Chapter 14. Three-by-Three Matrices and Determinants

Solutions to Exercises

Exercise 14.1 *Find the matrices R_x, R_y, R_z that rotate by π about the x, y, z axes respectively.*

Solution:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_z = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 14.2 *Find the matrices R_{xy}, R_{yz}, R_{zx} that reflect about the xy, yz, zx planes, respectively.*

Solution:

$$R_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{zx} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 14.3 *Find the determinants of the following matrices. Do not use a calculator and show all of your work.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad \det A = -2$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \quad \det A = 0$

(c) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \quad \det A = abc$

(d) $A = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix} \quad \det A = abc$

(e) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \det A = 0$

(f) $A = \begin{bmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{bmatrix} \quad \det A = 0.$

Exercise 14.4 *Make up two vectors* $\mathbf{u},\ \mathbf{v}$ *in* $\mathbb{R}^3$*, choose two scalars* a *and* b*, and let* $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$*. Compute the determinant of the matrix* A *whose columns are* $\mathbf{u}, \mathbf{v}, \mathbf{w}$*.* The answer does not depend on your choices.

Answer: For example, if $\mathbf{u} = (1,0,1),\ \mathbf{v} = (0,1,-1),\ \mathbf{w} = 2\mathbf{u} + 3\mathbf{v} = (2,3,-1)$, then

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = 1(-1+3) - 0 + 1(-2) = 0.$$

No matter what choices you make for $\mathbf{u}$ and $\mathbf{v}$, the determinant will be zero.

Exercise 14.5 *Draw the parallelopiped spanned by* $\mathbf{e}_1,\ \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$*. Then compute its volume.*

Solution: In lieu of an online picture, a description will have to suffice: The parallelopiped has vertices at

$$(0,0,0),\ (1,0,0),\ (1,1,0),\ (1,1,1),\ (2,1,0),\ (2,1,1),\ (2,2,1),\ (3,2,1).$$

The volume is

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

Exercise 14.6 *Find the inverses of the following matrices. Do not use a calculator and show all of your work.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{9} \begin{bmatrix} -14 & -11 & -3 \\ 10 & -13 & 6 \\ 1 & 5 & -3 \end{bmatrix}.$

Exercise 14.7 *There are six ways to permute the numbers* $1, 2, 3$*. For each permutation* σ*, let* A_σ *be the matrix which sends* $\mathbf{e}_i$ *to* $\mathbf{e}_{\sigma(i)}$.

(a) *Write down the six matrices* A_σ *and compute their determinants.*

(b) *Each permutation* σ *corresponds to the term* $a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$ *in the expanded determinant. What is the relation between the sign of this term and* $\det(A_\sigma)$*?*

(c) *Each permutation corresponds to a symmetry of an equilateral triangle with vertices labelled* $1, 2, 3$*. Three of the six symmetries are rotations of the triangle, and the other three are reflections. What is the relation between this dichotomy and* $\det(A_\sigma)$*?*

Solutions:

(a) Label the permutations according to their effect on the sequence 123. For example 312 indicates the permutation that sends 1 to 3, 2 to 1, 3 to 2. Then

$$123 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 312 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad 231 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$213 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 321 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad 132 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

The determinants in the first row are $+1$ and those in the second row are -1.

(b) The sign of the term $a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$ is equal to $\det(A_\sigma)$.

(c) The σs for which $\det(A_\sigma) = +1$ are rotations of the triangle. Those σ for which $\det(A_\sigma) = -1$ are flips of the triangle.