

Chapter 19. General Matrices
Solutions to Exercises

Exercise 19.1 Compute the product, if can be computed

$$(a) \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solutions:

(a) The product cannot be computed.

$$(b) \begin{bmatrix} 3 & 3 \\ 2 & 5 \\ 1 & 1 \end{bmatrix}$$

$$(c) \begin{bmatrix} 5 & 2 & 3 \\ 13 & 6 & 7 \end{bmatrix}$$

Exercise 19.2 Let $A = [a_{ij}]$ be an arbitrary $n \times n$ matrix, and let D be a diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

a) What are the ij entries of DA and AD ?

b) Suppose the d_i 's are distinct, and that $AD = DA$. What can you say about A ?

Solutions:

$$(a) (DA)_{ij} = d_i a_{ij}, \quad (AD)_{ij} = d_j a_{ij}.$$

(b) If $d_i a_{ij} = d_j a_{ij}$ and $i \neq j$ then $a_{ij} = 0$. So $AD = DA$ exactly when A is a diagonal matrix. In other words, the only matrices which commute with a diagonal matrix having distinct entries are other diagonal matrices.

Exercise 19.3 M2DCs, hence M3DCs like us, can see images of the hypercube in \mathbb{R}^4 , by using a matrix $A : \mathbb{R}^4 \rightarrow \mathbb{R}^2$. First of all, the hypercube has 16 corners (x_1, x_2, x_3, x_4) , $x_i = 0$ or 1. Two corners have an edge between them if they differ in only one component. For example, there is an edge between (1011) and (1001), but no edge between (1011) and (0111).

a) Choose two random vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in \mathbb{R}^2 , and let

$$A = \begin{bmatrix} 1 & 0 & u_1 & v_1 \\ 0 & 1 & u_2 & v_2 \end{bmatrix}.$$

(To get a good view, make sure no u_i or v_i is 0 or 1.)

b) Plot the image under A of the 16 corners of the hypercube. You will plot 16 points in the plane \mathbb{R}^2 , but label them by the coordinates of the corresponding points in \mathbb{R}^4 , to keep track of them. For example, the corner (0101) goes via A to $(0, 1) + (v_1, v_2) = (v_1, 1 + v_2)$. Label this point by (0101).

c) Connect pairs of your 16 points in \mathbb{R}^2 by line segments if their 4-tuple labels differ in only one component.

d) Behold the hypercube!

Solution: Of course, the picture depends on your choice of \mathbf{u} and \mathbf{v} . In all cases, you should see two rectangular cubes, with corresponding vertices connected by parallel lines.