario for each applicant is to get the same salary as the one under the minimal competitive price.

Now case 2 ($n > m$). By Claim 4, we know a_m gets offers only from firms $i = m, ..., n$. We claim that $\bar{s}_m = s^*_m$ holds and f_{m+1} offers to a_m a salary Y_{m+1m} with probability one. Suppose that $0 < \bar{s}_m < s^*_m = Y_{m+1m}$. Since f_{m+1} obtains a positive expected payoff, the equilibrium must be in mixed strategies. Thus we have

$$\begin{aligned} Y_{m+1m} - \bar{s}_m &= w_{m+1m}(s'_m)\,(Y_{m+1m} - s'_m)\,, \\ Y_{mm} - \bar{s}_m &= w_{mm}(s'_m)\,(Y_{mm} - s'_m)\,. \end{aligned}$$

Since $Y_{mm} > Y_{m+1m}$, we have

$$w_{mm}(s'_m) = \frac{Y_{mm} - \bar{s}_m}{Y_{mm} - s'_m} > \frac{Y_{m+1m} - \bar{s}_m}{Y_{m+1m} - s'_m} = w_{m+1m}(s'_m).$$

In order to be an equilibrium, we need $w_{mm}(\underline{s}_{mm}) > 0$. However, no firm f_j ($j = m+1, ..., n$) has incentives to make an offer to a_m with a salary not more than $\underline{s}'_{mm}$. This contradicts with $w_{mm}(\underline{s}_{mm}) > 0$. Thus, $\bar{s}_m < s^*_m$ cannot happen. Now, we assume $\bar{s}_m = s^*_m = Y_{m+1m}$. In this case, firm f_{m+1} can make only zero payoff by making such an offer, the winning probability of f_{m+1} by offering to a_m anything less than $\bar{s}_m = s^*_m$ must be zero (otherwise, f_{m+1} would not make $\bar{s}_m = s^*_m$ offer). Thus, we conclude that f_{m+1} makes an offer to a_m with salary s^*_m with probability one. Then, firm f_m makes an offer to a_m with salary s^*_m with a positive probability. The rest is the same as the case 1. We completed the proof.■

To see how mixed strategy equilibrium looks like, we provide a simple example.

Example 1. Consider a 3×3 **multiplicative output matrix**.[8]

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

By using Proposition 1, we can find a unique mixed strategy equilibrium. The upper bound vector $s^* = (s^*_1, s^*_2, s^*_3) = (3, 1, 0)$. Thus, we have

$$\begin{aligned} w_{11}(s_1)(9 - s_1) &= 6 \\ w_{21}(s_1)(6 - s_1) &= w_{22}(s_2)(4 - s_2) = 3 \\ w_{31}(s_1)(3 - s_1) &= w_{32}(s_2)(2 - s_2) = w_{33}(s_3)(1 - s_3) = 1 \end{aligned}$$

The equilibrium strategy profile G that satisfies the above is as follows: Firm f_1 makes an offer to a_1 only: $s_1 = 0$ with probability $\frac{1}{2}$, and $s_1 \in (0, 3]$ with

[8] Multiplicative output matrix has been used in the literature commonly.

density $\frac{3}{(6-s_1)^2}$. Firm f_2 makes an offer $s_1 \in (0,3]$ to a_1 with density $\frac{6}{(9-s)^2}$, an offer $s_2 = 0$ to a_2 with probability $\frac{1}{3}$ and $s_2 \in (0,1]$ with density $\frac{1}{(2-s)^2}$. Firm f_3 does not make an offer to a_1, makes an offer $s_2 \in (0,1]$ to a_2 with density $\frac{3}{(4-s)^2}$, and an offer $s_3 = 0$ to a_3 with probability $\frac{3}{4}$.

Given these strategies, we can calculate expected utility of applicants. First, by Proposition 1, firms get the same expected payoff as the ones under the minimal competitive equilibrium, thus $u(G) = (6,3,1)$. Second, given strategy profile G, firm f_i's winning probability $w_{ij}(s_j)$ when it makes an offer with salary s_j to a_j is as follows:

$$\begin{aligned} w_{11}(s_1) &= \frac{6}{9-s_1}, \\ w_{21}(s_1) &= \frac{3}{6-s_1}, \\ w_{22}(s_2) &= \frac{3}{4-s_2}, \\ w_{32}(s_2) &= \frac{1}{2-s_2}. \end{aligned}$$

Thus, we have

$$v_1(G) = \int_0^3 s_1 \frac{3}{(6-s_1)^2} \frac{6}{9-s_1} ds_1 + \int_0^3 s_1 \frac{3}{(9-s_1)^2} \frac{3}{6-s_1} ds_1 = 1.0478,$$

$$v_2(G) = \int_0^1 s_2 \frac{1}{(2-s_2)^2} \frac{3}{4-s_2} ds_2 + \int_0^1 s_2 \frac{3}{(4-s_2)^2} \frac{1}{2-s_2} ds_2 = 0.3918$$

$$v_3(G) = 0$$

The resulting expected salary vector is $v(G) = (1.0478, 0.3918, 0)$. By summing up all the payoffs, we obtain

$$\sum_{i=1}^{3} u_i(G) + \sum_{j=1}^{3} v_i(G) = 11.4396.$$

Under the minimal competitive equilibrium salaries, the total surplus is $\sum_{i=1}^{3} Y_{ii} =$ 14. and the efficiency loss is 2.5604 or 18.28%.

