Math 804 Analysis I
Final Exam Solutions
December 20, 2008

Each problem is worth 30 points, for a total of 150 points. The first three problems come from our textbook. You may consult any book or website for help. However, you may not discuss or consult or ask or speak to or email or communicate in any way about any of this exam with any living person other than the instructor. Your solutions are due at or before 5:00 pm Wednesday, December 17. Good luck!

1. Let A be a closed subset of $\mathbb{R}$. Define a function $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = \inf\{|x-a| : a \in A\}$.

(a) Prove that $|\alpha(x) - \alpha(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$ and use this to show that α is continuous on $\mathbb{R}$.

Proof: We first prove that for every $x \in \mathbb{R}$ there exists $a \in A$ such that $\alpha(x) = |x - a|$. Choose a sequence $(a_n) \subset A$ such that $|x-a_n| - \alpha(x) < 1/n$ for all n. Then $|x-a_n|$ converges to $\alpha(x)$, so $|x-a_n|$ is a bounded sequence, and since $|a_n| \leq |x - a_n| + |x|$, the sequence (a_n) is bounded. Hence it has a convergent subsequence a_{n_k}. Since A is closed, a_{n_k} converges to a point $a \in A$ and we have

$$0 \leq |x-a| - \alpha(x) = \lim_k |x - a_{n_k}| - \alpha(x) \leq \lim_k \frac{1}{n_k} = 0,$$

so $|x - a| = \alpha(x)$, as desired. Now, without loss of generality, we may assume $\alpha(x) < \alpha(y)$. Choose $a \in A$ such that $\alpha(x) = |x - a|$. Then we have

$$|x-y| = |(y-a)-(x-a)| \geq |y-a| - |x-a| \geq \alpha(y) - |x-a| = \alpha(y) - \alpha(x) = |\alpha(x) - \alpha(y)|.$$

(b) Show that $A = \{x \in \mathbb{R} : \alpha(x) = 0\}$.
(Thus, every closed subset of $\mathbb{R}$ is the zero-set of a continuous function.)

If $x \in A$ then $0 \leq \alpha(x) \leq |x - x| = 0$, so $\alpha(x) = 0$. For the other containment, suppose $\alpha(x) = 0$. Recall that A is closed, and that we proved there is $a \in A$ such that $\alpha(x) = |x - a|$. But then $|x - a| = 0$, so $x = a \in A$.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and that $|f'(x)| < 1$ for all $x \in \mathbb{R}$.

(a) Prove that f has at most one fixed-point.

Since f is differentiable on $\mathbb{R}$, it is also continuous on $\mathbb{R}$, so MVT applies. If f had two fixed points, say $a < b$, the MVT guarantees there is $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.$$

This contradicts the assumption $|f'| < 1$, so f has at most one fixed point.

(b) Show that $f(x) = x + 1/(1 + e^x)$ satisfies $|f'(x)| < 1$ for all $x \in \mathbb{R}$, but f has no fixed point.

We have

$$f'(x) = 1 - \frac{e^x}{1 + e^x}$$

and

$$0 < \frac{e^x}{1 + e^x} < 1,$$

so $|f'(x)| < 1$ for all x. Since $1/(1 + e^x)$ never vanishes, $f(x) \neq x$ for any $x \in \mathbb{R}$.

3. Let f be a continuous non-negative function on the closed bounded interval $[a, b]$.
Let $||f||_\infty = \sup\{f(x) :\ x \in [a, b]\}$.

(a) Prove that for all $\epsilon > 0$ there are $c < d$ with $[c, d] \subset [a, b]$, such that $||f||_\infty - \epsilon \leq f(x)$ for all $x \in [c, d]$.

Since f is continuous on the compact set $[a, b]$ it attains its maximum at some point $x_0 \in [a, b]$ and we have $||f||_\infty = f(x_0)$. Let $\epsilon > 0$ be given. Assume first that $x_0 \in (a, b)$. Since f is continuous, we can choose $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a, b)$ and $|f(x_0) - f(x)| < \epsilon$ if $|x - x_0| < \delta$. Since $f(x_0) = ||f||_{infty}$, this implies that $||f||_\infty - \epsilon \leq f(x)$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Assume next that $x_0 = a$. Since f is continuous, we can choose $\delta > 0$ such that $[x_0, x_0 + \delta) \subset [a, b)$ and $|f(x_0) - f(x)| < \epsilon$ if $0 \leq x - x_0 < \delta$. This implies that $||f||_\infty - \epsilon \leq f(x)$ for all $x \in [x_0, x_0 + \delta]$. The case where $x_0 = b$ is entirely similar.

(b) Use part (a) to prove that

$$\lim_{p \to \infty} \left[\int_a^b f^p \right]^{1/p} = ||f||_\infty.$$

(The term in the limit is usually denoted $||f||_p$, hence the notation $||f||_\infty$.)

