Linear Algebra Notes

Chapter 7

SOLUTIONS TO EXERCISES

Exercise 7.1. *You may have noticed that equation (7b) gives a formula for M^n, but it is not in the form of a matrix. Calculate the matrix M^n.*

Solution: You can use equation (7b) to find $M^n\mathbf{e}_1$, and $M^n\mathbf{e}_2$, or you could multiply matrices

$$M^n = B\begin{bmatrix}\lambda^n & 0\\ 0 & 1\end{bmatrix}B^{-1} = \frac{1}{b+c}\begin{bmatrix}b+\lambda^n c & b-\lambda^n b\\ c-\lambda^n c & c+\lambda^n b\end{bmatrix}.$$

Exercise 7.2. *Suppose $P = 1000$ students, and that each month, 80% of students in dorms switch to apts, and 60% of students in apts switch to dorms.*

(a) *Find the migration matrix M and its eigenvalues.* **Solution:**

$$M = \begin{bmatrix}.4 & .8\\ .6 & .2\end{bmatrix}, \qquad \lambda = -.4, \quad \mu = 1$$

(b) *Assume that, initially, 500 students are in dorms and the other half are in apts. Find the population distribution at months 1,2,3,4 (round to the nearest integer).*
Solution:

$$\mathbf{w}_1 = M\begin{bmatrix}500\\500\end{bmatrix} = \begin{bmatrix}600\\400\end{bmatrix}$$
$$\mathbf{w}_2 = M^2\begin{bmatrix}500\\500\end{bmatrix} = \begin{bmatrix}560\\440\end{bmatrix}$$
$$\mathbf{w}_3 = M^3\begin{bmatrix}500\\500\end{bmatrix} = \begin{bmatrix}576\\424\end{bmatrix}$$
$$\mathbf{w}_4 = M^4\begin{bmatrix}500\\500\end{bmatrix} = \begin{bmatrix}570\\430\end{bmatrix}$$

(c) *Find the stable equilibrium point $\mathbf{w}_\infty$.*
Solution:

$$\mathbf{w}_\infty = \begin{bmatrix}571.42\ldots\\ 428.57\ldots\end{bmatrix}$$

(d) *Suppose that initially all 1000 students are in the dorms. How many months will it take to reach the stable equilibrium point? (In other words, find the smallest n such that the entries of $M^n\mathbf{w}_0$ differ from those of $\mathbf{w}_\infty$ by less than 1.)*
Solution: *From equation (7b) we have*

$$M^n\begin{bmatrix}0\\P\end{bmatrix} = \mathbf{w}_\infty - \frac{\lambda^n Pb}{b+c}(1,-1),$$

so we need $|\lambda^n| < \frac{b+c}{Pb} = \frac{1.4}{800} = .00175$ *Since* $|\lambda| = .4$, *we need* $n > \frac{ln.00175}{\ln .4} = 6.9\ldots.$ *so we need* $n = 7$ *months.*

(e) *As Housing Director, you crave equilibrium (otherwise, either your facilities are strained, or your voicemail is clogged with complaints about students from residents of Allston-Brighton). From this point of view, is the initial distribution in (d) the worst-case scenario, or are there other initial distributions that could take even more months to reach equilibrium?*

Solution:

$$M^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \mathbf{w}_\infty + \frac{\lambda^n(cx_0 - by_0)}{b+c}(1, -1).$$

The worst case is when $|cx_0 - by_0|$ *is maximal. In our case,* $|b| \geq |c|$, *so the maximum occurs when* $y_0 = P$. *So (d) is the worst-case scenario.*