It may be interesting to compare our result with the one in Bulow and Levin (forthcoming). Bulow and Levin (forthcoming) analyzed the performance of a centralized matching procedure with a preplay stage of a game of salary determination. They showed that there is only a mixed strategy equilibrium, and the equilibrium payoffs for firms are higher yet the ones of applicants are mostly lower under the centralized matching procedure than under the minimal competitive equilibrium salaries. In this example we see

Firms	Bulow-Levin's Centralized	Min. Competitive Eq.	Our Game
u_1	6.67	6.00	6.00
u_2	3.67	3.00	3.00
u_3	1.00	1.00	1.00

Applicants	Bulow-Levin's Centralized	Min. Competitive Eq.	Our Game
v_1	1.56	3.00	1.05
v_2	0.73	1.00	0.39
v_3	0.02	0.00	0.00

Thus, applicants' expected payoffs in the centralized matching procedure in Bulow and Levin (forthcoming) are mostly worse than the minimal competitive equilibrium outcome, but the applicants' equilibrium payoffs in our one-shot offer-acceptance game are even worse than Bulow and Levin's. The sum of expected payoffs of all players in the centralized matching procedure is 13.65, so the efficiency loss is only 2.5% comparing with the 18.28% loss in our decentralized matching. This is because in the centralized matching procedure, each applicant is matched with some firm at least (although there can be mismatches). In contrast, our decentralized matching leave some applicants unmatched with any hospitals.■

4 Multi-Stage Game

In this section, we consider a multi-stage game with a specific structure. The process of making and accepting offers in the real world is very complex, and we need to simplify it by impose many assumptions. Moreover, we have to make detailed assumptions in order to specify a noncooperative game. Obviously, the equilibrium of a game depends on the set of assumptions we impose. Our strategy of choosing assumptions is (i) to keep our game as simple as possible, and (ii) to guarantee a fully-unraveling contracting to be one of possible subgame perfect equilibria. Later, we check how the result change if those assumptions are modified. We will start describing the game now. We assume that *each offer is an open offer that is valid until the end of the game.*

There are a large finite number of stages $\ell = 1, 2, ..., L$. Each firm f_i decides to which applicant to make an offer, at which stage to make an offer, and how much salary to pay her. We assume that *an offer is a take-it-or-leave-it offer: each firm f_i can make only one offer to a_j.*[9] This assumption says that once f_i is rejected by a_j, *Each firm is allowed to have at most one outstanding offer at a time.* This assumption may be viewed as the one that the cost of hiring multiple applicants for one position is prohibitively high. We assume that *if no offer is made by any firm in a stage, then the game ends at that stage, and unmatched firms and applicants at that time will not be matched in after market activities.* If this assumption is not placed, then the last stage becomes important (in the last stage, a mixed strategy equilibrium is played that has been discussed in the previous section). We also assume that *it costs a small amount to make an offer to an applicant.* This assumption discourages firms to make irrelevant offers for them.

[9] This rule was proposed by Roth and Xing (1994) as an equilibrium refinement. If f_i makes an offer and a_j rejects it, f_i cannot go back to a_j making an offer with a higher salary. That is, there is no room for negotiation in salary.

Each applicant a_j can accept or reject offers that she receives from firms. Once applicant a_j accepts an offer from firm f_i they are matched and are out of the game: their contract is final and cannot be renegotiated. Recall that each applicant just care about salaries offered by firms unless offered salaries are exactly the same. If an applicant a_i receives offers s_j^i and $s_j^{i'}$ with $s_j^i > s_j^{i'}$ from firms f_i and $f_{i'}$, respectively, then she prefers firm i to firm i'. If $s_j^i = s_j^{i'}$ then she prefers f_i to $f_{i'}$ when $i < i'$. We assume that *if an applicant receives two offers, then she reject one of the offers (the less preferable one) immediately.*

We first show a simple result that set upperbounds for applicants' payoffs (salaries). To state the result, we first introduce a concept: A **state** is a list of applicants who have not accepted an offer, firms whose offers have not been accepted, pending offers (each pending offer is a list of a firm, an applicant and a salary), and rejected offers (each rejected offer is a list of a firm and an applicant) among available firms and applicants . A **stationary Markov strategy** is a strategy for a player that maps a state to an action. That is, each firm just cares about information such as who is available for it with what condition (a pending salary), and which firms are its potential competitors. Without stationarity, making an irrelevant offer (the offer which would not be accepted in any case) may bring a totally different equilibrium in a subgame, which can affect the equilibrium outcome. A subgame perfect equilibrium with stationary Markov strategies is called a **stationary Markov perfect equilibrium**.[10]

Proposition 2. In a multi-stage game, in any pure strategy stationary Markov perfect equilibrium, applicant a_j's payoff is not more than s_j^* for all $j = 1, ..., m$.

Proof. First note that making or not making an offer that is immediately rejected would not affect an equilibrium path in the subgame afterwards. Given that making an offer is costly, a firm has no incentive to make such an offer.

Now, suppose that there is a stationary Markov perfect equilibrium in which a_j receives a salary s'_j that is strictly more than s_j^*. Then in some stage, she receives an offer s'_j from f_i which she will eventually accept. Note that when she receives the offer from f_i, no other firm makes her another offer simultaneously (by the above argument, and the assumption that an applicant rejects all but one offer in each stage). This implies that f_i makes an offer s'_j to a_j by fearing a competitor firm $f_{i'}$ that is willing to pay the amount. Otherwise, f_i tries to reduce the salary for a_j. Firm $f_{i'}$ will be matched with an applicant $a_{j'}$ with a salary $s'_{j'}$ in the equilibrium, so $s'_j \simeq Y_{i'j} - Y_{i'j'} + s'_{j'}$. By supermodularity, we have $Y_{i'j} - Y_{i'j'} \leq s_j^* - s_{j'}^*$ if $j < j'$, and $Y_{i'j'} - Y_{i'j} \leq s_{j'}^* - s_j^*$ if $j' < j$. Thus, $s'_{j'} > s_{j'}^*$ must follow. Repeating the same argument, we can conclude that there is a firm $f_{\bar{i}}$ that pays $s'_{\bar{j}} > s_{\bar{j}}^*$ for an applicant $a_{\bar{j}}$, although there is no competitor for $a_{\bar{j}}$ (the number of agents is finite). But if so, $f_{\bar{i}}$ wants to give a zero salary to $a_{\bar{j}}$. This is a contradiction. Hence, there is no stationary Markov perfect equilibrium in pure strategies with equilibrium salaries exceeding $\mathbf{s}^*$.■

[10] Without such a restriction on strategies, we may have very unintuitive subgame perfect equilibrium even if there are costs for making offers. See Example 2.

