Chapter 6

SOLUTIONS TO EXERCISES

Exercise 6.1.

(a) *Use this method to compute* F_{20}, F_{21} *and* F_{30} *without computing any other Fibonnaci numbers.*

(b) *Compute* F_{30} *again by starting with your values for* F_{20} *and* F_{21} *from part (a) and then computing* $F_{22},\ F_{23}, \ldots, F_{30}$ *using the recursive formula* $F_{n+1} = F_{n-1} + F_n$.

Solution:

a) $\lambda^{20}/\sqrt{5} = 6765.000029\ldots$, so $F_{20} = 6765$.
$\lambda^{21}/\sqrt{5} = 10945.999981\ldots$, so $F_{21} = 10946$.
$\lambda^{30}/\sqrt{5} = 832040.00000024\ldots$, so $F_{30} = 832040$.

b) $F_{22} = 17711,\ F_{23} = 28657,\ F_{24} = 46368,\ F_{25} = 75025,\ F_{26} = 121393,\ F_{27} = 196418,\ F_{28} = 317811,\ F_{29} = 514229,\ F_{30} = 832040$.

Exercise 6.2. *Show that*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lambda.$$

Solution:

$$\frac{F_{n+1}}{F_n} = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda^n - \mu^n} = \lambda \frac{1 - (\mu/\lambda)^{n+1}}{1 - (\mu/\lambda)^n} \to \lambda,$$

since $|\mu/\lambda| < 1$.

Exercise 6.3. *The eigenvectors* $\mathbf{u}, \mathbf{v}$ *of* Φ *live on the lines with equations* $y = \lambda x$, $y = \mu x$, *respectively. Consider the function*

$$f(x, y) = (y - \lambda x)(y - \mu x)$$

which is zero on these lines.

(a) *Show that* $f(x, y) = y^2 - xy - x^2$.

(b) *Given a point* (x, y), *define* (x', y') *by the equation*

$$\Phi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Show that

$$f(x', y') = -f(x, y).$$

(c) *Let* C_+ *be the graph of the equation* $y^2 - xy - x^2 = 1$, *and let* C_- *be the graph of the equation* $y^2 - xy - x^2 = -1$. *These graphs are hyperbolas, having the lines* $y = \lambda x,\ y = \mu x$ *as asymptotes. Use part (b) to show that the point* (F_n, F_{n+1}) *lives on* C_+ *if* n *is even and on* C_- *if* n *is odd.*

A picture to accompany this exercise will be drawn in class.

Solution:

a) Since $(x-\lambda)(x-\mu) = x^2 - x - 1$, we have $\lambda + \mu = 1$ and $\lambda\mu = -1$. Hence

$$(y-\lambda x)(y-\mu x) = y^2 - (\lambda+\mu)xy + \lambda\mu x = y^2 - xy - x^2.$$

b)

$$f(x', y') = (y')^2 - x'y' - (x')^2 = (x+y)^2 - (x+y)y - y^2 = -y^2 + xy + x^2 = -f(x, y).$$

c)

$$\begin{aligned} f(F_n, F_{n+1}) &= -f(F_{n-1}, F_n) \\ &= (-1)^2 f(F_{n-2}, F_{n-1}) \\ &= \cdots \\ &= (-1)^n f(F_0, F_1) \\ &= (-1)^n f(0, 1) \\ &= (-1)^n. \end{aligned}$$

Exercise 6.4. *The "Lucas numbers" are defined like the Fibonaccis, except the two starting values are $L_0 = 2$, $L_1 = 1$. Thus, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, etc. Using the method of this chapter, find a formula for L_n that does not require computing any other Lucas numbers. (Note the same matrix Φ is used, but the initial vector is different).*

Solution:

$$\begin{aligned} \begin{bmatrix} L_n \\ L_{n+1} \end{bmatrix} &= \Phi^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda^{n-1} - \mu^{n-1} & \lambda^n - \mu^n \\ \lambda^n - \mu^n & \lambda^{n+1} - \mu^{n+1} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \end{aligned}$$

So

$$L_n = \frac{1}{\sqrt{5}}[2(\lambda^{n-1} - \mu^{n-1}) + (\lambda^n - \mu^n)] = \frac{1}{\sqrt{5}}[\lambda^n(1 + \frac{2}{\lambda}) - \mu^n(1 + \frac{2}{\mu}).$$

Since

$$1 + \frac{2}{\lambda} = -(1 + \frac{2}{\mu}) = \sqrt{5},$$

we get

$$L_n = \lambda^n + \mu^n.$$