Math 216
Homework 4
Due Fri. Feb. 22

4.1 Let f and g be two functions. Express the n^{th} derivative of the product fg in terms of the derivatives of f and g. (Start with $n = 1$, which is the familiar product rule from Calculus. Then do $n = 2, 3 \ldots$, guess the pattern, and prove the formula using induction.)

4.2. Prove that if $p(x)$ is a polynomial of degree n then

$$\int p(x)e^x \, dx = [p(x) - p'(x) + p''(x) - \cdots + (-1)^n p^{(n)}(x)]e^x.$$

(The "degree" of a polynomial is the highest power of x appearing in it. The notation $p^{(n)}(x)$ means the n^{th} derivative of $p(x)$. Use integration by parts and induction.)

4.3. Let a, d, m be positive integers. Prove that if $d \mid m$ then $a^d - 1 \mid a^m - 1$.

4.4. Prove that if q is a prime dividing $2^p - 1$ then $q = 1 + 2pk$ for some integer k. Use this to determine whether $2^{17} - 1$ is prime.

4.5.
a) Find the remainder of 7^{1000} divided by 23.
b) Find the remainder of $2^{100000000000}$ divided by 101.
c) Find the remainder of 9728! divided by 19457. (See problem A10.)
d) Find the last digit of 9^n for any n. (Hint: 9=10-1)

4.6. The Fibonacci numbers are defined by the recursive formulas $f_0 = 0,\ f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Let F be the matrix

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Prove by induction that the powers of F give the Fibonacci numbers, as follows:

$$F^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}.$$

Then use this to find the last digit of F_{128}, by hand.

4.7. Prove by induction on n that

$$\sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} a_i b_j = \Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{j=1}^{n} b_j\Big).$$

4.8. Calculate the sum $\sum \frac{1}{k}$ over those positive integers k whose prime divisors belong to $\{2, 3, 5, 7\}$. (Including $k = 1$.)