Now, we show that there is a stationary Markov perfect equilibrium that attains salary vector $\mathbf{s}^*$. We will support the following equilibrium path.

> At stage $\ell = 1, 2, ..., n$, f_ℓ makes an offer to a_ℓ with salary s^*_ℓ, and a_ℓ immediately accepts it. At the end of stage n, the game ends.

Thus, after n stages, all matchings are made and the assortative matching realizes with the minimal competitive salary profile. Now, we will show that this path can be supported as a subgame perfect equilibrium. We have the following proposition.

Proposition 3. In a multi-stage game, there is a stationary Markov perfect equilibrium path such that at stage $\ell = 1, ..., n$, f_ℓ makes an offer with s^*_ℓ to a_ℓ and a_ℓ accepts the offer immediately.

Proof. We only need to show non-profitability of unilateral deviations from the equilibrium path described above. At stage ℓ, let $F' \subseteq F$ and $A' \subseteq A$ be active firms and applicants of the game, that is, players who have not exited the game by finalizing contracts. Note that some of F' might have given offers to applicants before stage ℓ, and some of them might have been rejected, and others might be outstanding at stage ℓ. Let $s^*(F', A')$ be the minimal competitive equilibrium salary vector for an assignment problem (F', A'), where production matrix used is $Y|_{F',A'}$ which is a restriction of the production matrix Y on (F', A'): this matrix is also strictly supermodular and strictly increasing, thus we can apply Lemma 1 for $s^*(F', A')$.

Now, we describe players' strategies in relevant states (subgames). Note that we will abuse notations a little bit. Strictly speaking, a state involves three components: available agents, pending offers, and rejected offers. However, the formal description is very cumbersome, and in most relevant states, we only need the information on available agents in order to describe the stationary Markov perfect equilibrium. Thus, we omit the latter two from each state, and treat the definition of state casually without losing the rigor of the argument.

First, we describe strategies in relevant states. We start with applicants. Each applicant $a_j(A')$ (not only the highest productivity applicant) accepts an offer with salary more than or equal to $s^*_j(F', A')$ immediately, if she receives only one such offer at a stage. If she receives multiple offers with salary more than or equal to $s^*_j(F', A')$ at a stage, then she chooses the highest salary offer and immediately accepts that offer (if there are multiple offers with the highest salaries, then she chooses an offer from a firm with a higher productivity). If a_j $(j > 1)$ receives an offer with a salary lower than $s^*_j(F', A')$ from f_i with $i < j$ (and it is the only one offer she receives at stage ℓ), then a_j immediately rejects the offer. Finally, if $a_j(A')$ receives an offer with salary less than $s^*_j(F', A')$ from f_i with $i \geq j$, then $a_j(A')$ keeps the best offer without accepting it and waits for another offer.

In contrast, firms' strategies in relevant states are simple. At each state, the highest ranked firm among F', $f_1(F')$, makes an offer to the highest-ranked applicant among the ones who have never rejected an offer from $f_1(F')$ with the minimal competitive salary.[11]

The above strategy profile generates the equilibrium path. At stage ℓ, $f_1(F')$ and $a_1(A')$ are matched, and at stage $\ell+1$, the game is played by $F' \backslash \{f_1(F')\}$ and $A' \backslash \{a_1(A')\}$, and the above strategies apply to this subgame, too. It is easy to see that the above strategies generates the simple path of firm f_ℓ making an offer to a_ℓ with salary s^*_ℓ for all $\ell = 1, ..., n$, and of applicants accepting offers immediately.

Given the above on-equilibrium strategies, the following cases may occur by having a unilateral deviation by a firm.

1. Firm $f_1(F')$ deviates at a stage. There are three cases.
   (a) $f_1(F')$ does not make an offer. In this case, the game ends, and $f_1(F')$ gets zero payoff. Thus, there is no such an incentive.
   (b) $f_1(F')$ makes an offer to $a_j(A')$ with $j \neq 1$. In this case, if the salary is more than or equal to $s^*_j(F', A')$ then $a_j(A')$ accepts the offer immediately. Thus, $f_1(F')$ and $a_j(A')$ are matched and they exit the game. However, $f_1(F')$'s payoff is lower than on-equilibrium outcome: $f_1(F')$ being matched with $a_1(A')$ with salary $s^*_1(F', A')$. It is because $\mathbf{s}^*(F', A')$ is a competitive salary vector. If the salary is less than $s^*_j(F', A')$ then the offer is rejected, and $f_1(F')$'s payoff would not be affected if he goes back to the original plan, making an offer to $a_1(A')$ with salary $s^*_1(F', A')$. However, making an additional offer is costly, so $f_1(F')$ is worse off by making such an irrelevant offer.
   (c) $f_1(F')$ makes an offer to $a_1(A')$ with salary s'_1 that is less than $s^*_1(F', A')$. (If more, then the offer will be accepted immediately, and $f_1(F')$ is worse off.) In this case, $a_1(A')$ does not accept the offer immediately, and $f_1(F')$ cannot make any offer at this stage due to the outstanding offer. In this subgame (here we are abusing the definition of state: the on-going state is a pair of (F', A') and a pending offer $(f_1(F'), a_1(A'), s'_1))$, the on-equilibrium path is described in the following way. $f_2(F')$ makes an offer to $a_1(A')$ with salary $\max\{s'_1, s^*_1(F' \backslash \{f_1(F')\}, A')$, and $a_1(A')$ accepts the offer from $f_2(F')$ immediately.[12] The rest of the game is played by $F' \backslash \{f_2(F')\}$ and $A' \backslash \{a_1(A')\}$, and the on-equilibrium strategies described in the beginning of the proof applies to this subgame, too. Given this, $f_1(F')$

