## Chapter 17. Orthogonal Matrices and Symmetries of Space
## Solutions to Exercises

We write $\mathbb{R}\mathbf{u}$ to denote the line through a nonzero vector $\mathbf{u}$. The angle $\theta$ of rotation is measured as $\mathbf{u}$ points at you.

**Exercise 17.1** *Find the axis and angle of rotation of the following matrices.*

$$\textbf{(a)}\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad \textbf{(b)}\ B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \textbf{(c)}\ C = \tfrac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

**Solutions:**

**(a)** The axis is $\ker(A - I) = \mathbb{R}(1, 1, -1)$, the angle is $\theta = \arccos(\frac{1}{2}) = 2\pi/3$.

**(b)** The axis is $\mathbb{R}(1, 1, 0)$, the angle is $\pi$.

**(c)** The axis is $\mathbb{R}(1, 1, 1)$, the angle is $\pi$.

**Exercise 17.2** *Find the axis and the cosine of the angle of rotation for*

$$A = \frac{1}{4}\begin{bmatrix} 2\sqrt{3} & -\sqrt{3} & 1 \\ 2 & 3 & -\sqrt{3} \\ 0 & 2 & 2\sqrt{3} \end{bmatrix}.$$

**Solution:**

The axis is $\mathbb{R}(1, 2 - \sqrt{3}, 1)$, and the angle satisfies $\cos\theta = \frac{4\sqrt{3}-1}{8}$.

**Exercise 17.3** *Show that for any number t, the matrix*

$$A_t = \frac{1}{1 + t + t^2}\begin{bmatrix} -t & t + t^2 & 1 + t \\ 1 + t & -t & t + t^2 \\ t + t^2 & 1 + t & -t \end{bmatrix}$$

*is a rotation, find its axis of rotation, and the cosine of the angle of rotation.*

Comments: (i) You can use the continuity argument to easily compute $\det A_t$.

(ii) This formula shows how to write down lots of rotations with rational entries: just take $t$ to be an integer.

**Solution:** First check that $A_t A_t^T = I$. To see that $\det(A_t) = +1$, note that

$$A_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

so that $\det(A_0) = +1$. By the continuity argument we then have $\det(A) = +1$ for all $t$. (Note that you don't have to actually compute $\det(A_t)$!) Then compute

$$A_t - I = \frac{1+t}{1+t+t^2}\begin{bmatrix} 1+t & -t & -1 \\ -1 & 1+t & -t \\ -t & -1 & 1+t \end{bmatrix}.$$

You can forget the scalar in front of this matrix when computing its kernel. The cross product of the first two rows is $(1+t+t^2)(1,1,1)$, so the axis is

$$\ker(A_t - I) = \mathbb{R}(1,1,1).$$

$$\cos\theta = \frac{1}{2}\left[\frac{-3t}{1+t+t^2} - 1\right] = -\frac{1}{2}\left[\frac{1+4t+t^2}{1+t+t^2}\right].$$

Comment: This matrix $A_t$ was found as follows. We try to find three vectors of the form $(a,b,c)$, $(c,a,b)$, $(b,c,a)$ which are orthogonal. This leads to the equation $ac+ab+bc = 0$, or $a = -bc/(b+c)$. Let $t = c/b$. Then $a = -tb/(1+t)$, and we want this to be an integer, so why not take $b = 1+t$. Then $a = -t$ and $c = tb = t+t^2$. The lucky break is that $a^2+b^2+c^2 = (1+t+t^2)^2$, so you don't get square roots when you scale to get unit vectors.

**Exercise 17.4** *Find the matrix that rotates about the axis through* $\mathbf{u} = \frac{1}{3}(-1,2,2)$ *by the angle* $\pi/4$*, measured as* $\mathbf{u}$ *points toward you.*

**Solution:** Use the formula $A = I + (\sin\theta)U + (1-\cos\theta)U^2$, where

$$U = \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}, \qquad U^2 = \frac{1}{9}\begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix}$$

to get

$$\begin{aligned} A &= I + \frac{\sqrt{2}}{6}\begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} + \frac{2-\sqrt{2}}{18}\begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix} \\ &= \frac{1}{18}\begin{bmatrix} 2+8\sqrt{2} & -4-4\sqrt{2} & -4+8\sqrt{2} \\ -4+8\sqrt{2} & 8+5\sqrt{2} & 8-\sqrt{2} \\ -4-4\sqrt{2} & 8-7\sqrt{2} & 2+5\sqrt{2} \end{bmatrix}. \end{aligned}$$

**Exercise 17.5** *What rotation matrices do you get from the formula in the solution to the previous exercise if $c = \pm 1$?*

**Solution:** (not graded) Here we have $a = b = 0$, so

$$A = I + \sin\theta \begin{bmatrix} 0 & \mp 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1 - \cos\theta) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \pm\sin\theta & 0 \\ \pm\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which makes sense, since our axis is the $z$-axis.

