Math 320 Analysis
Exam 1
October 12, 2007

This exam has five questions, worth 20 points each, for a total of 100 points.

1. Give precise and complete statements of the following (no partial credit on this problem).

(a) The definition of a convergent sequence (s_n).

A sequence (s_n) converges to a real number s if for every $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that whenever $n > N$ we have $|s_n - s| < \epsilon$.

(b) A countable set (assume only that I know what a function is).

A set A is countable if there is a function $f : \mathbb{N} \to A$ which is one-to-one and onto. This means that for every $a \in A$ there is a unique $n \in \mathbb{N}$ such that $f(n) = a$.

(c) The supremum of a subset $A \subset \mathbb{R}$.

The supremum of a subset $A \subset \mathbb{R}$ is a number s such that $s \geq a$ for all $a \in A$ and s is the smallest such number.

(d) The Axiom of Completeness.

Every nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

2. Give examples of the following. Proofs are not required on this problem.

(a) A convergent sequence (s_n).

Oh, lots, like $s_n = 1/n$, $s_n = 1$, $s_n = 2^{-n}$, etc.

(b) A subset $A \subset \mathbb{Q}$ for which $\sup(A) = \sqrt{5}$.

$A = \{t \in \mathbb{Q} : \ t^2 < 5\}$.

(c) Nonempty subsets $A_1 \supset A_2 \supset \cdots A_n \supset A_{n+1} \supset \cdots$ of $\mathbb{R}$ whose intersection is empty.

$$\cap_{n=1}^{\infty}(0, \frac{1}{n}).$$

(d) A countable subset of $(0, 1)$.

Two examples:

$$\{\frac{1}{n+1}:\ n\in\mathbb{N}\},\quad (0,1)\cap\mathbb{Q}.$$

3. Let (s_n) be a convergent sequence of real numbers, and let $c\in\mathbb{R}$. Prove that $\lim(cs_n)=c\lim(s_n)$.

Proof: Let $s=\lim(s_n)$. If $c=0$ then $cs_n=0$ for all n, so $\lim(cs_n)=0=c\lim(s_n)$. Assume $c\neq 0$. Let $\epsilon>0$. There is $N\in\mathbb{R}$ such that for all $n>N$ we have $|s_n-s|<\epsilon/|c|$. Then for all $n>N$ we have

$$|cs_n-cs|=|c||s_n-s|<|c|\frac{\epsilon}{|c|}=\epsilon.$$

Hence (cs_n) converges to cs. ■

4. Let S be the set of all sequences (s_n) where $s_n\in\{0,1\}$ and only finitely many of the s_n are nonzero. Decide, with proof, whether S is countable or uncountable.

S is countable.

Proof: Let S_n be the set of sequences in S whose terms beyond the n^{th} term are all 0. Then S_n is finite, in fact $|S_n|=2^n$. Also, every sequence in S is contained in some S_n. Therefore S is a countable union of finite sets, hence is countable. ■

5. Let $T=\{t\in\mathbb{R}:\ t^3<2\}$. Prove that $\sup T$ exists and is a cube root of 2.

Proof: If $t\in T$ then $t^3<2<2^3$ so $t<2$. Hence T is bounded above. Hence the number $\alpha=\sup(T)$ exists.

Suppose $\alpha^3<2$. For every $n\in\mathbb{N}$, we have

$$\left(\alpha+\frac{1}{n}\right)^3=\alpha^3+\frac{3\alpha^2}{n}+\frac{3\alpha}{n^2}+\frac{1}{n^3}\leq\alpha^3+\frac{3\alpha^2+3\alpha+1}{n}.$$

If we choose n so that

$$\frac{3\alpha^2+3\alpha+1}{n}<2-\alpha^3,$$

then we have

$$\left(\alpha+\frac{1}{n}\right)^3<2,$$

so $\alpha + \frac{1}{n} \in T$, contradicting the fact that α is an upper bound for T.

Suppose $\alpha^3 > 2$. For every n, we have

$$\left(\alpha - \frac{1}{n}\right)^3 = \alpha^3 - \frac{3\alpha^2}{n} + \frac{3\alpha}{n^2} - \frac{1}{n^3} \geq \alpha^3 - \frac{3\alpha^2 + 1}{n}.$$

If we choose n so that

$$\frac{3\alpha^2 + 1}{n} < \alpha^3 - 2,$$

then we have

$$\left(\alpha - \frac{1}{n}\right)^3 > 2,$$

so $\alpha - \frac{1}{n} \notin T$, contradicting the fact that α is the *least* upper bound for T. Hence $\alpha^3 = 2$. ■