Linear Algebra Notes

Chapter 5

SOLUTIONS TO EXERCISES

Exercise5.1. *A hexagon has six vertices, starting at* $(1,0)$ *and rotating by multiples of* $\pi/3$*.*

(a) *Find the coordinates of the remaining five vertices.*
(b) *There are six reflections that map the hexagon to itself. Draw the reflecting lines of these reflections and find their matrices.*

Solution:
a) The rotation by $\pi/3$ is given by the matrix

$$A = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.$$

The vertices of the hexagon are

$$\begin{aligned} \mathbf{e}_1 &= (1,0) \\ A\mathbf{e}_1 &= (1/2, \sqrt{3}/2) \\ A^2\mathbf{e}_1 &= (-1/2, \sqrt{3}/2) \\ A^3\mathbf{e}_1 &= (-1,0) \\ A^4\mathbf{e}_1 &= (-1/2, -\sqrt{3}/2) \\ A^5\mathbf{e}_1 &= (1/2, -\sqrt{3}/2) \end{aligned}$$

b) There are six reflections, about lines making angles

$$\theta = 0,\ \pi/6,\ \pi/3,\ \pi/2,\ 2\pi/3,\ 5\pi/6$$

with the x axis, so by our formula, the matrices for these reflections are, respectively,

$$\begin{aligned} \theta = 0 &: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \theta = \pi/6 &: \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}, \\ \theta = \pi/3 &: \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \\ \theta = \pi/2 &: \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \theta = 2\pi/3 &: \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, \\ \theta = 5\pi/6 &: \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}. \end{aligned}$$

Exercise 5.2. *Suppose A is reflection matrix about a line with angle θ, as above, and A' is a reflection about a line with angle ϕ. Then $A'A$ is a rotation matrix. What is the angle of rotation of $A'A$? Check your answer by taking A, A' to be two reflections of the hexagon, as in exercise 5.1b.*

Solution:

$$\begin{aligned} A'A &= \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2(\phi-\theta) & -\sin 2(\phi-\theta) \\ \sin 2(\phi-\theta) & \cos 2(\phi-\theta) \end{bmatrix}, \end{aligned}$$

using the trig identities, so the angle of rotation of $A'A$ is $2(\phi - \theta)$, ie, twice the angle between the lines, measured from L to L'.

Exercise 5.3. *Find two reflection matrices that do not commute with each other.*

Solution: There are many answers, for example,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are two non-commuting reflections.

Exercise 5.4. *Suppose A and A' are two distinct reflections that commute with each other. What is the relation between their reflecting lines? (Hint: Compare AA' and $A'A$, which were computed in exercise 5.2.)*

Solution: To have $A'A = AA'$, we need to have

$$\cos 2(\phi-\theta) = \cos 2(\theta-\phi), \quad \text{and} \quad \sin 2(\phi-\theta) = \sin 2(\theta-\phi).$$

The first equation holds for all θ and ϕ. The second only holds when $\sin 2(\phi-\theta) = 0$. This means $\phi - \theta$ is a multiple of $\pi/2$. Since the reflections are supposed to be distinct, it must be an odd multiple of $\pi/2$, ie, the lines must be perpendicular.