Math 210 Linear Algebra   Exam 2   April 11, 2007

This exam has seven questions. The first four are worth 10 points each and the last three are worth 20 points each, for a total of 100 points.

1. Find the kernel of the matrix $A = \begin{bmatrix} 0 & 4 & 2 \\ 6 & 2 & 7 \\ 3 & 4 & 5 \end{bmatrix}$.

Solution: $\mathbb{R}(2, 1, -2)$.

2. Find the eigenvalues of the matrix in problem 1.

Solution: $P_A(x) = x^3 - 7x^2 - 48x$, evals: $0,\ \frac{1}{2}(7 \pm \sqrt{241})$.

3. Find an eigenvector for the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -3 & 1 & 1 \\ 4 & -1 & -1 \end{bmatrix}$.

Solution: $P_A(x) = (x-1)^3$, evec: (1,-2,3). Up to scalar, this is the only evec.

4. (a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) Use A^{-1} to solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 1 \\ x + y + 2z &= 0 \\ x + y + z &= 1. \end{aligned}$$

Solution: $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and the solution to the system is $(0, 2, -1)$.

5. (a) Find a nonzero vector on the intersection of the two planes

$$x + 2y + 2z = 0, \qquad 2x + 2y + z = 0.$$

Solution: $(2, -3, 2)$

(b) Find the equation of the plane spanned by $\mathbf{u} = (1, 2, 2)$ and $\mathbf{v} = (2, 2, 1)$.

Solution: $2x - 3y + 2z = 0$.

(c) What is the geometric relation between the plane in part (b) and the two planes in part (a)?

Solution: The plane in (b) is perpendicular to both of the planes in (a).

6. Show that $A = \frac{1}{7}\begin{bmatrix} -2 & 3 & 6 \\ 6 & -2 & 3 \\ 3 & 6 & -2 \end{bmatrix}$ is a rotation matrix. Find the axis and cosine of the angle of reflection.

Solution: Compute $AA^T = I$ and $\det(A) = +1$. This shows that A is a rotation. The axis is $\ker[A - I] = \mathbb{R}(1, 1, 1)$ and the cosine of the rotation angle θ is $\cos(\theta) = \frac{1}{2}[\mathrm{tr}(A) - 1] = -\frac{6}{7}$.

7. Find the vector obtained by rotating $\mathbf{e}_1 = (1, 0, 0)$ about the axis through $(1, 2, 3)$, counterclockwise by $90°$ as $(1, 2, 3)$ points at you.

Solution: Let's work with a general unit vector $\mathbf{u} = (a, b, c)$. Rotation about $\mathbf{u}$ by $90°$ sends a general vector $\mathbf{v}$ to

$$\mathbf{v} + \mathbf{u} \times \mathbf{v} + \mathbf{u} \times (\mathbf{u} \times \mathbf{v}).$$

We compute

$$\mathbf{u} \times \mathbf{e}_1 = (0, c, -b), \qquad \mathbf{u} \times (\mathbf{u} \times \mathbf{e}_1) = (-b^2 - c^2, ab, ac),$$

so $\mathbf{e}_1$ rotates to

$$\mathbf{e}_1 + \mathbf{u} \times \mathbf{e}_1 + \mathbf{u} \times (\mathbf{u} \times \mathbf{e}_1) = (1 - b^2 - c^2, ab + c, ac - b) = (a^2, ab + c, ac - b).$$

In our problem, we have $\mathbf{u} = \frac{1}{\sqrt{13}}(1, 2, 3)$, so $\mathbf{e}_1$ rotates to the vector

$$\left(\frac{1}{13}, \frac{2}{13} + \frac{3}{\sqrt{13}}, \frac{3}{13} - \frac{2}{\sqrt{13}}\right).$$