Chapter 15. Solutions to Exercises

Exercise 15.1 Find the kernels of the following matrices

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ **(c)** $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **(e)** $A = \begin{bmatrix} 8 & 15 & 7 \\ 2 & 9 & 5 \\ -10 & -3 & 1 \end{bmatrix}$ **(f)** $A = \begin{bmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{bmatrix}$.

Solutions:

(a) $\ker A = \mathbf{0}$ is trivial, **(b)** $\ker A$ is the plane $x + y + z = 0$,
(c) $\ker A$ is the line through $(1, 1, 1)$, **(d)** $\ker A$ is the line through $\mathbf{e}_1$,
(e) $\ker A$ is the line through $(6, -13, 21)$,
(f) $\ker A$ is the line through $(-1, 2, -1)$, no matter what x is.

Exercise 15.2 Let

$$A_t = \begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{bmatrix}.$$

(a) Compute $\det A_t$, and factor it. (Note that $\det A_t$ is a polynomial in t.)

(b) For each root $t = \lambda$ of $\det A_t$, find $\ker A_\lambda$.

(c) What is the relation between the dimension of $\ker A_\lambda$, and the multiplicity of λ as a root of $\det A_t$?

Solutions:

(a) $\det A_t = (t-1)^2(t+2)$.
(b) $\ker A_1$ is the plane $x + y + z = 0$, $\ker A_{-2}$ is the line through (1,1,1).
(c) The dimension of $\ker A_\lambda$ equals the multiplicity of λ as a root.

Exercise 15.3 Let $\mathbf{x} = (a_1, a_2, a_3)$ and $\mathbf{y} = (b_1, b_2, b_3)$ be two nonzero vectors, and let A be the matrix whose ij entry is $a_{ij} = a_i b_j$. Find the kernel of A.

Solution: $\ker A$ is the plane perpendicular to $\mathbf{y}$, with equation $b_1x+b_2y+b_3z = 0$.

Exercise 15.4 Let A and B be the following rotation matrices

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}.$$

It turns out that AB is again a rotation matrix. Find the axis of rotation of AB. Simplify the answer as much as you can.

Hint: As in Example 5, the axis is the kernel of $AB - I$.

Solution: Assume that $A \neq I \neq B$. We compute

$$AB - I = \begin{bmatrix} \cos\alpha - 1 & -\sin\alpha\cos\beta & \sin\alpha\cos\beta \\ \sin\alpha & \cos\alpha\cos\beta - 1 & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta - 1 \end{bmatrix}.$$

Since $A \neq I$ we have $\sin\alpha \neq 0$ so the last two rows of $AB - I$ are non-proportional. Take their cross product. The axis of AB is the line through

$$\mathbf{u} = ((1+\cos\alpha)(1-\cos\beta),\ \sin\alpha(1-\cos\beta),\ \sin\alpha\sin\beta)\,.$$

Any nonzero vector on the axis is an acceptable answer. However, since $A \neq I \neq B$ we can divide by $\sin\alpha(1-\cos\beta)$, which gives a simpler vector on the axis of AB:

$$\left(\frac{1+\cos\alpha}{\sin\alpha},\ 1\ ,\ \frac{\sin\beta}{1-\cos\beta}\right) = \left(\frac{1+\cos\alpha}{\sin\alpha},\ 1\ ,\ \frac{1+\cos\beta}{\sin\beta}\right).$$