Math 216
Homework 2
Due Fri. Feb. 1

2.1 Prove by induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2.2. Prove by induction on n that

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} \qquad (n \geq 0)$$

2.3. Prove (either by induction on k, or something else) that

$$1 - x + x^2 - x^3 + \cdots - x^{2k-1} + x^{2k} = \frac{x^{2k+1} + 1}{x + 1} \qquad (k \geq 0)$$

2.4. a) Use integration by parts and a trig identity to show that

$$\int \sin^n x \, dx = \frac{1}{n}\left[-\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx\right] \quad (n \geq 1)$$

b) Use a) to show that

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx \quad (n \geq 1)$$

c) Use b) to prove, by induction on k, that

$$\int_0^{\pi/2} \sin^{2k} x \, dx = \frac{\pi}{2} \cdot \frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots (2k)} \qquad (k \geq 1).$$

2.5. The alternating sum of the rows of Pascal's triangle look like $1-1 = 0$, $1-2+1 = 0$, $1-3+3+1 = 0, \cdots$. Make a conjecture for general n and prove it. (The expansion of $(1+x)^n$ is helpful.)

2.6. Prove that there are infinitely many primes of the form $6k + 5$. (Follow the method of 1.3. State and prove the necessary lemmas.)

2.7. A chessboard is an 8×8 square. Place one grain of rice on the first square, two grains on the second square, four on the third square, eight on the fourth square, and continue in this way until every square on the board has rice. Let N be the total number of grains of rice on the board.

a) Use 2.2 to write N in a simple way, without dots.

b) Use a) to factor N into a product of Fermat numbers. Then give the complete factorization of N as a product of primes.

c) Assume that each grain of rice is one centimeter long. Place the N grains of rice end to end, and calculate the length of this trail of rice. Express your answer in light years. (A light year is 9.46091×10^{12} centimeters.)