[11] Here, we use the term "the minimum competitive salary" instead of $s^*_{1j}(F', A')$, since there are possibly pending offers and rejected offers in general. If some offers are pending, remove firms that made these pending offers. If some offers are rejected (say, $f_i(F')$'s offer has been rejected by $a_j(A')$, then let $Y_{ij}(F', A') = 0$. Then, we can still calculate the minimal competitive salary vector by using an algorithm in Demange, Gale and Sotomayor (1986), although $Y(F', A')$ does not satisfy supermodularity. However, in order to verify our strategy profile to be a stationary Markov perfect equilibrium, we do not need such a machinery. See the argument for $a_1(A')$'s incentive to accept an offer from $f_1(F')$ on equilibrium.

[12] More precisely, $f_2(F')$ needs to make an offer to $a_1(A')$ with a salary slightly higher than s'_1 (due to the tie-breaking rule).

is matched with $a_2(A')$ with salary $s_1^*(F'\backslash\{f_2(F')\}, A'\backslash\{a_1(A')\}) = s_2^*(F', A')$, and it is apparently not beneficial to $f_1(F')$.

2. Firm $f_i(F')$ ($i \neq 1$) also makes an offer to an applicant in addition to $f_1(F')$ making an offer to $a_1(A')$. There are two cases.

   (a) $f_i(F')$ made an offer to $a_j(A')$ with $j > i$ at the last stage. If the salary was not more than $s_j^*(F', A')$, then the offer was rejected immediately. If it was more than or equal to $s_j^*(F', A')$, then the offer is accepted immediately, but $f_i(F')$ was worse off than the on-equilibrium outcome. In any case, this case does not leave any outstanding offer to the next stage. Actually, $f_i(F')$ is worse off by deviating from the on-equilibrium strategy.

   (b) $f_i(F')$ made an offer to $a_j(A')$ with $j \leq i$. In this case, as long as the offered salary is at least $s_j^*(F', A')$, this offer will be outstanding at stage ℓ (otherwise, it is accepted, and $f_i(F')$ is worse off). If $i - j \geq 2$ (thus, $f_i(F')$ is irrelevant in determining on-equilibrium salary of $a_j(A')$: $f_j(F')$ and $f_{j+1}(F')$ would compete for $a_j(A')$ potentially by offering $s_j^*(F', A')$ in expectation that the rejected $f_i(F')$ makes an offer to $a_i(A')$ following the on-equilibrium strategy.[13] Thus, firm $f_i(F')$'s payoff is not affected by this irrelevant offer (except for a cost to make an additional offer). If $i = j$ or $i = j+1$, the resulting salary structure could be affected. Consider the case $i = j$. If the offer is above $s_j^*(F', A')$, then $f_i(F')$ is worse off. If the offer is less than that, then $f_{i+1}(F')$ matches (or pays even more than that if $f_{i+2}(F')$ is happy to pay the salary that $f_i(F')$ offers: in this case $f_{i+2}(F')$ is the real competitor for $f_{i+1}(F')$, so $f_{i+1}(F')$ needs to pay more to get $a_j(A')$), and $a_j(A')$ accepts $f_{i+1}(F')$'s offer. As the result, $f_i(F')$ is worse off. Lastly, consider the case $i = j + 1$. If the offered salary is more than or equal to $s_j^*(F', A')$, then it is accepted but $f_i(F')$ is worse off. If the salary is less than $s_j^*(F', A')$, then the offer will be outstanding. At a stage when $f_j(F')$'s turn comes, $f_j(F')$ will offer exactly the same salary as $f_i(F')$ did, and $a_j(A')$ accepts $f_j(F')$'s offer. Being rejected, $f_i(F')$ will make an offer to $a_i(A')$ with salary $s_i^*(F', A')$, ending up with the same payoff as on the equilibrium path (except for a cost for an additional offer).

Thus, in any case, a unilateral deviation from the equilibrium strategy does not improve firm's payoff. Finally, we need to check if an applicant has an incentive to deviate from her equilibrium strategy unilaterally. First consider $a_1(A')$. Suppose that $a_1(A')$ gets an on-equilibrium offer $s_1^*(F', A')$ from $f_1(F')$, and suppose to the contrary that she rejects the offer. Then, since $f_1(F')$ can no longer make an offer to $a_1(A')$, so it goes after $a_2(A')$. Now, firm $f_2(F')$ will have a chance to get $a_1(A')$, and its primary competitor is $f_3(A')$. Thus, $f_2(F')$

[13] If $f_i(F')$ is better off by deviating at the stage of making an offer to $a_i(A')$, then she can also be better off by deviation at that stage alone.

