II. THE ZETA FUNCTION (The Book, chapter 6)

11. Product formula for the zeta function — Recall that for $s > 1$, the series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

converges to some value $< \frac{s}{s-1}$. For example,

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots < 2.$$

Theorem 11.1. *The zeta function also has a product form*

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

where the right side, a product over all primes p, is defined as the limit

$$\lim_{n \to \infty} \prod_{1 \le i \le n} \frac{1}{1 - \frac{1}{p_i^s}},$$

where $p_1, p_2, \ldots$ is the list of primes in increasing order.

Proof. Let S_n be the set of $k \in \mathbb{N}$ whose prime divisors are among $p_1, \ldots, p_n$. Then

$$\zeta(s) = \lim_{n \to \infty} \sum_{k \in S_n} \frac{1}{k^s}.$$

Now every $k \in S_n$ is a product $k = p_1^{e_1} \cdots p_n^{e_n}$ for a unique set of exponents $e_1, \ldots, e_n$. Hence

$$\frac{1}{k^s} = \frac{1}{p_1^{se_1}} \cdots \frac{1}{p_n^{se_n}},$$

so

$$\begin{aligned}\sum_{k \in S_n} \frac{1}{k^s} &= (1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \cdots)(1 + \frac{1}{p_2^s} + \frac{1}{p_2^{2s}} + \cdots) \cdots (1 + \frac{1}{p_n^s} + \frac{1}{p_n^{2s}} + \cdots) \\ &= \prod_{1 \le i \le n} \frac{1}{1 - \frac{1}{p_i^s}}.\end{aligned}$$

Taking the limit as $n \to \infty$ proves the Theorem. $\square$

12. Values of the zeta function — Euler determined the actual sum of the series $\zeta(2d)$ for any $d \in \mathbb{N}$. We only only give a complete proof for $\zeta(2)$.

Theorem 12.1. $\zeta(2) = \frac{\pi^2}{6}$.

First proof, incomplete. This is Euler's method. He starts with the product formula

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right). \tag{12a}$$

We did not prove formula (12a) in this course. It follows from the cotangent formula in chapter 19, which we did not cover. At least it is evident that both sides of (12a) are zero exactly at the non-zero integers, and both sides are 1 when $x = 0$.

Euler computed $\zeta(2)$ by considering the coefficient of x^2 on both sides of (12a). From the Taylor expansion of $\sin x$, we get

$$\frac{\sin \pi x}{\pi x} = \sum_{n=0}^{\infty} (-1)^n \frac{(\pi x)^{2n}}{(2n+1)!}.$$

So the coefficient of x^2 in (11a) is $-\pi^2/6$. On the other hand, the coefficient of x^2 in the right side of (12a) is $-\sum_{n=1}^{\infty} \frac{1}{n^2} = -\zeta(2)$, so Euler concludes that $\zeta(2) = \pi^2/6$. $\square$

Second proof, complete. It depends on a lemma.

Lemma12.2. *For any $m \in \mathbb{N}$, we have*

(1)

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \cdots + \cot^2\left(\frac{m\pi}{2m+1}\right) = \frac{2m(2m-1)}{6},$$

(2)

$$\csc^2\left(\frac{\pi}{2m+1}\right) + \csc^2\left(\frac{2\pi}{2m+1}\right) + \cdots + \csc^2\left(\frac{m\pi}{2m+1}\right) = \frac{2m(2m+2)}{6}.$$

We will prove this lemma later. Here is how it is used to compute $\zeta(2)$. First note that all the angles in Lemma12.2 are contained in the interval $(0, \pi/2)$. For $y \in (0, \pi/2)$, we have

$$0 < \sin y < y < \tan y,$$

which implies

$$\cot^2 y < \frac{1}{y^2} < \csc^2 y.$$

Hence, for every $m \in \mathbb{N}$, we have

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) < \sum_{k=1}^{m} \frac{1}{\left(\frac{k\pi}{2m+1}\right)^2} < \sum_{k=1}^{m} \csc^2\left(\frac{k\pi}{2m+1}\right).$$

By Lemma 12.2, we get

$$\frac{2m(2m-1)}{6} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(2m+2)}{6},$$

hence

$$\frac{\pi^2}{6} \cdot \frac{2m(2m-1)}{(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{\pi^2}{6} \cdot \frac{2m(2m+2)}{(2m+1)^2}.$$

Take the limit as $m \to \infty$. The outer terms converge to $\frac{\pi^2}{6}$, hence the inner sum does as well.

Proof of 12.2. First of all, part (2) of 12.2 follows from part (1) using the identity $\csc^2 \theta = 1 + \cot^2(\theta)$. Proving part (1) is the heart of the matter. First we explain the idea. Suppose you have a polynomial $p(t)$ of degree m, whose roots are $\alpha_1, \alpha_2, \ldots, \alpha_m$. This means that

$$p(t) = c(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_m),$$

where c is the coefficient of t^m in $p(t)$. The coefficient of t^{m-1} is

$$c(-\alpha_1 - \alpha_2 - \cdots - \alpha_m) = -c(\alpha_1 + \alpha_2 + \cdots + \alpha_m).$$

This means that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = -\frac{\text{coefficient of } t^{m-1}}{\text{coefficient of } t^m}. \tag{12b}$$

Thus, if you have complicated sum $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ on your hands, but you can find the coefficients of a polynomial having $\alpha_1, \ldots, \alpha_m$ as its roots, then you can find the sum using (12b).

