A Trigonometric Lemma

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In our evaluation of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

we used the following fact from Trigonometry:

Lemma: For any $m \in \mathbb{N}$, we have

$$\cot^2\left(\frac{\pi}{2m+1}\right) + \cot^2\left(\frac{2\pi}{2m+1}\right) + \cdots + \cot^2\left(\frac{m\pi}{2m+1}\right) = \frac{2m(2m-1)}{6}.$$

Proof: We will prove this Lemma here, but you are not required to know this. First we explain the idea of the proof. Suppose you have a polynomial $p(t)$ of degree m, whose roots are $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_m$. This means that

$$p(t) = c(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_m),$$

where c is the coefficient of t^m in $p(t)$. The coefficient of t^{m-1} is

$$c(-\alpha_1 - \alpha_2 - \cdots - \alpha_m) = -c(\alpha_1 + \alpha_2 + \cdots + \alpha_m).$$

This means that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = -\frac{\text{coefficient of } t^{m-1}}{\text{coefficient of } t^m}. \tag{1}$$

Thus, if you have complicated sum $\alpha_1 + \alpha_2 + \cdots + \alpha_m$ on your hands, but you can find the coefficients of a polynomial having $\alpha_1,\ \ldots,\ \alpha_m$ as its roots, then you can find the sum using (1).

In our situation, $\alpha_k = \cot^2(\frac{k\pi}{2m+1})$, for $k = 1, \ldots, m$. To find $p(t)$, we let $n = 2m+1$ and start with the identity

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n,$$

then expand the right side using the binomial theorem

$$\cos nx + i \sin nx = \sum_{\ell=0}^{n} \binom{n}{\ell} \cos^{n-\ell} x (i \sin x)^{\ell}.$$

We equate the imaginary part (coefficient of i) on both sides. The ℓ^{th} term in the right side has an i when $\ell = 2j+1$, and $j = 0, 1, \ldots, m$, and $(i)^{2j+1} = i(-1)^j$, so

$$\sin nx = \sum_{j=0}^{m} (-1)^n \binom{n}{2j+1} \cos^{n-2j-1} x \sin^{2j+1} x,$$

so

$$\begin{aligned} \frac{\sin nx}{\sin^n x} &= \sum_{j=0}^{m} (-1)^j \binom{n}{2j+1} \cos^{n-2j-1} x \sin^{2j+1-n} x \\ &= \sum_{j=0}^{m} (-1)^j \binom{n}{2j+1} \cot^{n-2j-1} x. \end{aligned}$$

Recall that $n = 2m+1$. If we set $x = \frac{k\pi}{2m+1}$, the left side becomes zero, so

$$0 = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} \cot^{2m-2j} \left(\frac{k\pi}{2m+1} \right).$$

This shows that $\alpha_k = \cot^2(\frac{2k\pi}{2m+1})$ is a root of the polynomial

$$p(t) = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} t^{m-j}.$$

By (11b), we have

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = -\frac{(-1)^1 \binom{2m+1}{3}}{(-1)^0 \binom{2m+1}{1}} = \frac{2m(2m-1)}{6}.$$

This completes the proof of the Lemma. ■