

Linear Algebra
 Chapter 9
 Multiple Eigenvalues and Nilpotent Matrices
 Solutions to Exercises

Exercise 9.1. For each of the following matrices A , find the eigenvalue λ , and a matrix B such that $B^{-1}AB = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

(a) $A = \begin{bmatrix} 6 & 1 \\ -1 & 8 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

(To make life easier for the grader, please choose $\mathbf{v} = \mathbf{e}_1$ in each problem. Your B 's will all be the same, having 1 in the upper right corner.)

Solutions:

a) 7, $B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

b) 0, $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

c) 1, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d) 1, $B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

In general, $B = \begin{bmatrix} (a-d)/2 & 1 \\ c & 0 \end{bmatrix}$, if A is not already upper triangular.

Exercise 9.2. Show that every eigenvector of $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is proportional to \mathbf{e}_1 . (Hint: Just calculate.)

Solution: Set

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Get

$$\begin{bmatrix} \lambda x + y \\ \lambda y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix},$$

so $y = 0$, and $(x, 0) = x\mathbf{e}_1$.

Exercise 9.3. Let A be a matrix with multiple eigenvalue λ , and let \mathbf{u} be an eigenvector of A . Assume that A is not a scalar matrix. Prove the following statements.

(a) Every eigenvector of A is proportional to \mathbf{u} . (Hint: Proposition 1.)

(b) $A_0\mathbf{u} = (0, 0)$

(c) If \mathbf{v} is a vector for which $A_0\mathbf{v} = (0, 0)$, then \mathbf{v} is an eigenvector of A .

(Taken together, these facts mean that A has exactly one eigenline, which consists precisely of the vectors sent to $(0,0)$ by A_0 .)

Solutions:

a) If \mathbf{v} is any eigenvector of A , then $B^{-1}\mathbf{v}$ is an eigenvector of $B^{-1}AB$. But by exercise 9.2, all eigenvectors of $B^{-1}AB$ are proportional to \mathbf{e}_1 . So $B^{-1}\mathbf{v} = t\mathbf{e}_1$ for some scalar t . Multiplying both sides by B , we get

$$\mathbf{v} = tB\mathbf{e}_1 = t\mathbf{u},$$

so \mathbf{v} is proportional to \mathbf{u} .

b) Apply both sides of the matrix equation $A = \lambda I + A_0$ to \mathbf{u} . We get

$$A\mathbf{u} = \lambda\mathbf{u} + A_0\mathbf{u}.$$

But $A\mathbf{u} = \lambda\mathbf{u}$, so

$$\lambda\mathbf{u} = \lambda\mathbf{u} + A_0\mathbf{u}.$$

Subtract $\lambda\mathbf{u}$ from both sides, and get $A_0\mathbf{u} = 0$.

c) If $A_0\mathbf{v} = (0,0)$, then $A\mathbf{v} = (\lambda I + A_0)\mathbf{v} = \lambda\mathbf{v} + (0,0) = \lambda\mathbf{v}$.

Exercise 9.4. Suppose $A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for some $n \geq 0$.

(a) What are the eigenvalues of A ? (Hint: Apply A repeatedly to both sides of the equation $A\mathbf{u} = \lambda\mathbf{u}$.)

(b) Compute A^2 . (Hint: Use Proposition 3, and the value of λ from part (a).)

Solutions:

a) $\lambda^n = 0$, so $\lambda = 0$.

b) $A = A_0$, so $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, by Prop. 3.

Exercise 9.5. Suppose $\lambda = 0$ is the only eigenvalue of A . Does this imply that A is nilpotent?

Solution: Yes, because $A = A_0$, and $A_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Exercise 9.6. Show that the nilpotent matrices are precisely those of the form

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \quad \text{with} \quad a^2 + bc = 0.$$

(You have to show two things: First, that every nilpotent matrix has this form. Second, that every matrix of this form is nilpotent.)

Solution: First suppose A is nilpotent. Then $\lambda = 0$ is the only eigenvalue of A , by exercise 9.4a). Hence $P_A(x) = x^2$, so $\text{tr } A = \det A = 0$. The former means $d = -a$, the latter means $-a^2 - bc = 0$.

Second, suppose

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \quad \text{with} \quad a^2 + bc = 0.$$

then $P_A(x) = x^2$ so $\lambda = 0$ is the only eigenvalue, so A is nilpotent, by the affirmative answer to exercise 9.5