

Chapter 18. A Tour of Four Dimensions
Solutions to Exercises

Exercise 18.1 Find the ranks and the kernels of the following matrices.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Solutions:

$$\text{rank}(A) = 3, \quad \ker A = \mathbb{R}(1, 0, 0, 0).$$

$\text{rank}(B) = 2$, $\ker B$ is the intersection of any two row hyperplanes, but is most simply given by the intersection $x + y = z + w = 0$.

Exercise 18.2 Find the inverses of the following matrices.

$$A = \begin{bmatrix} 0 & 0 & 0 & d \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}, \quad abcd \neq 0, \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Solutions:

$$A^{-1} = \begin{bmatrix} 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & c^{-1} \\ d^{-1} & 0 & 0 & 0 \end{bmatrix}, \quad B^{-1} = \frac{1}{3} \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

Exercise 18.3 Use the cross product to find the equation of the hyperplane spanned by the vectors

$$\mathbf{u} = (1, 1, 1, 1), \quad \mathbf{v} = (1, 2, 1, 2), \quad \mathbf{w} = (4, 3, 2, 1).$$

Solution: The cross product is $\mathbf{u} \times \mathbf{v} \times \mathbf{w} = (2, -2, -2, 2)$, so the equation is

$$x - y - z + w = 0.$$

Exercise 18.4 Find a nonzero 3-minor and use the cross-product to find the kernel of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}.$$

Solutions:

The top three rows $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ have nonzero 3-minors. We compute

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} = 0, \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix} = -1 \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix} = -2 \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix} = -1.$$

Using the cross-product formula we get

$$\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 = (0, 1, -2, 1) \quad \text{and} \quad \ker A = \mathbb{R}(0, 1, -2, 1).$$

Comment: there are different choices for a row of 3-minors, but they should all lead to the line through $(0, 1, -2, 1)$.

Exercise 18.5 Let P be the plane given as the intersection of two hyperplanes

$$ax + by + cz + dw = 0, \quad a'x + b'y + c'z + d'w = 0,$$

with non-proportional normal vectors $\mathbf{n} = (a, b, c, d)$ and $\mathbf{n}' = (a', b', c', d')$. Let N be the plane spanned by the vectors \mathbf{n} and \mathbf{n}' . What is the geometric relationship between the planes P and N ? Justify your answer. Hint: Dot product.

Solution:

A vector $\mathbf{v} = (x, y, z, w)$ is on P exactly when it is orthogonal to both \mathbf{n}, \mathbf{n}' , which is equivalent to being orthogonal to N . Hence the planes P and N are orthogonal.

Exercise 18.6 The characteristic polynomial $P_A(x) = \det(xI - A)$ has the same pattern as before: The coefficient of x^{4-k} in $P_A(x)$ is $(-1)^k$ times the sum of the diagonal k -minors of A . Compute $P_A(x)$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & d \\ 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \end{bmatrix}.$$

Solution: $P_A(x) = x^4 - ax^3 - bx^2 - cx - d$.

Exercise 18.7 The eigenvalues of A are the roots λ of the characteristic polynomial $P_A(x)$ and the λ -eigenspace is $E(\lambda) = \ker(\lambda I - A)$. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

You will find that ± 1 are two of the eigenvalues. Compute $E(1)$ and $E(-1)$.

Solutions:

$$P_A(x) = x^4 - 1, \quad E(1) = \mathbb{R}(1, 1, 1, 1), \quad E(-1) = \mathbb{R}(1, -1, 1, -1).$$