Math 216
Practice D For Final Exam

Use all previous homework and practice problems for Practice. Here are more problems. They are not all "sample exam questions" but are designed to help you grasp concepts that will be on the exam. The exam will have 10 questions, based on material in the hw, practice and earlier exams, with 2-3 questions on material after exam 2.

D1. List all of the injections $f : \{1, 2\} \longrightarrow \{1, 2, 3\}$. Then calculate the number of injections (without giving an actual list of them) from a set with k elements to a set with n elements.

D2. Previous problem, with "injection" replaced by "surjection".

D3. Give an example of a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ which is injective, but not surjective. Then give an example which is surjective but not injective.

D4. In problem C9, we saw that the set of all bijections from $\mathbb{N}$ to itself is uncountable. Let S_∞ be the set of all bijections $f : \mathbb{N} \longrightarrow \mathbb{N}$ for which $f(n) = n$ for all but finitely many $n \in \mathbb{N}$. Prove that S_∞ is countable.

D5. In C12 we have seen that the set of all subsets of $\mathbb{N}$ is uncountable. Prove that the set of all finite subsets of $\mathbb{N}$ is countable.

D6. Define $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$.

(a) Describe $f(\mathbb{R})$.
(b) Define an equivalence relation on $\mathbb{R}$ by $t \sim s$ if $t - s$ is an integer. Let S denote the set of equivalence classes. Construct a bijection between S and the unit circle.

D7. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$. Define $f : S \longrightarrow [-1, 1]$ by $f(x, y) = x$. Consider the equivalence relation on S defined by f, as in problem 7.4. Draw a picture of S showing how it is partitioned into equivalence classes.

D8. Let C_3 be the graph with vertex set $V = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (triples of 0's and 1's), with an edge between two vertices (x, y, z) and (x', y', z') if they differ in exactly one entry. Draw C_3. You should get a cube.

D9. Let $\mathbb{Z}_2^d = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (d-factors). Let C_d be the graph with vertex set $\mathbb{Z}_2^d$ and an edge between two vertices if they differ in exactly one entry. C_d is called the *hypercube* graph, in dimension d.

(a) Draw C_4.
(b) Determine the number of vertices and edges in C_d for any d.
(c) (Easy) Show that C_d is not planar for $d \geq 6$.
(d) (More fun) Show that C_4 is not planar. (One route: Assume it is planar, and use Euler to count the number of faces. Then show there are no 3-cycles in C_4, then imitate the proof of $e < 3n - 6$ to get a contradiction.)

D10. Let S be the set of all subsets of $\mathbb{Z}_2^d$.

(a) Define a bijection $f : S \longrightarrow \mathbb{Z}_2^d$.
(b) Define a graph with vertex set S such that your function f in (a) is a graph isomorphism, where $\mathbb{Z}_2^d$ is regarded as the vertex set of the graph C_d defined above.

D11. Prove that the real cube root of 2 is irrational.

D12. Find the three cube roots of 1 (see problem C2) and then find the three cube roots of 2. (Hint: The quotent of two cube roots of 2 is a cube root of 1.

D13. Prove that the three roots of $x^3 + x + 1 = 0$ are irrational.