In our situation, $\alpha_k = \cot^2(\frac{k\pi}{2m+1})$, for $k = 1, \ldots, m$. To find $p(t)$, we let $n = 2m + 1$ and start with the identity

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n,$$

then expand the right side using the binomial theorem

$$\cos nx + i \sin nx = \sum_{\ell=0}^{n} \binom{n}{\ell} \cos^{n-\ell} x (i \sin x)^{\ell}.$$

We equate the imaginary part (coefficient of i) on both sides. The ℓ^{th} term in the right side has an i when $\ell = 2j + 1$, and $j = 0, 1, \ldots, m$, and $(i)^{2j+1} = i(-1)^j$, so

$$\sin nx = \sum_{j=0}^{m} (-1)^n \binom{n}{2j+1} \cos^{n-2j-1} x \sin^{2j+1} x,$$

so

$$\begin{aligned} \frac{\sin nx}{\sin^n x} &= \sum_{j=0}^{m} (-1)^j \binom{n}{2j+1} \cos^{n-2j-1} x \sin^{2j+1-n} x \\ &= \sum_{j=0}^{m} (-1)^j \binom{n}{2j+1} \cot^{n-2j-1} x. \end{aligned}$$

Recall that $n = 2m+1$. If we set $x = \frac{k\pi}{2m+1}$, the left side becomes zero, so

$$0 = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} \cot^{2m-2j}(\frac{k\pi}{2m+1}).$$

This shows that $\alpha_k = \cot^2(\frac{2k\pi}{2m+1})$ is a root of the polynomial

$$p(t) = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} t^{m-j}.$$

By (11b), we have

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = -\frac{(-1)^1 \binom{2m+1}{3}}{(-1)^0 \binom{2m+1}{1}} = \frac{2m(2m-1)}{6}.$$

This completes the proof of Lemma 12.2, and Theorem 12.1. $\square$

By comparing coefficients of higher powers of x, Euler found

$$\zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}, \ldots$$

and eventually Euler discovered the general formula, for $d \in \mathbb{N}$,

$$\zeta(2d) = (-1)^{d-1} 2^{2d-1} B_{2d} \frac{\pi^{2d}}{(2d)!},$$

where B_{2d} are the so-called "Bernoulli numbers"; they are the coefficients in the Taylor expansion

$$\frac{x}{e^x - 1} = \sum_{n=1}^{\infty} B_n \frac{x^n}{n!}.$$

In other words, B_n is the n^{th} derivative of $\frac{x}{e^x-1}$, evaluated at $x = 0$. In particular, $\zeta(2d) = r_d \pi^{2d}$, where $r_d \in \mathbb{Q}$. (See chapter 19.)

No one knows much about $\zeta(2d+1)$, except that infinitely many of them, including $\zeta(3)$, are irrational. It is natural to guess that $\zeta(3) = r\pi^3$, where $r \in \mathbb{Q}$. But this r would have an extremely large denominator. To 100 digits, we have

$$\frac{\zeta(3)}{\pi^3} = 0.038768179602916798941119890318721149806234568 0\\ 395525792231267621237771370122868552718512987939065771 4\ldots$$

with no sign of repeating. Taking 100 terms in the continued fraction for $\frac{\zeta(3)}{\pi^3}$, we get the rational approximation (accurate to more than 100 decimal places, but not exact)

$$\frac{\zeta(3)}{\pi^3} \sim \frac{115260206291788849276355948983278215293180020494 14281}{297306212136710759637702242559318365694579816662482057},$$

and any rational number which is closer to $\frac{\zeta(3)}{\pi^3}$, including $\frac{\zeta(3)}{\pi^3}$ itself, if it is rational, must have larger denominator than this one.

We can put all the ζ values into a power series

$$Z(t) = \sum_{n=0}^{\infty} \zeta(n+2) t^n.$$

Since $\zeta(s) < s/(s-1)$, it follows that $\zeta(n+2) < 2$, so the series $Z(t)$ converges for $0 < t < 1$ because it is less than the geometric series $2 \sum t^n$. Now the coefficient of t^n in a power series is $n!$ times the n^{th} derivative of the series at $t = 0$. Hence

$$\zeta(n+2) = n! Z^{(n)}(0).$$

Thus, the function $Z(t)$ determines the values $\zeta(2+n)$. For example, $Z(0) = \zeta(2)$, $Z'(0) = \zeta(3)$, etc.