**Exercise 17.6** *Formula $A = I + (\sin\theta)U + (1 - \cos\theta)U^2$ for a specified rotation also works in two dimensions. Verify this by finding a $2 \times 2$ matrix $U$ such that*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = I + (\sin\theta)U + (1 - \cos\theta)U^2.$$

**Solution:** $U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

**Exercise 17.7** *Let $R_{\mathbf{u}}$ be the reflection matrix*

$$R_{\mathbf{u}} = \begin{bmatrix} 1 - 2a^2 & -2ab & -2ac \\ -2ab & 1 - 2b^2 & -2bc \\ -2ac & -2bc & 1 - 2c^2 \end{bmatrix},$$

*about the plane $ax + by + cz = 0$, where $\mathbf{u} = (a, b, c)$ is a unit vector. Show that the matrix $A = -R_{\mathbf{u}}$ is a rotation and determine its axis and angle of rotation.* Hint: this can be done without writing down any matrices, and very little calculation.

**Solution:** First, $-I$ and $R_{\mathbf{u}}$ are orthogonal matrices so $A = -R_{\mathbf{u}}$ is orthogonal. Since $\det(-I) = -1$, we have $\det(A) = -\det(R_{\mathbf{u}}) = +1$, so $A$ is a rotation. Since $R_{\mathbf{u}}$ sends $\mathbf{u}$ to $-\mathbf{u}$, we have $A\mathbf{u} = \mathbf{u}$, so the axis of $A$ is the line through $\mathbf{u}$. Finally, $R_{\mathbf{u}}^2 = I$, so $A^2 = I$. This means the angle of rotation is $\pi$.

**Exercise 17.8** *In two dimensions, an orthogonal matrix is either a rotation or a reflection. This is no longer true in three dimensions.*

**(a)** *Show that the matrix*

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

*is an orthogonal matrix that is neither a reflection nor a rotation.*

**(b)** *In three dimensions, we have $\det(-I) = -1$, so any orthogonal matrix with determinant $-1$ is of the form $-A$, where $A$ is a rotation. For what kinds of rotations $A$ is $-A$ a reflection?*

**Solutions:**

**(a)** First, note $\det A = -1$, so $A$ is not a rotation. Then check that $A^2 \neq I$, so $A$ is not a reflection either.

**(b)** If $A$ is a rotation and $-A$ is a reflection then $A^2 = (-A)^2 = I$, so $A$ must be a rotation by $\pi$, as we saw in exercise 17.7. Conversely, if $A$ is rotation by $\pi$ then the eigenvalues of $A$ are $1, e^{i\pi}, e^{-i\pi} = 1, -1, -1$. Hence $-A$ has eigenvalues $-1, 1, 1$, so $-A$ is a reflection about its 1-eigenspace. In summary: The rotations for which $-A$ is a reflection are precisely the rotations by $\pi$.

**Exercise 17.9** *How many orthogonal matrices have all their entries integers? How many of these are rotations? What are the axes of these rotations? The connection between this problem and the octahedron will be discussed in class.*

**Solutions:** Since the columns are orthonormal, each column has exactly one nonzero entry, which must be $\pm 1$. There are 3 choices for the location of $\pm 1$ in the first column, then 2 choices in the second column, and 1 choice in the third column, so the number of integer orthogonal matrices is

$$(2 \times 3)(2 \times 2)(2 \times 1) = 48.$$

Changing all the signs changes the determinant (since 3 is odd) so half of these, i.e., 24 are rotations. The axes are the thirteen lines through the vectors

$$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_1 \pm \mathbf{e}_2,\ \mathbf{e}_1 \pm \mathbf{e}_3,\ \mathbf{e}_2 \pm \mathbf{e}_3,\ (1,1,1),\ (1,1,-1),\ (1,-1,1),\ (-1,1,1).$$

Comment: The six possible columns of these matrices are the vertices of an octahedron. The 48 matrices are all the symmetries of the octahedron.