More Linear Algebra Study Problems

1. *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

Find the kernel and image of A. Where do these live?

Solution: The kernel of A is the hyperplane in $\mathbb{R}^6$ with equation

$$x_1 + 2x_2 + \cdots + 6x_6 = 0.$$

The image is the line in $\mathbb{R}^4$ through the vector $(1, 1, 1, 1)$.

2. *Find the 3×3 matrix that reflects about the plane $x + y + z = 0$.* Hint: use the formula from the second exam: $A\mathbf{v} = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{u} \rangle \mathbf{u}$, where $\mathbf{u}$ is a unit vector normal to the plane. You can check your answer by computing the eigenspaces of your matrix.

3. *Find the angle and axis of rotation of the matrix*

$$A = \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 4 & 7 & 4 \\ 1 & 4 & -8 \end{bmatrix}.$$

Think about how you would check your answer.

4. *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Check your answer. Then use your answer to solve the system

$$\begin{aligned} x + y &= 1 \\ x + 2y + z &= 1 \\ y + 2z &= 1 \end{aligned}$$

5. *Find the ranks and the kernels of the following matrices.*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

$\operatorname{rank}(A) = 3$, $\ker A = \mathbb{R}\mathbf{e}_4$, $\operatorname{rank}(B) = 2$, $\ker(B)$ is the plane spanned by $(2, -3, 0, 1), (1, -2, 1, 0)$.

6. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Check your answer.

7. Find, if possible, planes in $\mathbb{R}^4$ which meet the hyperplane $x + y + z + w = 0$ in

a) a point;
b) a line;
c) a plane.

Hint 1: One case is impossible. There are many possible answers for the other two.

Hint 2: Let $\mathbf{x}, \mathbf{y}$ be vectors in $\mathbb{R}^4$ and let the sum of their entries be $d = \sum x_i$ and $e = \sum y_i$. Show that $e\mathbf{x} - d\mathbf{y}$ is in the hyperplane.

8. *A 4×4 matrix A with real entries satisfies $A^4 = I$. What are the possible characteristic polynomials of A?*

9. *Find a 4×4 non-diagonal matrix with eigenvalues* $1, 2, 3, 4$. (Hint: Exercise 18.6.)

10. A three-page web has modified link matrix $M = \frac{1}{6} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$.

Find the importance vector.