Math 216
Homework 5
Due Fri. March 22

5.1 Prove by induction that $(\cos x + i \sin x)^n = \cos nx + i \sin nx$. You will need the addition formulas for sine and cosine. See p. 120 in the Book.

5.2. Use the binomial expansion and 5.1 to express $\cos nx$ as a polynomial in $\cos x$. (The case $n = 2$ is the familiar formula $\cos 2x = 2\cos^2 x - 1$.)

5.3. Use the formula from the middle of p. 32

$$0 = \binom{2m+1}{1} \cot^{2m} x - \binom{2m+1}{3} \cot^{2m-2} x + \cdots \qquad (x = \frac{k\pi}{2m+1})$$

along with the quadratic formula to compute $\cot^2(\pi/5)$ and $\cot^2(2\pi/5)$ in terms of $\sqrt{5}$. Check your answers using equation (1) p. 32.

5.4. (The Golden Triangle) Take an isosceles triangle with angles α, α, $\frac{\alpha}{2}$. Let the short side have length s and long side have length ℓ. Let $x = \frac{\ell}{s}$. Show the following:
a) $\alpha = \frac{2\pi}{5}$, b) $\frac{1}{x} = 2\cos\alpha$, c) $x = 2\cos\frac{\alpha}{2}$ (hint for c: Law of Sines).
Then find a cubic equation in x, and solve it for x, thus computing $\cos\frac{2\pi}{5}$.

5.5. Compute the sum

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}.$$

Hint: Break the sum for $\zeta(2)$ into even and odd terms, and "solve" for the odd terms.

5.6. In this problem and the next, assume, along with Euler, the (actually correct) formula

$$\frac{\sin \pi x}{\pi x} = \big(1 - \frac{x^2}{1^2}\big)\big(1 - \frac{x^2}{2^2}\big)\big(1 - \frac{x^2}{3^2}\big)\cdots$$

Set $x = \pi/2$ to derive the *Wallis formula* for π:

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots.$$

5.7. Replace x by x^2 in the above formula for $\frac{\sin \pi x}{\pi x}$, to find the value of

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

5.8. (A lemma we'll need in class) Let $f(x)$ be a polynomial of degree $2n$, and let

$$F(x) = \frac{1}{\pi} f(x) - \frac{1}{\pi^3} f''(x) \pm \cdots + \frac{(-1)^n}{\pi^{2n+1}} f^{(2n)}(x).$$

Prove by induction on n that

$$\int_0^1 f(x) \sin \pi x \, dx = F(1) - F(0).$$