Math 216
Homework 6
Due Fri. April 5

6.1. Use the power series for e^x, $\cos x$ and $\sin x$ to prove *Eulers formula*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

6.2 Irrational numbers can be approximated by rational numbers as follows. For any number x, let $\lfloor x \rfloor$ be the greatest integer $\leq x$. Let α be irrational, and let $a_0 = \lfloor \alpha \rfloor$, $r_0 = \alpha - a_0$, $a_1 = \lfloor r_0^{-1} \rfloor$, $r_1 = r_0^{-1} - \lfloor r_0^{-1} \rfloor$, $a_2 = \lfloor r_1^{-1} \rfloor$, $r_2 = r_1^{-1} - \lfloor r_1^{-1} \rfloor$, etc. Then the fractions

$$a_0, \quad a_0 + \frac{1}{a_1}, \quad a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \quad etc.$$

are good (in fact the best, in some sense) approximations to α. Work this out for π, up to the fraction with a_3. (See p. 29 in the Book.)

6.3. Expand $\frac{1}{1+x^2}$ in a geometric series, and integrate $\int_0^1$ term by term. Then use the formula $\int \frac{1}{1+x^2} = \arctan x$ to prove *Leibniz formula*

$$\pi = 4[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots].$$

6.4. Prove that

$$\pi = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2$$

(HInt: The right side is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy.$$

Do this last integral in polar coordinates. (See your calculus book if you forgot this!)

6.5. Prove that the irrational numbers are uncountable. (Hint: The set of real numbers is the union of the rational numbers and the irrational numbers.)

6.6. Find a bijection $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$.

6.7. Find a bijection $f : \mathbb{R} \longrightarrow (0, 1)$.

6.8. Let $a < b$, $c < d$ be four real numbers. Find a bijection $f : (a, b) \longrightarrow (c, d)$.

6.9. Find a bijection $f : [0, 1) \longrightarrow (0, 1)$. (Hint: perhaps modify the function on p.89).

6.10. Prove that the set

$$S = \{.a_1a_2a_3\cdots \ : \ a_n \in \{0, 1\}\}$$

is uncountable.