Math 202
Exam 1 SOLUTIONS

1. (10) Find the equation of the tangent plane to the graph of $f(x,y) = x^3 - 2xy^2$ at the point $(1,1,-1)$.

The gradient of $z - f(x,y)$ is $(-3x^2 + 2y^2, 4xy, 1)$, which when evaluated at $(1,1,1)$ gives $(-1,4,1)$. The tangent plane is $-1(x-1) + 4(y-1) + (z+1) = 0$, or $x - 4y - z = 2$.

2. (10) Find a vector in the direction of maximal increase of $f(x,y,z) = xy + \sin z$ at the point $(1,0,0)$, and find the rate of increase of f in this direction.

$\nabla f = (y, x, \cos z)$, so $\nabla f(1,0,0) = (0,1,1)$ is the direction of greatest increase, and the magnitude of increase is $||(0,1,1)|| = \sqrt{2}$.

3. (10) Find parametric equations for the lines on the saddle $z = xy$ through the point $(1,2,2)$

Let the line be $(1+at, 2+bt, 2+ct)$. Plugging in to $z = xy$ gives $2 + ct = 2 + (2a+b)t + abt$, so $ab = 0$ and $c = 2a + b$. Taking $a = 1, b = 0$ we get the line $(1+t, 2, 2+2t)$. Taking $a = 0, b = 1$ we get the line $(1, 2+t, 2+t)$.

4. (15) Find parametric equations for the line of intersection of the two planes

$$x + y + z = 2, \qquad y = 1.$$

The cross product of the normal vectors is $(1,1,1) \times (0,1,0) = (-1,0,1)$. A point on the plane is $(1,1,0)$. The line is $(1,1,0) + t(-1,0,1)$.

5. (15) Find the point on the plane $z = x + y$ which is closest to the point $(1,1,1)$.

Minimize the square of the distance, which is given by $f(x,y) = (x-1)^2 + (y-2)^2 + (x+y-3)^2$. Thus, $f_x = 2(x-1) + 2(x+y-1) = 0$ and $f_y = 2(y-1) + 2(x+y-1) = 0$. Solving for x, y, we get $x = y = \frac{2}{3}$. The closest point is $(\frac{2}{3}, \frac{2}{3}, \frac{4}{3})$.

6. (30) In this problem, $f(x,y) = (x^2 - y^2)(x-1)$.

a) Draw the level curve $f = 0$, and give approximate sketches of other level curves for positive and negative values of f.

b) Draw arrows of the gradient of f on your level curves. The length of the arrows need not be exact.

c) Find the critical points of f, and classify them as max, min, saddle, or degenerate.

The level curve $f = 0$ consists of three lines $x = y, x = -y, x = 1$. These divide the plane into 7 regions. f is positive in the region containing the point $(2,0)$. The triangle is negative, and the regions around the triangle alternate in signs. ∇f points outward in the unbounded positive regions, inward in the unbounded negative regions, and inward in the triangle. The critical points are $(0,0), (1,\pm 1)$ (all saddles) and $(\frac{2}{3}, 0)$ (minimum, inside the triangle).

7. (10) The temperature at point (x,y,z) in the water is $f(x,y,z) = xyz$. A particle swims in the water, and its velocity at the point $p = (1,1,1)$ is $\mathbf{v} = (1,2,3)$. What is the rate of change of the particle's temperature at p? (Assume that the particle instantly adopts the temperature of its location.)

The temperature of the particle is $f(p + t\mathbf{v})$. $\nabla f = (yz, xz, xy)$. The velocity of the path $p + t\mathbf{v}$ is just $\mathbf{v}$. The rate of change is $\nabla f(p) \cdot \mathbf{v} = (1,1,1) \cdot (1,2,3) = 6$.