LINEAR ALGEBRA
CHAPTER 10 SOLUTIONS TO EXERCISES

Exercise 10.1 For each of the following matrices A, find the canonical form, the eigenvalues, r and θ.

(a) $A = \begin{bmatrix} 7 & -10 \\ 4 & -5 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 5 & -5 \\ 2 & -1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Solutions:

(a) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $\lambda = 1 \pm 2i$, $r = \sqrt{5}$, $\theta = \arctan 2$.

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, $\lambda = 2 \pm i$, $r = \sqrt{5}$, $\theta = \arctan \frac{1}{2}$.

(c) $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $\lambda = \pm 2i$, $r = 2$, $\theta = \frac{\pi}{2}$.

(d)A is its own canonical form, $\lambda = a \pm bi$, $r = \sqrt{a^2 + b^2}$, $\theta = \arctan \frac{b}{a}$.

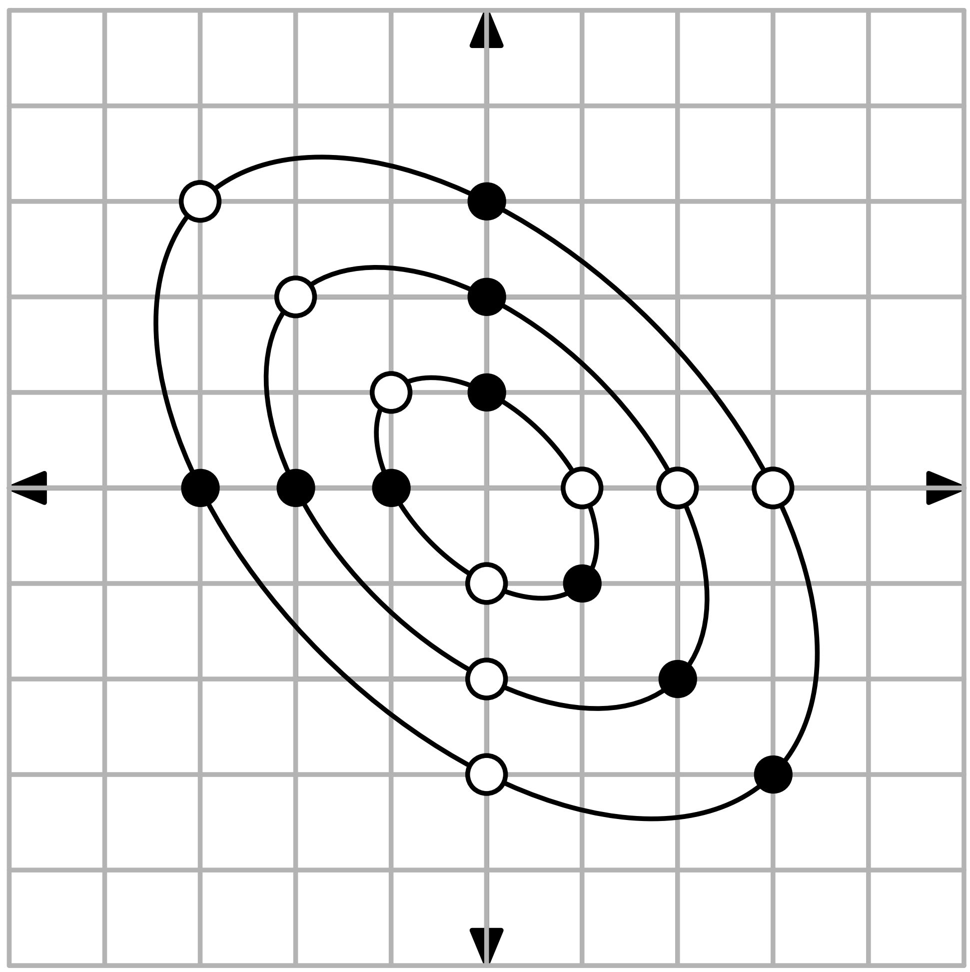
Exercise 10.2 Find the eigenvalues of

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Solution: $\lambda = \cos\theta \pm i \sin\theta = e^{\pm i\theta}$.

