

### Chapter 13. Solutions to Exercises

**Exercise 13.1** Let  $\mathbf{u} = (x, y, z)$ ,  $\mathbf{v} = (x', y', z')$ . Use the dot product to verify that  $\langle \mathbf{u}, \mathbf{u} \times \mathbf{v} \rangle = 0$ .

*Solution:* We use the dot product to test for orthogonality. Let

$$\mathbf{u} = (x, y, z), \quad \mathbf{v} = (x', y', z').$$

Then

$$\mathbf{u} \times \mathbf{v} = (yz' - zy', zx' - xz', xy' - yx'),$$

so

$$\langle \mathbf{u}, \mathbf{u} \times \mathbf{v} \rangle = x(yz' - zy') + y(zx' - xz') + z(xy' - yx') = 0,$$

since each term appears twice with opposite signs. ■

**Exercise 13.2** Let  $\mathbf{u} = (1, -2, 1)$ ,  $\mathbf{v} = (2, 1, 1)$ . Find  $\langle \mathbf{u}, \mathbf{v} \rangle$  and  $\mathbf{u} \times \mathbf{v}$  and the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

*Solution:* We have

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2 - 2 + 1 = 1, \quad \mathbf{u} \times \mathbf{v} = (-2 - 1, 2 - 1, 1 + 4) = (-3, 1, 5).$$

The angle between  $\mathbf{u}$  and  $\mathbf{v}$  is

$$\arccos \left( \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u}| |\mathbf{v}|} \right) = \arccos \frac{1}{6}.$$

**Exercise 13.3** Find a nonzero vector on the line of intersection of the two planes

$$x - 2y + z = 0, \quad 2x + y + z = 0.$$

*Solution:* The cross-product of the normal vectors is on the line of intersection. These normal vectors are  $\mathbf{u}$  and  $\mathbf{v}$  from the previous problem. So  $\mathbf{u} \times \mathbf{v} = (-3, 1, 5)$  is a vector on the line of intersection. To check this answer, note that  $(-3, 1, 5)$  satisfies both plane equations.

**Exercise 13.4** Find the angle between the diagonal of a cube and one of the edges of the cube. It doesn't matter which cube you use, but you may wish to use the cube whose vertices are the eight points  $(x, y, z)$  with  $x, y, z$  either 0 or 1.

*Solution:* The diagonal vector is  $\mathbf{u} = (1, 1, 1)$  and an edge vector is  $\mathbf{e}_1$ . The angle is

$$\arccos\left(\frac{\langle \mathbf{u}, \mathbf{e}_1 \rangle}{|\mathbf{u}||\mathbf{e}_1|}\right) = \arccos\frac{1}{\sqrt{3}}.$$

**Exercise 13.5** Take two vectors

$$\mathbf{u} = (x, y, z) \quad \text{and} \quad \mathbf{v} = (x', y', z').$$

Since  $\mathbf{u} \times \mathbf{v}$  is a vector, its length  $|\mathbf{u} \times \mathbf{v}|$  is a number. The dot product  $\langle \mathbf{u}, \mathbf{v} \rangle$  is also a number. The relation between these numbers is:

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2.$$

Prove this equation by expanding both sides in terms of  $x, y, z, x', y', z'$ .

*Solution:* First expand the left side:

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle^2 + |\mathbf{u} \times \mathbf{v}|^2 &= (xx' + yy' + zz')^2 + (yz' - zy')^2 + (zx' - xz')^2 + (xy' - yx')^2 \\ &= [(xx')^2 + (yy')^2 + (zz')^2 + 2xx'yy' + 2xx'zz' + 2yy'zz'] \\ &\quad + [(yz')^2 - 2yz'zy' + (zy')^2 \\ &\quad + (zx')^2 - 2zx'xz' + (xz')^2 \\ &\quad + (xy')^2 - 2xy'yx' + (yx')^2] \\ &= (xx')^2 + (yy')^2 + (zz')^2 + (yz')^2 + (zy')^2 + (zx')^2 + (xz')^2 + (xy')^2 + (yx')^2. \end{aligned}$$

Next, expand the right side:

$$\begin{aligned} |\mathbf{u}|^2|\mathbf{v}|^2 &= (x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) \\ &= x^2x'^2 + x^2y'^2 + x^2z'^2 + y^2x'^2 + y^2y'^2 + y^2z'^2 + z^2x'^2 + z^2y'^2 + z^2z'^2 \\ &= (xx')^2 + (yy')^2 + (zz')^2 + (yz')^2 + (zy')^2 + (zx')^2 + (xz')^2 + (xy')^2 + (yx')^2. \end{aligned}$$

So the two sides are equal. ■

**Exercise 13.6** Use the equation  $\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u}||\mathbf{v}| \cos \theta$  and Exercise 13.5 to prove the length formula

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

*Solution:* From Exercise 13.5 we have

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2 - \langle \mathbf{u}, \mathbf{v} \rangle^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - |\mathbf{u}|^2|\mathbf{v}|^2 \cos^2 \theta \\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1 - \cos^2 \theta) \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 \sin^2 \theta. \end{aligned}$$

Recall that  $\theta$  is the acute angle between  $\mathbf{u}$  and  $\mathbf{v}$ , so that  $\sin \theta \geq 0$ . Taking the square-root of both sides gives the result. ■

**Exercise 13.7** Consider the tetrahedron with vertices

$$A = (a, 0, 0), \quad B = (0, b, 0), \quad C = (0, 0, c), \quad \text{and} \quad O = (0, 0, 0).$$

For any three points  $X, Y, Z$  in space, write  $[XYZ]$  for the area of the triangle with vertices  $X, Y, Z$ . Use the Equation:  $\text{Area}(\Delta) = \frac{1}{2}|\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}|$  for the area of a triangle  $\Delta$  with vertices  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  (notation as in text) to prove that

$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2.$$

This is the **three dimensional Pythagorean theorem**.

*Solution:* From the area formula, we have

$$[ABC]^2 = \frac{1}{4}|A \times B + B \times C + C \times A|^2.$$

We compute

$$\begin{aligned} A \times B &= (a, 0, 0) \times (0, b, 0) = (0, 0, ab), \\ B \times C &= (0, b, 0) \times (0, 0, c) = (bc, 0, 0), \\ C \times A &= (0, 0, c) \times (a, 0, 0) = (0, ca, 0), \end{aligned}$$

so

$$\begin{aligned} [ABC]^2 &= \frac{1}{4}[(ab)^2 + (bc)^2 + (ca)^2] \\ &= \left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ca}{2}\right)^2 \\ &= [OAB]^2 + [OBC]^2 + [OCA]^2. \end{aligned}$$