can offer $a_1(A')$ with a salary $s'_{21} = Y_{31}(F', A') - Y_{33}(F', A') + s^*_3(F', A')$. With this offer, $f_2(F')$ obtains payoff (if accepted) $Y_{21}(F', A') - s'_{21} = Y_{21}(F', A') - Y_{31}(F', A') + Y_{33}(F', A') - s^*_3(F', A')$. By strict supermodularity, this is higher than $Y_{22}(F', A') - s^*_2(F', A') = Y_{22}(F', A') - Y_{32}(F', A') + Y_{33}(F', A') - s^*_3(F', A')$. Thus, firm $f_2(F')$ indeed makes an offer to $a_1(A')$ instead of $a_2(A')$ (if s'_{21} would be accepted). Now, let us focus on $a_1(A')$'s payoff. If $a_1(A')$ accepted the offer from $f_1(F')$ then she gets $s^*_1(F', A')$, and if she accepta the offer from $a_2(A')$, then she gets $s'_{21} = Y_{31}(F', A') - Y_{33}(F', A') + s^*_3(F', A')$. However, by strict supermodularity and strict increasingness, it is easy to see

$$
\begin{aligned}
s^*_1(F', A') &= Y_{21}(F', A') - Y_{22}(F', A') + s^*_2(F', A') \\
&= Y_{21}(F', A') - Y_{22}(F', A') + Y_{32}(F', A') - Y_{33}(F', A') + s^*_3(F', A') \\
&> Y_{31}(F', A') - Y_{33}(F', A') + Y_{32}(F', A') - Y_{33}(F', A') + s^*_3(F', A') \\
&> Y_{31}(F', A') - Y_{33}(F', A') + s^*_3(F', A') \\
&= s'_{21}.
\end{aligned}
$$

This implies that $a_1(F', A')$ does not have an incentive to reject an offer $s^*_1(F', A')$ from $f_1(F')$.[14] It is easy to see that it does not make sense for $a_j(A')$ to reject an offer with a salary more than or equal to $s^*_j(F', A')$. If an offer is made by $f_i(F')$ with $i < j$, and if the salary is lower than $s^*_j(F', A')$, then there is no reason to keep such an offer. By waiting, she will receive an offer from $f_j(F', A')$ with salary $s^*_j(F', A')$. Thus, by rejecting such an offer immediately does not alter her payoff. Finally, if an offer is made by $f_i(F')$ with $i = j$ or $j + 1$ with salary less than $s^*_j(F', A')$, then she has no incentive to reject such an offer. Rejecting an offer is a weakly dominated strategy since it reduces the number of relevant competitors. This proves that applicants also have no incentives to deviate from the strategy profile unilaterally.■

We will refer to this equilibrium as a **simple equilibrium** of the (one period) game, and each player's equilibrium strategy in a simple equilibrium is called a **simple equilibrium strategy**.

Before closing this subsection, we note that the simple equilibrium is not the only stationary Markov perfect equilibrium of the one period game. It is one of many stationary Markov perfect equilibria. Consider the following example.

Example 2. Consider the case where $m = n = 2$.

$$
\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}
$$

Consider the following strategy profile. At stage 1, f_2 makes a zero salary offer to a_2, and at stage 2, f_1 makes a zero salary offer to a_1. Except for small costs

[14] Here, we assumed that $a_1(A')$ would accept an offer s'_{21} from $f_2(F')$ in the case where she rejected an offer $s^*_1(F', A')$ from $f_1(F')$. It is easy to see if $a_1(A')$ rejects this offer as well, then she would be worse off even more.

needed to make offers, firm f_2 is indifferent between making an offer to f_2 with $s_2 = 0$ and making an offer to f_1 with whatever salary $s_1 \in [0, Y_{11} - Y_{12}]$ at stage 1. As long as an offer is made in stage 1 by f_2, then the game continues, and if a_1 rejects the offer by receiving a competing offer from f_1 in stage 2, f_2 can get a_1 with $s_1 = 0$ anyway in stage 3. However, if f_2 makes an offer to a_1 first, then it needs to pay the cost associated with making an offer twice instead of once if f_2 makes an offer to a_2 from the beginning.

If we do not assume that there is a small cost to firms associated with making an offer, then more equilibria show up. At stage 1, f_2 offers a salary of $\delta > 0$ to a_1. At stage 2, f_1 makes an offer with the same salary to a_1. Each applicant holds an offer until she receives both offers (or the period ends), and if she receives multiple offers with identical salaries then she chooses a higher productivity firm. By the applicant's strategy, f_2's offer is rejected, and f_2 makes an offer to a_2 with zero salary. This offer is accepted. This is certainly a subgame perfect equilibrium - neither firm can do better for itself by deviating at any stage. (Observe that f_2 can make any offer to a_1 as long as it is positive and not more than $(Y_{21} - Y_{22})$. Firm f_2 can hire a_2 at zero salary in any case).

Note that in the above argument, δ can exceed s_1^* as long as $Y_{11} - \delta > Y_{12}$. This can happen since $s_1^* = Y_{21} - Y_{22}$. Thus, without stationary-Markov and cost-to-offer assumptions, we cannot have our Proposition 2. Moreover, if there are three firms and three applicants, we can cook up a subgame perfect equilibrium path in which f_1 and f_2 make an offer to a_1 with a such $\delta > s_1^*$, and then f_3 makes a zero salary offer to a_3, and finally f_2 makes a zero salary offer to a_2. If f_2 deviates in the first stage by not making an offer, then the following equilibrium path is that f_2 makes an offer to a_2 with salary s_2^*, and then f_3 makes a zero salary offer to a_3. Thus, dependent on f_2's seemingly irrelevant choice in stage 1, the real outcome can be affected if we do not impose a stationary-Markov restriction even if making an offer involves a positive cost.■

5 Equilibrium with Full-Unraveling

In this section, we introduce multi-periods. There is a finite number of periods $t = 1, 2, ..., T$, where T is the terminal period (the date of graduation or the date when all the transcripts become available). In the beginning period (period 1), every agent (including a worker herself) is uncertain about each worker's true ability: what ability each worker eventually obtain as a work force. As each period goes by, this uncertainty gradually resolves, and in period T each agent will obtain perfect information about each worker's ability. In each period $t \in \{1, ..., T\}$ there is a large finite number of stages. Within the same period no new information is revealed, thus the level of uncertainty on each worker's ability stays the same at each stage within the same period. We introduce sufficiently many stages in each period in order to describe firms' competition of making offers and firms' contingent plans after some of their offers have been rejected by applicants.

