III. IRRATIONAL NUMBERS (The Book, chapter 6)

16. A taxonomy of Numbers — A **rational** number is a quotient of two integers $\frac{a}{b}$, with $b \neq 0$. An **irrational** number is a real number which is not rational. It takes a bit of work to define real numbers precisely, but we will view them as decimals, i.e., as limits of rational numbers. Let $\mathbb{R}$ be the set of real numbers. Thus, $\mathbb{R} - \mathbb{Q}$ is the set of irrational numbers. Let $\mathbf{C} = \{a + bi : \ a, b \in \mathbb{R}\}$ be the set of complex numbers. An **algebraic** number is an $\alpha \in \mathbf{C}$ which is the root of a polynomial with rational coefficients. The standard notation for the set of algebraic numbers is.

$$\bar{\mathbb{Q}} = \{\alpha \in \mathbf{C} : \ \alpha \text{ is algebraic}\}.$$

Any rational number $r \in \mathbb{Q}$ is the root of $x - r$, so rational numbers are algebraic. As irrational examples, i is a root of $x^2 + 1$, and $\sqrt{2}$ is a root of $x^2 - 2$, and so i and $\sqrt{2}$ are algebraic. In the proof of Lemma 12.2, we found a rational polynomial $p(t)$ whose roots are $\cot^2 k\pi/(2m+1)$ for $k = 1, \ldots m$. So these cotangent values are algebraic. In fact any trigonometric function evaluated at a rational multiple of π is algebraic. It is also true, though we will not prove it, that the sum and product of two algebraic numbers is again algebraic. So there are lots of algebraic numbers. But we shall see that in fact "most" numbers are not algebraic.

Non-algebraic numbers are called **transcendental numbers**. Thus, $\mathbf{C} - \bar{\mathbb{Q}}$ denotes the set of transcendental numbers. The numbers π, e are transcendental, but this is a bit too hard for us to prove in this course. It is always hard to prove that a number is transcendental. Nevertheless, we shall see that "most" numbers are transcendental, even though only a handful have been provably found.

It is often less hard to prove that a given number is irrational. You assume it is rational, and try to derive a contradiction from this assumption. The level of difficulty varies, depending on the given number. Here is the easiest one.

16.1. *$\sqrt{2}$ is irrational.*

Proof. Assume $\sqrt{2}$ is rational. Say $\sqrt{2} = \frac{a}{b}$ with $a, b \in \mathbb{N}$. Then $2b^2 = a^2$. Any prime dividing a square does so an even number of times. So 2 divides a^2 an even number of times, while 2 divides $2b^2$ an odd number of times. This is a contradiction. □

The same idea shows that $\sqrt{d}$ is irrational for any non-square $d \in \mathbb{N}$. Instead of 2, you use a prime p dividing d an odd number of times.

17. The number e — The famous number

$$e = 2.71828182845904523536028747135266249775724709369996\ldots$$

can be defined in many ways:

- $$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

- $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

- $e = f(1)$, where $f(x)$ is the unique function satisfying $f'(x) = f(x)$, $f(0) = 1$.
- e is the point > 1 on the x-axis, above which the area under the graph of $\frac{1}{x}$ equals one.
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$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ddots}}}}}}}}}$$

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$$e = \lim_{n\to\infty} \Big[\prod_{p \le n} p\Big]^{1/n},$$

where the product is that of all primes $\le n$.

Theorem 17.1. *The number e is irrational.*

Proof. Assume $e = \frac{a}{b}$, with $a,\ b \in \mathbb{N}$. Choose an integer $n > b$ and consider the number

$$N = n![e - \sum_{k=1}^{n} \frac{1}{k!}].$$

On one hand, N is an integer, because $n!e = n!\frac{a}{b}$ and $b \le n$, and each $k \le n$ in the sum.

On the other hand,

$$0 < N = n! \sum_{k=n+1}^{\infty} \frac{1}{k!} < \sum_{j=1}^{\infty} \frac{1}{(n+1)^j} = \frac{1}{n} < 1.$$

But there are no integers in $(0, 1)$, so this is a contradiction, and e must be irrational. $\square$

In fact, e is transcendental, but we will not prove this completely. We only prove that e does not satisfy an equation of the form $x^s - r = 0$, for $r \in \mathbb{Q}$, and $s \in \mathbb{N}$. To completely prove that e is transcendental, we would have to replace $x^s - r$ by an arbitrary polynomial with $\mathbb{Q}$-coefficients.

This proof, and the proof of irrationality of π in the next section, will rely on the properties of a magic polynomial.

Lemma 17.2. *For every $n \in \mathbb{N}$, there is a polynomial $f_n(x)$ of degree $2n$ with the following properties.*

(1) $f_n(x) = \frac{1}{n!}\sum_{k=n}^{2n} c_k x^k$, *with* $c_k \in \mathbb{Z}$.
(2) $0 < f_n(x) < \frac{1}{n!}$ *if* $0 < x < 1$.
(3) $f_n^{(k)}(0)$ *and* $f_n^{(k)}(1)$ *are integers for all* $k \ge 0$.

