Linear Algebra
Chapter 1
Solutions to Exercises

Exercise 1.1. *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Compute AB, BA. You should get different answers.

Solution:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix}.$$

Exercise 1.2. *Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as in 1.1, and suppose $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a+c = d$, $2c = 3b$. Show that $AB = BA$.*

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix},$$

while

$$BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}.$$

Using the two given equations $a + c = d$, $2c = 3b$, one sees that $AB = BA$.

Exercise 1.3. *Compute*

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

You should get the same answer both times.

Solution: Both products are I.

Exercise 1.4. *Using the trig identities*

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi, \quad \sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta,$$

show that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \sin\phi\cos\theta & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}, \quad \blacksquare$$

using the trig identities.

Exercise 1.5. *The powers of a matrix are computed as* $A^2 = AA$*,* $A^3 = AAA$*, etc. Let* $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$*. Compute* $A^2,\ A^3,\ A^4,\ A^{100}$*.*

Solution:

$$A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^3 = I, \quad A^4 = A, \quad A^{100} = AA^{99} = A(A^3)^{33} = A(I)^{33} = A.$$

Exercise 1.6. *Find the inverses of*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Solution:

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}.$$

Exercise 1.7. *Let*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Show that

$$A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

Solution: First note that $\det A = \sin^2\theta + \cos^2\theta = 1$, so

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

To get the desired expression for A^{-1}, recall that $\cos\theta$ is an even function and $\sin\theta$ is an odd function.

Exercise 1.8. *There are only two numbers that are their own inverses, namely 1 and -1. Find five matrices that are their own inverses.*

Solution: There are infinitely many such matrices, besides the obvious ones like

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

For example, take any nonzero number b, and consider the matrices

$$\begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & b \\ \frac{1}{b} & 0 \end{bmatrix}.$$