Recall there are T periods, and a sufficiently large number of stages L in each period. We assume that *each offer is valid* ***until the end of the period*** *of the offer is made.* We also assume that *at one stage if there is no offer made by any firm then the period will be terminated and move to the next period.* This assumption is adopted since we introduced arbitrarily many stages in each period in order to describe offer-acceptance/rejection dynamics in a very short time.[15] If an applicant (a firm) is not matched even after period T ends, her payoff is set to be zero. The outcome of the game is an assignment between applicants and firms together with their payoff vectors. In the same fashion, an outcome of each period is an assignment of applicants with firms and payoff vectors who exited the game by contracting.

Following Roth and Xing (1994) we identify a worker w in period T as an agent with a certain ability from now on. In contrast, in periods $t < T$ workers' abilities are not known with certainty, and workers are identified only as applicants $a_1^t, ...,$ and a_m^t, where a_j^t represents an applicant $j \in \{1, ..., m\}$ in period $t \in \{1, 2, ..., T\}$. Applicants will become one of m workers, $a_1, ..., a_m$ stochastically. Thus in each period there is a probability matrix $P^t = \{p_{jk}^t\}$ with p_{jk}^t being the probability that agent a_j^t will eventually become worker w_k in period T. Let $A^t = \{a_1^t, ..., a_m^t\}$ be the set of applicants in period t. Obviously, $A^T = A$.

We assume that $P^t = (P)^{T-t}$ for $t = 1, ...T-1$, where transitional probability matrix P is specified as:

$$P = \begin{pmatrix} 1-(m-1)\epsilon & \epsilon & \cdots & \epsilon \\ \epsilon & 1-(m-1)\epsilon & \cdots & \epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \cdots & 1-(m-1)\epsilon \end{pmatrix}$$

where $0 < \epsilon < \frac{1}{m}$.

Thus, as each period passes the level of uncertainty gradually resolves according to the following process $\mathbf{a}^t = P\mathbf{a}^{t+1}$ for $t = 1, ..., T-1$ where $\mathbf{a}^t = (a_1^t, ...a_m^t)$. At each time $t < T$, any applicant a_i^t is more likely to become an applicant a_i^{t+1} (with probability $1-(m-1)\epsilon$), but there are chances that a_i^t becomes other types (with probability ϵ). The assumption $0 < \epsilon < \frac{1}{m}$ guarantees that $1-(m-1)\epsilon$ is greater than ϵ.

Note that this process implies that uncertainty resolves over time. In period T, a_i^T becomes w_i with probability one, in period $T-1$, a_i^{T-1} becomes w_i with probability $1-(m-1)\epsilon$; and in period $T-2$, a_i^{T-1} becomes w_i with probability $(1-(m-1)\epsilon)^2 + (m-1)\epsilon^2$. One can easily show the following:

$$\begin{aligned} (1-(m-1)\epsilon)^2 + (m-1)\epsilon^2 &= 1-2(m-1)\epsilon + (m-1)^2\epsilon^2 + (m-1)\epsilon^2 \\ &= 1-2(m-1)\epsilon + m(m-1)\epsilon^2 \\ &= 1-(m-1)\epsilon - (m-1)\epsilon(1-m\epsilon) \\ &< 1-(m-1)\epsilon. \end{aligned}$$

[15] If we drop this assumption, yet keep finiteness of stages in one period, then it might generate a last-minute rush of making offers in a period (see Niederle and Roth, 2002 for example). We discuss this possibility in the end of the paper.

The uncertainty level changes from period to period, so does the output matrix. Under our uncertainty resolving process P, the expected outputs of firm-applicant pairs at period $T-1$ are described by,

$$\begin{aligned} Y^{T-1} &= Y^T P \\ &= \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nm} \end{pmatrix} \begin{pmatrix} 1-(m-1)\epsilon & \epsilon & \cdots & \epsilon \\ \epsilon & 1-(m-1)\epsilon & \cdots & \epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \cdots & 1-(m-1)\epsilon \end{pmatrix}. \end{aligned}$$

More generally, we can write $Y^t = Y(P)^{T-t}$ for any $t = 1, ..., T-1$, where $(P)^{T-t}$ is a multiplication of P by $T-1$ times. It is easy to see that Y^t is a matrix of which arguments are all positive, since Y is so, too.

Lemma 2. If Y^{t+1} is strictly supermodular and strictly increasing then $Y^t = PY^{t+1}$ is also strictly supermodular and strictly increasing.

Proof. We use $Y^t_{ij} = (1-(m-1)\epsilon)Y^{t+1}_{ij} + \epsilon \sum_{k \neq j} Y^{t+1}_{ik}$ repeatedly. First we show that Y^t is strictly increasing.

$$\begin{aligned} Y^t_{ij} - Y^t_{i,j+1} &= [(1-(m-1)\epsilon)Y^{t+1}_{ij} + \epsilon \sum_{k \neq j} Y^{t+1}_{ik}] - [(1-(m-1)\epsilon)Y^{t+1}_{i,j+1} + \epsilon \sum_{k \neq j+1} Y^{t+1}_{ik}] \\ &= (1-(m-1)\epsilon)(Y^{t+1}_{ij} - Y^{t+1}_{ij+1}) - \epsilon(Y^{t+1}_{ij} - Y^{t+1}_{ij+1}) \\ &= (1-m\epsilon)(Y^{t+1}_{ij} - Y^{t+1}_{ij+1}) > 0 \end{aligned}$$

The last inequality follows, since $\frac{1}{m} > \epsilon$ implies $(1-m\epsilon) > 0$. Thus, if Y^{t+1} is strictly increasing, then Y^t is strictly increasing as well. Second we show strict supermodularity.

$$\begin{aligned} & (Y^t_{ij} - Y^t_{ij+1}) - (Y^t_{i+1j} - Y^t_{i+1j+1}) \\ = \ & (1-m\epsilon)[(Y^{t+1}_{ij} - Y^{t+1}_{ij+1}) - (Y^{t+1}_{i+1j} - Y^{t+1}_{i+1j+1})] > 0 \end{aligned}$$

The last equality follows since Y^t is strict supermodular.■

The following lemma shows that the optimal assignment matrix does not change over periods.