Proof. The polynomial (*is it unique?*) is

$$f_n(x) = \frac{1}{n!}x^n(1-x)^n.$$

Parts (1) and (2) are clear. For $0 \le k < n$ we have $f_n^{(k)}(0) = 0$ since the lowest power of x is x^n. For $k > n$, $f_n^{(k)}(0) = \frac{k!c_k}{n!} \in \mathbb{Z}$. Finally, since $f_n(x) = f_n(1-x)$, we have $f_n^{(k)}(x) = (-1)^k f_n^{(k)}(1-x)$, so $f_n^{(k)}(1) = (-1)^k f_n^{(k)}(0) \in \mathbb{Z}$. □

Now we can prove the irrationality of e^s.

Theorem 17.3. *The number e^s is irrational for every $s \in \mathbb{N}$.*

Proof. Assume that $e^s = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. We will consider the integral

$$\int_0^1 f_n(x)e^{sx}\,dx,$$

and show that it has contradictory properties for large n.

We proved in homework 1 that for any polynomial $f(x)$ of degree $2n$ that

$$\int_0^1 f(x)e^{sx}\,dx = \Big[\frac{1}{s}f(x) - \frac{1}{s^2}f'(x) + - \cdots + \frac{1}{s^{2n+1}}f^{(2n)}(x)\Big]e^{sx}\Big|_0^1.$$

For $f = f_n$, we have $f^{(k)}(0) = f^{(k)}(1) \in \mathbb{Z}$ for all k, by 17.2(3). Since $e^s = \frac{a}{b}$, we see that the number

$$N = bs^{2n+1}\int_0^1 f_n(x)e^{sx}\,dx$$

is an integer.

On the other hand, for $0 < x < 1$ we have $f_n(x) < \frac{1}{n!}$ by 17.2(2), and $e^{sx} < e^s$, so

$$N \le bs^{2n+1}\frac{1}{n!}\int_0^1 e^s\,dx = be^s\frac{s^{2n+1}}{n!}.$$

But the ratio $\frac{s^{2n+1}}{n!} \to 0$ as $n \to \infty$. Hence we can make N close to zero by taking n large, and then N is not an integer. This contradiction completes the proof that e^s is irrational. □

18. The number π — The famous number

$$\pi = 3.14159265358979323846264338327950288419716939937 5106\cdots$$

can be defined in many ways.

- π is the area of a circle of radius one.
- The ratio of circumference over diameter is the same in any circle; this constant is π.
- $$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}.$$

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$$\pi = 2 \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdots(2n)(2n)}{3\cdot 3\cdot 5\cdot 5\cdots(2n+1)(2n+1)}$$

• π is the total volume under the graph of $e^{-(x^2+y^2)}$, above the xy plane.

• π is the smallest positive zero of the unique function h satisfying $h'' = -h$, $h(0) = 0,\ h'(0) = 1$.

To prove that π is irrational, we will prove the stronger result that π^2 is irrational. In other words, π is not the root of an equation $x^2 - r = 0$, with $r \in \mathbb{Q}$. In fact, π is transcendental, but we do not prove this.

Theorem 18.1. *The number π^2 is irrational. Hence π is irrational*

Proof. In the last definition of π given above, the function $h(x)$ is of course $\sin x$. Hence 1 is the smallest positive zero of $\sin \pi x$. We consider the integral

$$\int_0^1 f_n(x)\sin \pi x \; dx,$$

where $f_n(x)$ is a polynomial as in 17.2. Since both factors in the integrand are positive on $(0,1)$, the integral is positive. Repeated integration by parts, using the fact that the second derivative of $\sin x$ is $-\sin x$, shows that

$$\int_0^1 f_n(x)\sin \pi x \; dx = F(0) + F(1),$$

where

$$F = \frac{1}{\pi} f_n - \frac{1}{\pi^3} f_n'' \cdots + \frac{(-1)^n)}{\pi^{2n+1}} f_n^{(2n)}.$$

Assume that $\pi^2 = \frac{a}{b}$, with $a, b \in \mathbb{N}$. Then multiplying by πa^{2n} will clear the denominators in F, so by 17.2(3), the number

$$N = \pi a^{2n} \int_0^1 f_n(x)\sin \pi x \; dx$$

is a positive integer.

On the other hand, $f_n(x)\sin \pi x < \frac{1}{n!}$ for $0 < x < 1$, so

$$N < \pi \frac{a^{2n}}{n!} \longrightarrow 0$$

as $n \to \infty$. This contradiction completes the proof that π^2 is irrational.

Now suppose π is rational. Then π^2 would be rational as well, which we have just proved not to be the case. Hence π is irrational. □

It is not known if π^e is rational or irrational.