Math 210 exam 1 review solutions

1. a) $P_{A^{-1}}(x) = (x - \frac{1}{2})(x - \frac{1}{3})$.
   b) $(x-1)^2, (x+1)^2, x^2 - 1$.
   c) 2 and 3.

2. $A\mathbf{e}_2 = A(\mathbf{e}_1 + \mathbf{e}_2) - A\mathbf{e}_1 = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_1 = -2\mathbf{e}_1 - \mathbf{e}_2$, so $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$.

3. First, BA is an orthogonal matrix, because $(BA)^T = A^T B^T = A^{-1}B^{-1} = (AB)^{-1}$. Since $\det(BA) = \det(B)\det(A) = -1 \cdot 1 = -1$, in fact BA is a reflection matrix. That means $(BA)^T = BA$. On the other hand, B is a reflection too, so $B^T = B$, so $(BA)^T = A^T B^T = A^{-1}B$. Comparing the two expressions for $(BA)^T$), we get $A^{-1}B = BA$, so $A^{-1} = BAB^{-1}$. (This formula says any reflection conjugates a rotation to the rotation by the same angle in the opposite direction.)

4. You know $\cos\theta = \frac{1}{\sqrt{10}}$, $\sin\theta = \frac{3}{\sqrt{10}}$, and $\cos 2\theta = 2\cos^2\theta - 1$, $\sin 2\theta = 2\sin\theta\cos\theta$, so the formula for a reflection about line with angle θ says

$$A = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}.$$

5. a) $\det A = 0$
   b) $\mathrm{tr}(A) = 0$
   c) Any nilpotent matrix, i.e., any conjugate of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6. Let $B = [\mathbf{u}\ \mathbf{v}]$, compute $B^{-1}\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so $\mathbf{w} = 2\mathbf{u} + \mathbf{v}$.

7. The reflection matrix formula says the product of two reflections is rotation by twice the angle between the lines. In this case, the product is rotation by $\pi/2$, namely $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

8. We have $x_{n+1} = x_n + 2x_{n-1}$, like Fibonacci, but now the matrix is $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, and $A^n\,[0 \quad 1] = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$. Finding $\lambda, \mu, \mathbf{u}, \mathbf{v}, B$ as usual, you get

$$A^n = B\begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1} = \frac{1}{3}\begin{bmatrix} 2^n + 2(-1)^n & 2^n + (-1)^{n+1} \\ 2^{n+1} + 2(-1)^{n+1} & 2^{n+1} + (-1)^n \end{bmatrix},$$

so $x_n = \frac{2^n + (-1)^{n+1}}{3}$.

9. a) $M = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix}$, $\lambda = .6$, $\mu = 1$ and $\mathbf{u} = (1, -1)$, $\mathbf{v} = (1, 1)$ respectively.
   b) The point (200,200) is already stable, so the population doesn't ever change.
   c) For a general initial distribution $\mathbf{w}_0$, we have

$$M^n\mathbf{w}_0 = (\frac{x_0 - y_0}{2})\lambda^n\mathbf{u} + (\frac{x_0 + y_0}{2})\mathbf{v}.$$

There are two $\mathbf{w}_0$'s taking longest to reach the stable point: $(400, 0)$ and $(0, 400)$.

10. a) $\lambda = \frac{1}{2}(3 + \sqrt{17}),\ \mu = \frac{1}{2}(3 - \sqrt{17})$, $B = \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}$.

b) $B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

c) $\lambda = \frac{1}{2}(5 + \sqrt{33}),\ \mu = \frac{1}{2}(5 - \sqrt{33})$, $B = \begin{bmatrix} 2 & 2 \\ \lambda & \mu \end{bmatrix}$.

d) $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.