Math 202 Multivariable Calculus
FINAL EXAM
May 8, 2000

This exam has 150 points. There are NINE questions, TWO ON THE OTHER SIDE, with various point values, as shown.

1. (10) Calculate the line integral $\int_{\mathbf{c}} e^x dy$, where $\mathbf{c}$ is the line segment from $(1,-1)$ to $(2,1)$.
SOLUTION: $2(e^2 - e)$.

2. (10) Calculate the flux integral $\int_S \mathbf{F}\cdot\mathbf{N}\; dS$, where $\mathbf{F} = (x, y, z)$, S is the unit sphere centered at $(0,0,0)$, and N is the outward unit normal vector to S.
SOLUTION 1: $\mathbf{F}\cdot\mathbf{N} = 1$ on S, so

$$\int_S \mathbf{F}\cdot\mathbf{N}\; dS = \int_S 1\; dS = \text{Area} = 4\pi.$$

SOLUTION 2: $\nabla \cdot F = 3$, so by Divergence Theorem

$$\int_S \mathbf{F}\cdot\mathbf{N}\; dS = 3\cdot \text{volume inside } S = 3\cdot\frac{4\pi}{3} = 4\pi.$$

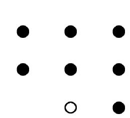
3. (15) Calculate the surface area of the Helical Ribbon, parametrized by

$$\mathbf{r}(u,v) = (u\cos v, u\sin v, v), \qquad 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

You will need the integral formula $\int \sqrt{1+x^2}\; dx = \frac{1}{2}[x\sqrt{1+x^2} + \log(x + \sqrt{1+x^2})]$.
SOLUTION: $\mathbf{r}_u \times \mathbf{r}_v = (\sin v, -\cos v, 1)$, $||\mathbf{r}_u \times \mathbf{r}_v|| = \sqrt{1+u^2}$, and the area is

$$\int_0^{2\pi}\int_0^1 \sqrt{1+u^2}\; dudv = \pi[\sqrt{2} + \log(1+\sqrt{2})].$$

4. (15) Calculate the line integral $\int_{\mathbf{c}} x\; dy$. Here $\mathbf{c}$ is the following curve, where each segment between dots has length one, and the hollow dot $\circ$ is the point $(1,0)$.



SOLUTION Completing the circuit from $(2,0)$ to $(1,0)$ would give the area, which is 3. The integral over this missing step is 1, so $3 = \text{Area} = \int_{\mathbf{c}} x\; dy - 1$, so $\int_{\mathbf{c}} x\; dy = 4$.

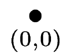
Shame on you for integrating over each step!

5. (20) Calculate the line integral

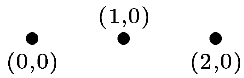
$$\oint_{\mathbf{c}} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

over each of the following curves .

a)



SOLUTION: The path does not go around $(0,0)$, so the integral is zero. b)



SOLUTION: The path is not closed. If you complete the circuit by going along the line segment from $(1,0)$ to $(2,0)$, you get a closed path around $(0,0)$, whose integral gives 2π. The integral over the segment is zero, so the integral over the original curve is 2π as well.

6. (20) Calculate the average value of $f(x,y,z) = z^{100}$ over the unit ball B_1, centered at $(0,0,0)$.

SOLUTION:

$$\frac{3}{4\pi}\int_0^1\int_0^\pi\int_0^{2\pi}(r\cos\phi)^{100}r^2\sin\phi d\theta d\phi dr = \frac{3}{101\cdot 103}.$$

7. (20) a) Calculate the curl of the vector field $\mathbf{F} = \frac{1}{4}(xz^2 - xy^2, yx^2 - yz^2, zy^2 - zx^2)$.
SOLUTION: (yz, zx, xy). b) Use Stokes theorem to calculate

$$\iint_S yz\ dydz + zx\ dzdx + xy\ dxdy,$$

where S is the top half of the unit sphere centered at $(1,1,0)$.

SOLUTION: We must integrate $\mathbf{F}$ around the boundary of S. The boundary is parametrized by $\mathbf{c}(t) = (1+\cos t, 1+\sin t, 0)$, $0 \le t \le 2\pi$. Note $z = 0$ on $\mathbf{c}$. We get $(c = \cos t,\ s = \sin t)$

$$\oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{T}\ ds = \frac{1}{4}\int_0^{2\pi}(1+c)(1+s)^2 s + (1+s)(1+c)^2 c = \pi.$$

8. (20) Consider the mapping $R(u,v) = (u\cosh v, u\sinh v)$, defined on the rectangle $R^* : 0 \le u \le 1,\ -a \le v \le a$ in the uv-plane. In the xy plane, let R be the image R^* under this mapping.

a) Draw a picture of R, including the images of horizontal and vertical lines in R^* under $R(u,v)$.

SOLUTION: The horizontal lines go to hyperbolas with asymptotes $y = \pm x$ the vertical lines go to lines thru the origin.

b) Calculate $\frac{\partial(x,y)}{\partial(u,v)}$.

SOLUTION: $\frac{\partial(x,y)}{\partial(u,v)} = u$

c) Give the integration formula for a function $f(x,y)$ on R.

SOLUTION:

$$\int_R f\ dR = \int_{-a}^{a}\int_0^1 f(u\cosh v, u\sinh v)u\ dudv.$$

d) Use your integration formula to calculate the area of R.

SOLUTION:

$$\text{Area} = \int_R f \; dR = \int_{-a}^{a} \int_0^1 u \; dudv = a.$$

9. (20) Let $r = (x^2 + y^2 + z^2)^{1/2}$, and let $\mathbf{F} = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$.

a) Calculate the gradient of $f = \frac{1}{r}$.

SOLUTION:

$$\nabla f = -(\frac{x}{r^{3/2}}, \frac{y}{r^{3/2}}, \frac{z}{r^{3/2}}).$$

b) Calculate the line integral $\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{T} \; ds$, where $\mathbf{c}(t) = (t\cos t, t\sin t, t+1), \quad 0 \leq t \leq 2\pi$.

SOLUTION:

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{T} \; ds = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t) \cdot \mathbf{c}'(t) \; dt = \int_0^{2\pi} \frac{2t+1}{\sqrt{2t^2+2t+1}} \; dt = \sqrt{8\pi^2 + 4\pi + 1} - 1.$$