Exercise 10.3 Take the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ from the example above. Plot the vectors $\beta_1, A\beta_1, A^2\beta_1$ with a hollow dot ($\circ$). Plot $\beta_2, A\beta_2, A^2\beta_2$ with a solid dot ($\bullet$). Connect the dots with an ellipse (which will be slanted). Repeat this starting with $t\beta_1,\ t\beta_2$ for various scalars t. For each t you will get an ellipse, and these ellipses will concentrically fill up the plane. The matrix A is a flattened rotation around these ellipses.

Solutions:



Exercise 10.4 In this exercise we find the equations of the ellipses in exercise 10.2. The idea is very similar to exercise 6.3. Let A be as in exercise 10.2, and let $\lambda, \bar{\lambda}$ be the eigenvalues of A. Let

$$f(x, y) = (y - \lambda x)(y - \bar{\lambda} x).$$

- "(a)" Show that $f(x, y) = x^2 + xy + y^2$.
- "(b)" Given a point (x, y), define (x', y') by the equation

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

  Show that

$$f(x', y') = f(x, y).$$

- "(c)" If you fix a number $k > 0$, the graph of $x^2 + xy + y^2 = k$ is an ellipse C_k. Show that if (x, y) is on the ellipse C_k, then (x', y') is on the same ellipse C_k.

Solutions: see exercise 6.3

Exercise 10.5 Suppose the glucose/insulin equations are

$$\begin{aligned} x_{n+1} &= (1-c)x_n - 0.4y_n \\ y_{n+1} &= cx_n + 0.9y_n, \end{aligned}$$

where $0 \le c \le 1$. Here c is the proportion of glucose absorbed, and cx_n is also the amount of insulin production stimulated by the presence of x_n units of glucose. For what values of c will the system oscillate? If it oscillates, is it stable or unstable?

Solutions: Compute $\Delta = 1.8c - c^2 - .01$. To oscillate, ie, have complex eigenvalues, we must have $\Delta > 0$. The only zero of $1.8c - c^2 - .01$ in the interval $[0,1]$ is $.9 - \sqrt{.8} = .00557\ldots$, so the system oscillates when $0 \le c < .00557\ldots$. Since $\det(A) = .9 - .5c$, we have $\det A \le .9$, so $r = \sqrt{\det(A)}$ is always < 1, meaning the system is stable.

Exercise 10.6 If the spiral goes into the origin, then we say the origin is **stable.** *If the spiral goes away from the origin, we say the origin is* **unstable.** *Look at each of the following matrices, and without writing any calculations at all, say if the origin is stable or unstable. (You are allowed to do one calculation (per matrix) in your head, but you are not allowed to write it down.)*

$$\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{3} & -1 \\ 2 & -\frac{1}{3} \end{bmatrix}.$$

Solutions: *unstable, stable, unstable.*

Exercise 10.7 *Can you find a matrix with all positive entries, and having complex eigenvalues?*

Solution: *No. the quantity Δ can be written as $\Delta = -(a-d)^2 - 4bc$. This must be positive to have complex eigenvalues, so $-4bc$ must be positive, so b and c must have opposite signs.*

Exercise 10.8 *Prove equation (10b) by multiplying the matrices.*

Solutions: *We have*

$$B = \frac{1}{2}\begin{bmatrix} 2b & 0 \\ d-a & -\sqrt{\Delta} \end{bmatrix}, \qquad B^{-1} = -\frac{1}{b\sqrt{\Delta}}\begin{bmatrix} -\sqrt{\Delta} & 0 \\ a-d & 2b \end{bmatrix},$$

Multiplying, we get

$$B^{-1}AB = \frac{-1}{2b\sqrt{\Delta}}\begin{bmatrix} -b\sqrt{\Delta}(a+d) & b\Delta \\ b[2a(a-d)+4bc-(a-d)^2+2d(d-a)] & -b\sqrt{\Delta}(a+d) \end{bmatrix}$$

Simplifying, and recalling that $\Delta = -4bc - (a-d)^2$, you get

$$B^{-1}AB\frac{1}{2}\begin{bmatrix} \operatorname{tr} A & -\sqrt{\Delta} \\ \sqrt{\Delta} & \operatorname{tr} A \end{bmatrix}.$$