Math 210 Linear Algebra
Exam 1
February 21, 2006

This exam has six questions, with point totals as shown, for a total of 100 points. Electronic calculators are not allowed, only human ones. Show all of your calculations.

1. (16) Find the eigenvalues and corresponding eigenvectors of $\begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$.

SOLUTION: The eigenvalues are $0, 1$. The eigenvectors are $(2, 3),\ (1, 2)$.

2. (16) Find the 2×2 matrix A that reflects about the line $y = 2x$. The entries in your answer should be rational numbers. Hint: First find the eigenvectors of A. More generally, suppose the line is $y = mx$. Then the evecs are

$$\mathbf{u} = (1, m), \qquad \mathbf{v} = (-m, 1),$$

so

$$A = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

So for $m = 2$, the answer is

$$A = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}.$$

3. (20) Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

a) Find the eigenvalues of A.

b) Describe, with words or a picture, how A moves vectors around in the plane.

SOLUTION: Eigenvalues $1 \pm i$.

$$A = \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}.$$

So A rotates by $\pi/4$ and scales by $\sqrt{2}$, moving vectors in an outward spiral.

4. (16) Suppose A is a matrix with eigenvalues $\lambda = 3,\ \mu = -3$, and corresponding eigenvectors $\mathbf{u} = (2, 1),\ \mathbf{v} = (1, 1)$. Find A.

SOLUTION:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & -12 \\ 6 & -9 \end{bmatrix}$$

5. (20) Numbers x_n are given recursively by $x_0 = 0,\ x_1 = 1,\ x_{n+1} = 2x_n + x_{n-1}$.

a) Find the matrix A such that $A \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$.

b) Express the entries of A^n in terms of x_{n-1}, x_n, x_{n+1}.

c) Find the eigenvalues λ, μ and the corresponding eigenvectors $\mathbf{u},\ \mathbf{v}$ of A.

d) Find a formula for x_n, in terms of λ and μ.

A tip: In your work, use the symbols λ, μ, not the numbers these symbols represent.

SOLUTION: $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

$$A^n = \begin{bmatrix} x_{n-1} & x_n \\ x_n & x_{n+1} \end{bmatrix}$$

$$\lambda = 1 + \sqrt{2},\ \mu = 1 - \sqrt{2}, \qquad \mathbf{u} = (1, \lambda),\ \mathbf{v} = (1, \mu).$$

$$x_n = \frac{\lambda^n - \mu^n}{2\sqrt{2}}.$$

6. (12) In this problem , write TRUE if the assertion holds for all 2×2 matrices, and FALSE if it does not hold for all 2×2 matrices. That's all you have to write. Scoring: +3 if correct, zero if not.

a) If A is invertible, the eigenvectors of A^{-1} are same as those for A. TRUE

b) For any two matrices A, B, we have $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$. FALSE

c) If A has only one eigenvalue then A is nilpotent. FALSE

d) If A has real entries and nonzero complex eigenvalues then $\det(A) > 0$. TRUE