Lemma 3. Suppose that X^* is the optimal assignment matrix for $Y^T = Y$. Then, X^* is also optimal for $Y^t = Y(P)^{T-t}$ for any $t = 1, ..., T-1$ for any $\epsilon < \frac{1}{m}$.

Proof. We first calculate each term of Y^{T-1}.

$$
\begin{aligned}
& Y^{T-1} \\
= & \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nm} \end{pmatrix} \begin{pmatrix} 1-(m-1)\epsilon & \epsilon & \cdots & \epsilon \\ \epsilon & 1-(m-1)\epsilon & \cdots & \epsilon \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon & \epsilon & \cdots & 1-(m-1)\epsilon \end{pmatrix} \\
= & \begin{pmatrix} Y_{11}(1-(m-1)\epsilon)+\epsilon\sum_{j\neq 1} Y_{1j} & \cdots & Y_{1n}(1-(m-1)\epsilon)+\epsilon\sum_{j\neq n} Y_{1j} \\ \vdots & \ddots & \vdots \\ Y_{n1}(1-(m-1)\epsilon)+\epsilon\sum_{j\neq 1} Y_{nj} & \cdots & Y_{nn}(1-(m-1)\epsilon)+\epsilon\sum_{j\neq n} Y_{nj} \end{pmatrix}
\end{aligned}
$$

Suppose that X^* is optimal assignment for $Y^T = Y$, and let $\mu^* : F \to A$ be a bijection (a matching function) generated from X^* such that $\mu^*(i) = j$ if and only if $x^*_{ji} = 1$. By the definition of optimal assignment, $X^*Y \geq XY$ holds for any assignment matrix X, or $\sum_{i\in F} Y_{i\mu^*(i)} \geq \sum_{i\in F} Y_{i\mu(i)}$ for any matching function $\mu : F \to A$.

By the definition of Y^{T-1}, the following holds for any matching function μ:

$$
\begin{aligned}
\sum_{i\in F} Y^{T-1}_{i\mu(i)} &= \sum_{i\in F}\left[Y_{i\mu(i)}(1-(m-1)\epsilon)+\epsilon\sum_{j\neq\mu(i)} Y_{ij}\right] \\
&= \sum_{i\in F}\left[Y_{i\mu(i)}(1-(m-1)\epsilon)+\epsilon\sum_{j\in A} Y_{ij} - \epsilon Y_{i\mu(i)}\right] \\
&= \sum_{i\in F}\left[Y_{i\mu(i)}(1-m\epsilon)+\epsilon\sum_{j\in A} Y_{ij}\right] \\
&= (1-m\epsilon)\sum_{i\in F} Y_{i\mu(i)} + \epsilon\sum_{i\in F}\sum_{j\in A} Y_{ij}.
\end{aligned}
$$

Since $\sum_{i\in F} Y_{i\mu^*(i)} \geq \sum_{i\in F} Y_{i\mu(i)}$ holds for any matching function $\mu : F \to A$, we have

$$
\begin{aligned}
\sum_{i\in F} Y^{T-1}_{i\mu^*(i)} &= (1-m\epsilon)\sum_{i\in F} Y_{i\mu^*(i)} + \epsilon\sum_{i\in F}\sum_{j\in A} Y_{ij} \\
&\geq (1-m\epsilon)\sum_{i\in F} Y_{i\mu(i)} + \epsilon\sum_{i\in F}\sum_{j\in A} Y_{ij} \\
&= \sum_{i\in F} Y^{T-1}_{i\mu(i)}.
\end{aligned}
$$

Recall that $Y_{T-t} = Y_{T-t+1}P$ for any $t = 1, ..., T-1$. By induction, we can conclude that for any $t = 1, ..., T-1$, $\sum_{i\in F} Y^{T-t}_{i\mu^*(i)} \geq \sum_{i\in F} Y^{T-t}_{i\mu(i)}$ for any assignment function μ, or $X^*Y^{T-t} \geq XY^{T-t}$ for any assignment matrix X.■

Now, we will show the possibility of strategic unraveling using simple equilibrium. We impose one technical assumption on the output matrix. It actually governs the behavior of both applicants and firms in multi-period game. In order to introduce our assumption, we define the following concepts. When $\mu(i) = j$, applicant a_j's salary s_{ij} **respects applicant a_j's merit** if $s_j \geq Y_{ij} - \frac{1}{m}\sum_{j'=1}^{m} Y_{ij'}$. That is, firm f_i pays applicant a_j a salary more than the difference between the output obtained by this partnership and the average output that can be obtained by partnerships between firm f_i and all applicant $a_{j'}$s. A salary vector **s respects applicants' merits** if every applicant's merit is respected by her salary. *We will assume that the minimal competitive salary s^* respects applicants' merits.* We call this assumption **merit salary**. This condition may seem to be imposed on endogenously determined equilibrium salary vector. However, since the minimal competitive salaries can be written as $s^*_{i+(m-n)} = \sum_{j'=m-n+1}^{i+(m-n)-1}(Y_{j'-(m-n),j'+1} - Y_{j'-(m-n),j'})$ for each $i = 1, ..., n$, this condition is actually imposed on output matrix Y. This condition is obviously not satisfied by every Y with strict supermodularity and strict increasingness. However, *it is satisfied in a commonly used* ***multiplicative production matrix*** *Y specified by $Y_{ij} = i \times j$ when $m = n$.* An important implication of the merit salary assumption is as follows. We denote a minimal competitive salary for applicant a_i in period t (thus production matrix is Y^t) by s_i^{*t}.

