Math 320 Analysis
Exam 2 Solutions
November 19, 2007

This exam has five questions, worth 20 points each, for a total of 100 points.

1. Give precise and complete definitions of the following (no partial credit on this problem).

(a) A limit point of a subset $A \subset \mathbb{R}$.

A point $x \in \mathbb{R}$ is a limit point of A if there exists a sequence (a_n) in A, with no $a_n = x$, such that $a_n \to x$.

(b) An open subset of $\mathbb{R}$.

A subset $A \subset \mathbb{R}$ is open if for all $a \in A$ there exists $\epsilon > 0$ such that if $x \in \mathbb{R}$ and $|x - a| < \epsilon$ then $x \in A$.

(c) A compact subset of $\mathbb{R}$.

A subset $K \subset \mathbb{R}$ is compact if any sequence (x_n) in K has a subsequence converging to an element of K.

(d) A continuous function $f : A \to \mathbb{R}$.

A function $f : A \to \mathbb{R}$ is continuous if for every $x \in \mathbb{R}$ and every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $y \in A$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

2. Give examples of the following. Proofs are not required on this problem.

(a) An infinite subset $A \subset \mathbb{R}$ consisting only of isolated points.

Examples include $\mathbb{N}$ and $\{\frac{1}{n} : n \in \mathbb{N}\}$.

(b) A subset $A \subset \mathbb{R}$ and an open cover of A with no finite subcover.

Take $A = (0, 1)$ and $O_n = (\frac{1}{n}, 1 - \frac{1}{n})$.

(c) A compact set containing no interval of positive length.

The Cantor set!

(d) A function which is discontinuous at every point in $\mathbb{Q}$.

Dirichlet's function $f(x) = 1$ for $x \in \mathbb{Q}$, $f(x) = 0$ for $x \in \mathbb{R} - \mathbb{Q}$.

3. Use the Nested Interval Property to prove the Axiom of Completeness.

Proof: Let $A \subset \mathbb{R}$ contain an element a_1 and be bounded above by b_1. Let $I_1 = [a_1, b_1]$. Given $I_n = [a_n, b_n]$ with $a_n \in A$ and b_n an upper bound for A, define $I_{n+1} = [a_{n+1}, b_{n+1}]$ as follows. Let c be the midpoint of $[a_n, b_n]$. If c is an upper bound for A, let $I_{n+1} = [a_n, c]$. If c is not an upper bound for A, choose $a_{n+1} \in A$ with $a_{n+1} > c$ and let $I_{n+1} = [a_{n+1}, b_n]$. By the N.I.P., the intersection $\cap I_n$ is nonempty. Since the lengths of the I_n go to zero, this intersection consists of a single element s. I claim that $s = \sup(A)$.

Suppose there is $a \in A$ such that $s < a$. Choose n such that $b_n - a_n < a - s$. Then $s - a_n < a - b_n < 0$, contradicting the fact that $s \in [a_n, b_n]$. Hence s is an upper bound for A. Let $\epsilon > 0$ and choose n such that $a_n - b_n < \epsilon$. Then $s - a_n < a_n - b_n < \epsilon$, so $s - \epsilon < a_n$. Hence s is the least upper bound for A. ■

4. Prove that the Cantor set is compact.

Proof: The Cantor set is defined as the intersection $C = \cap_{n=1}^{\infty} C_n$, where each C_n is a union of 2^n closed sub-intervals of $[0, 1]$. Since a finite union of closed sets is closed, each C_n is closed. Any intersection of closed sets is closed, hence C is closed. Finally, $C \subset [0, 1]$, so C is bounded. By the Heine-Borel theorem, C is compact. ■

5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $a \in \mathbb{R}$. Prove that the set $\{x \in \mathbb{R} : \ f(x) = a\}$ is closed.

Proof: Let x_n be a convergent sequence such that $f(x_n) = a$ for all n. Say that $x_n \to x$. Since f is continuous, we have $f(x_n) \to f(x)$. But $f(x_n) = a$ for all n, so $f(x) = a$. Hence the set $\{x \in \mathbb{R} : \ f(x) = a\}$ contains all of its limit points. ■