Chapter 16. Solutions to Exercises

Exercise 16.1 *Find the eigenvalues and eigenvectors of the following matrices.*

(a) $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **(c)** $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1/2 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -2/5 & 1/5 & 0 \end{bmatrix}$ **(e)** $A = \begin{bmatrix} -3 & -3 & 15 \\ -14 & 16 & 10 \\ -7 & -1 & 23 \end{bmatrix}$

Solutions: Unless otherwise specified, we give the eigenvectors and eigenvalues in the same order. When the eigenspace is a line we give a nonzero vector on the line; your answers may differ from those below by a scalar.

(a) Eigenvalues: $1, 2, 3$. Eigenvectors: $(2, 1, 1), (1, 1, 1), (1, 1, 2)$.

(b) Eigenvalues: $0, 0, 0$. Eigenvectors: $E(0)$ is the plane $y + z = 0$.

(c) Eigenvalues: $1, 0, -1$. Eigenvectors: $(2, 1, 1), (1, 1, 1), (1, 1, 2)$

(d) Eigenvalues: $0, 0, 0$. Eigenvector: (-1,-2,1)

(e) Eigenvalues: $18, 18, 0$. Eigenvectors: $E(18)$ is the plane $7x + y - 5z = 0$, $E(0)$ is line thru $(3, 2, 1)$.

Exercise 16.2 *Let A be a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Express the coefficients of the characteristic polynomial $P_A(x)$ in terms of $\lambda_1, \lambda_2, \lambda_3$.*

Solution: The coefficients of $P_A(x)$ are the various sums of diagonal determinants:

$$P_A(x) = x^3 - \operatorname{tr}(A)x^2 + (\det A_{11} + \det A_{22} + \det A_{33})x - \det(A). \quad (1)$$

But also, since the roots of $P_A(x)$ are $\lambda_1, \lambda_2, \lambda_3$, we have

$$P_A(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

Multiplying this out, we get

$$P_A(x) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)x - \lambda_1\lambda_2\lambda_3. \quad (2)$$

Comparing the coefficients in (1) and (2), we find that

$$\begin{aligned} \mathrm{tr}(A) &= \lambda_1 + \lambda_2 + \lambda_3 \\ \det A_{11} + \det A_{22} + \det A_{33} &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ \det(A) &= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

Exercise 16.3 *A 3×3 migration matrix A has all entries between 0 and 1, and the sum of each column is 1. Thus A looks like*

$$A = \begin{bmatrix} 1-x-y & u & v \\ x & 1-u-z & w \\ y & z & 1-v-w \end{bmatrix}.$$

Let P denote the initial population. Find the stable population distribution.
Hint: The stable population vector is fixed by A. Your answer will be a vector involving x, y, z, w, u, v.

Solutions:

We need a vector on the line

$$\ker[A - I] = \ker \begin{bmatrix} -x-y & u & v \\ x & -u-z & w \\ y & z & -v-w \end{bmatrix}$$

whose sum of coordinates is P. Take the cross-product of two rows, divide by the coordinate sum and multiply by P. Using the first two rows, we get

$$\frac{P(uw + zv + uv\ ,\ xv + xw + yw\ ,\ xz + yu + yz)}{(uw + zv + uv + xv + xw + yw + xz + yu + yz)}.$$