If we rearrange the sum for $Z(t)$, we get

$$\begin{aligned} Z(t) &= \sum_{n=0}^{\infty} t^n \sum_{k=1}^{\infty} \frac{1}{k^{n+2}} \\ &= \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{k^{n+2}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{n=0}^{\infty} \left(\frac{t}{k}\right)^n \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \frac{1}{1 - \frac{t}{k}} \\ &= \sum_{k=1}^{\infty} \frac{1}{k(k-t)}. \end{aligned}$$

So the function $Z(t)$ is merely a perturbation of $\zeta(2)$, with t as the perturbation variable.

13. A formula for π — The formula $\zeta(2) = \pi^2/6$ gives a formula for π in terms of prime numbers. We know that $\pi > 3$, so $\pi = 3A$ for some $A > 1$. From the product formula for $\zeta(2)$ we have

$$\begin{aligned} \frac{\pi^2}{6} &= \zeta(2) \\ &= \prod_p \frac{1}{1 - \frac{1}{p^2}} \\ &= \prod_p \frac{p^2}{p^2 - 1} \\ &= \frac{2^2}{2^2 - 1} \cdot \frac{3^2}{3^2 - 1} \cdot \prod_{p>3} \frac{p^2}{p^2 - 1} \\ &= \frac{3}{2} \prod_{p>3} \frac{p^2}{p^2 - 1} \end{aligned}$$

Solving for π, we get the formula

$$\pi = 3 \prod_{p>3} \frac{p}{\sqrt{p^2-1}}. \tag{13a}$$

Note that each term in the product is > 1, so the product is > 1.

As a numerical check, we compute the product up to the one thousandth prime.

$$3 \cdot \frac{5}{\sqrt{5^2-1}} \cdot \frac{7}{\sqrt{7^2-1}} \cdot \frac{11}{\sqrt{11^2-1}} \cdots \frac{7919}{\sqrt{7919^2-1}} = 3.14157\ldots$$

Since $\pi = 3.14159\ldots$ this is not an efficient way to compute π. The interesting thing about (13a) is the connection between the geometrically defined number π and the collection of all prime numbers.

14. $\zeta(s)$ as volume — Consider the function $\frac{1}{1-xy}$, defined over the square $0 \le x, y \le 1$. It looks like a terrifying ski-slope. The volume under the graph is

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{1}{1-xy}\, dxdy &= \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n \, dxdy \\
&= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 x^n y^n \, dxdy \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \zeta(2).
\end{aligned}$$

In three dimensions, we can integrate the function $\frac{1}{1-xyz}$ over the cube $0 \le x, y, z \le 1$ by a similar calculation, and get $\zeta(3)$. More generally, we have

$$\int_0^1 \cdots \int_0^1 \frac{dx_1 \cdots dx_n}{1 - x_1 x_2 \cdots x_n} = \zeta(n).$$

15. Density of the square-free numbers — We have classified the sets of natural numbers $S \subset \mathbb{N}$ and into two categories, big and small, according as $\sum_{k \in S} \frac{1}{k}$ is infinite or finite, respectively. A more refined measurement comes from comparing the growth of the series for S (which may or may not converge) with the growth of the series for $\mathbb{N}$ (which is the divergent harmonic series), using $\zeta(s)$.

Write $S = \{k_1, k_2, \ldots\}$ listed in increasing order. We define the **density** of S to be the limit

$$d(S) = \lim_{s \to 1} \frac{\sum_{k \in S} k^{-s}}{\sum_{k=1}^{\infty} k^{-s}}.$$

It could happen that the limit does not exist. So the $d(S)$ is only defined when the limit does exist. For example, if S is small, then the numerator $\sum_{k \in S} \frac{1}{k^s}$ is bounded as $s \to 1$, while the denominator is unbounded (since $\zeta(1) = \infty$), so $d(S) = 0$ if S is small. On the other hand, if $S = \mathbb{N}$ then the numerator equals the denominator, so $d(\mathbb{N}) = 1$. In general, the density $d(S)$, if it exists, is some number in $[0, 1]$.

Since it is easier to divide products, we often try to use the product formula

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

to compute densities. This works if the numerator $\sum_{k \in S} k^{-s}$ also has a product formula over the primes.

For example, let S be the set of square-free numbers in $\mathbb{N}$. (These are the numbers without square-factors; they appeared in section 11.) Every $k \in S$ is a product of a subset of the primes. Hence

$$\sum_{k \in S} k^{-s} = \prod_p (1 + p^{-s}).$$

We can now compute

$$\begin{aligned} d(S) &= \lim_{s \to 1} \frac{\prod_p (1 + p^{-s})}{\prod_p (1 - p^{-s})^{-1}} \\ &= \lim_{s \to 1} \prod_p (1 - p^{-2s}) \\ &= \lim_{s \to 1} \zeta(2s)^{-1} \\ &= \zeta(2)^{-1} \\ &= \frac{6}{\pi^2}. \end{aligned}$$

Since $\frac{6}{\pi^2} = .607\ldots$, this means that if you choose a random $k \in \mathbb{N}$, it has about a 60 percent chance of being square-free.