Lemma 4. The merit salary assumption implies that $Y_{ii}^t - s_i^{*t} \geq Y_{ii}^{t+1} - s_i^{*t+1}$ for all $i = 1, ..., n$ and all $t = 1, ..., T-1$.

Proof. Direct calculations prove this:

$$\begin{aligned} Y_{ii}^t - s_i^{*t} &= Y_{ii}^t - \sum_{j=i}^{n-1}(Y_{j+1j}^t - Y_{j+1j+1}^t) \\ &= (1-m\epsilon)(Y_{ii}^{t+1} - \sum_{j=i}^{n-1}(Y_{j+1j}^{t+1} - Y_{j+1j+1}^{t+1})) + \epsilon\sum_{j=1}^{m} Y_{ij}^{t+1}, \end{aligned}$$

and

$$Y_{ii}^{t+1} - s_i^{*t+1} = Y_{ii}^{t+1} - \sum_{j=i}^{n-1}(Y_{j+1j}^{t+1} - Y_{j+1j+1}^{t+1}).$$

The difference is

$$
\begin{aligned}
& (Y_{ii}^{t} - s_i^{*t}) - (Y_{ii}^{t+1} - s_i^{*t+1}) \\
= \quad & -m\epsilon(Y_{ii}^{t+1} - \sum_{j=i}^{n-1}(Y_{j+1j}^{t+1} - Y_{j+1j+1}^{t+1})) + \epsilon\sum_{j=1}^{m} Y_{ij}^{t+1} \\
= \quad & m\epsilon\left[\frac{1}{m}\sum_{j=1}^{m} Y_{ij}^{t+1} - \left(Y_{ii}^{t+1} - \sum_{j=i}^{n-1}(Y_{j+1j}^{t+1} - Y_{j+1j+1}^{t+1})\right)\right] \\
\geq \quad & 0.
\end{aligned}
$$

The last inequality is a direct consequence of the merit salary assumption.■

Note that early contract involves uncertainty in contracted applicants' qualities. This uncertainty reduces attractiveness of a highly-ranked applicant on the one hand, but it also reduces competitive salaries. Thus, in general, high-productivity firms may or may not like early contracting. The above lemma says that if merit salary condition is satisfied, all firms, even the highest productivity firms, prefer early contracts.

Proposition 4. Suppose that the merit salary assumption is satisfied. Then, there is at least one stationary Markov perfect equilibrium with full unraveling with the resulting period 1 salary vector being $\mathbf{s}^{*1}$.

Proof. Consider a strategy profile such that each agent plays her simple equilibrium strategy in every period (salary offers are based on the output matrix in the period). Lemma 4 guarantees that in each period $t = 1, ..., T-1$, no firm f^i $(i = 1, ..., n)$ has no incentive not to make an offer to any applicant when its turn comes. It is because if no offer is made by any firm in a stage in period t, then period t ends immediately and the next period $t+1$ arrives with new information: then a new subgame in period $t+1$ starts (and a simple equilibrium is played). Since a simple equilibrium strategy is played in each period, no other deviation by any agent is beneficial (Proposition 3). Thus, such a strategy profile is a stationary Markov perfect equilibrium.■

Example 3. To illustrate the theorem, here is a numerical example of the equilibrium described above. It is a 3×3 multiplicative output matrix:

$$
Y = \begin{pmatrix} 9 & 6 & 1 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{pmatrix}
$$

In a one period game with no uncertainty the minimal salary vector is $s^* = (3, 1, 0)$ and the profit vector is $u = (6, 3, 1)$, thus the total output is 14.

Now consider a two period game with uncertainty in the first period $\varepsilon = .15$. Thus the output matrix in the first period becomes:

$$
Y' = \begin{pmatrix} 7.65 & 6 & 1.45 \\ 5.1 & 4 & 2.9 \\ 2.55 & 2 & 1.45 \end{pmatrix}
$$

As the outcome of the stationary Markov perfect equilibrium described in the theorem, we obtain an allocation such that all the game participants are matched in the first period and the resulting salary vector is $s^{*1} = (1.65, 0.55, 0)$ the profit vector becomes $u^1 = (6, 3.45, 1.45)$ and the total output is 13.1. Thus, unraveling increases expected profits (for f_2 and f_3), decreases the salaries (for a_1 and a_2) and the total output falls down by 0.9.

6 Conclusion

Bulow and Levin (forthcoming) asserted that salaries are suppressed under the centrally planned matching mechanism comparing with the one under a decentralized market. However, they used the minimal competitive salary vector (a Vickrey auction salary vector) as the outcome of decentralized market. However, in the real labor market, decentralized market means a collection of bilateral offers and applicants' accept/reject decisions. In this paper, we tried to compare the equilibrium salary vectors in such situations by specifying games in tractable manners. We have shown three results. First, under a simultaneous move game, the resulting (expected) salary vector is dominated by the minimal competitive salary vector (Proposition 1). Second, under a sequential move game with open offers, any stationary Markov perfect equilibrium attains at most the minimal competitive salaries (Proposition 2). Although there is a stationary Markov perfect equilibrium that attains the minimal competitive salaries (Proposition 3), there are many other stationary Markov perfect equilibria including a zero salary vector equilibrium (Example 2). Third, even if a stationary Markov perfect equilibrium in each period attains the minimal competitive salary vector, due to incentives for unravelling, applicants' expected salaries can be suppressed by early contracting.

We recognize that assumptions we chose are overly simplistic in comparison with the real labor markets (especially, an open offer assumption). Relaxation of assumptions are subject for the future research.

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