Linear Algebra Notes

Chapter 2

SOLUTIONS TO EXERCISES

Exercise 2.1. *Show that scalar matrices commute with all other matrices.*

Solution:

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax & ay \\ az & aw \end{bmatrix},$$

while

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} xa & ya \\ za & wa \end{bmatrix}.$$

These are the same.

Exercise 2.2. *Suppose $a \neq d$. Show that the only matrices commuting with $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ are diagonal matrices.*

Solution: Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$AB = \begin{bmatrix} ax & ay \\ dz & dw \end{bmatrix}, \quad \text{and} \quad BA = \begin{bmatrix} ax & dy \\ az & dw \end{bmatrix}.$$

We'll have $AB = BA$ exactly when $dy = ay$ and $dz = az$. Since $a \neq d$, this can only happen if $y = z = 0$. This means B is diagonal.

Exercise 2.3. *Show that*

$$(AB)^T = (B^T)(A^T).$$

Note the reversal of order in the products.

Solution: Say

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix},$$

so

$$(AB)^T = \begin{bmatrix} aa' + bc' & ca' + dc' \\ ab' + bd' & cb' + dd' \end{bmatrix}.$$

On the other hand,

$$(B^T)(A^T) = \begin{bmatrix} a' & c' \\ b' & d' \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a'a + c'b & a'c + c'd \\ b'a + d'b & b'c + d'd \end{bmatrix}.$$

We see that $(AB)^T$ and $(B^T)(A^T)$ are the same.

Exercise 2.4.
a) Show that the product of two rotation matrices is again a rotation matrix.
b) Show that the product of two reflection matrices is a rotation matrix.
c) The product of a reflection matrix times a rotation matrix is what kind of matrix?

Solution: First of all, if A and B are either reflections or rotations, we have

$$(AB)(AB)^T = AB(B^T)(A^T) = A(BB^T)A^T = AA^T = I.$$

So AB is again a reflection or a rotation. We now look at determinants, using the formula

$$\det(AB) = (\det A)(\det B).$$

For (a), we have $\det(AB) = (+1)(+1) = +1$, so AB is a rotation.
For (b), we have $\det(AB) = (-1)(-1) = +1$, so AB is a rotation.
For (c), we have $\det(AB) = (-1)(+1) = +1$, so AB is a reflection.

Exercise 2.5. *Find a matrix A such that $A^2 = I$, but which is not a reflection matrix.*

Solution: There are many such matrices, for example

$$\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \qquad \text{for any nonzero number } b.$$

Exercise 2.6. *Find a rotation matrix whose entries are nonzero rational numbers. (A rational number is a quotient of two integers. $\frac{1}{\sqrt{2}}$ is not a rational number. Hint: Consider the 3-4-5 right triangle.)*

Solution: Take any right triangle with integer sides x, y, z. Then $x^2 + y^2 = z^2$, so

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1.$$

Then

$$\begin{bmatrix} \frac{x}{z} & -\frac{y}{z} \\ \frac{y}{z} & \frac{x}{z} \end{bmatrix}$$

is a rotation matrix with rational entries.