First, we have

$$\int_a^b f^p \leq ||f||_\infty^p (b - a).$$

Second, by part (a) and the non-negativity of f, we have

$$\int_a^b f^p \geq \int_c^d f^p \geq \int_c^d (||f||_\infty - \epsilon)^p = (||f||_\infty - \epsilon)^p (d - c).$$

Putting these inequalities together and taking p^{th} roots, we get

$$(||f||_\infty - \epsilon)(d - c)^{1/p} \leq \left[\int_a^b f^p \right]^{1/p} \leq ||f||_\infty (b - a)^{1/p}.$$

Taking the limit as $p \to \infty$ gives

$$||f||_\infty - \epsilon \leq \lim_{p\to\infty} \left[\int_a^b f^p\right]^{1/p} \leq ||f||_\infty.$$

Since ϵ was arbitrary, this gives the result.

4. Find the Bernstein polynomials $B_n(x, \kappa)$ for $n = 3, 4, 9$, where $\kappa(x)$ is the Cantor ternary function. Draw their graphs using a computer.

The polynomials are

$$\begin{aligned} B_3(x) &= \frac{x}{2}(2x^2 - 3x + 3) \\ B_4(x) &= \frac{x}{3}(2x^2 - 3x + 4) \\ B_9(x) &= \frac{x}{4}(58x^8 - 261x^7 + 612x^6 - 924x^5 + 882x^4 - 504x^3 + 168x^2 - 36x + 9) \end{aligned}$$

Their graphs are given on the next page.

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Plot[B3[x], {x, 0, 1}]
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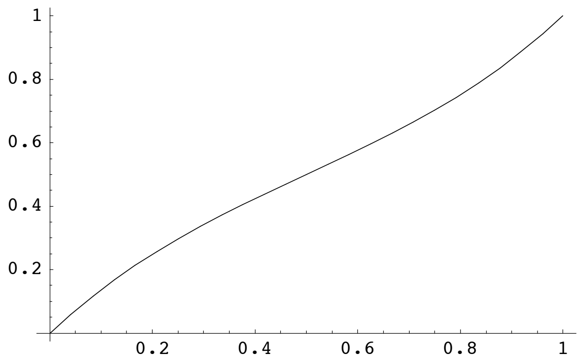


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In[20]:= Plot[B4[x], {x, 0, 1}]
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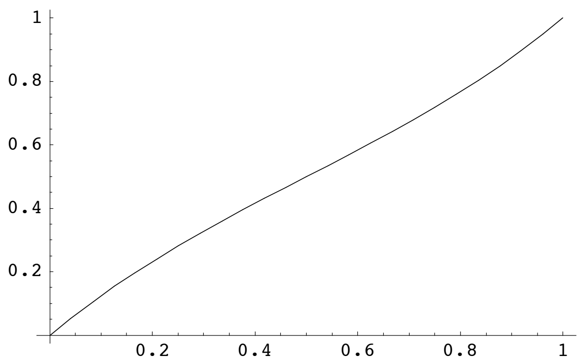


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In[21]:= Plot[B9[x], {x, 0, 1}]
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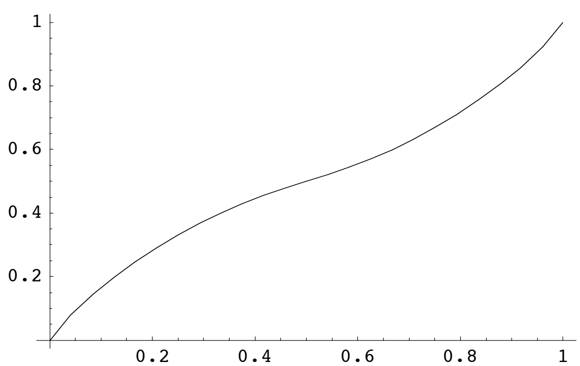

5. Let S be a subset of $\mathbb{R}$ and let $C(S)$ be the set of bounded continuous functions on S. For $f \in C(S)$, define $||f||_\infty = \sup\{|f(x)| : \ x \in S\}$ and for $f, g \in C(S)$ define $d(f, g) = ||f - g||_\infty$.

(a) Prove that $(C(S), d)$ is a metric space.

It is clear that $d(f, g) \geq 0$. Equality holds iff $\sup\{|f(x) - g(x)| : \ x \in S\} = 0$, which means $f = g$. Since $|f(x) - g(x)| = |g(x) - f(x)|$, we have $d(f, g) = d(g, f)$. Finally, given three functions f, g, h, we have

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \qquad \forall x \in S,$$

so

$$\sup\{|f(x) - h(x)| : \ x \in S\} \leq \sup\{|f(x) - g(x)| : \ x \in S\} + \sup\{|g(x) - h(x)| : \ x \in S\},$$

so $d(f, h) \leq d(f, g) + d(g, h)$.

(b) Prove that the set $B = \{f \in C(S) : \ ||f||_\infty \leq 1\}$ is closed in $C(S)$.

We will show that B^c is open. Suppose $f \in C(S)$ and $||f||_\infty > 1$. Let $\delta = ||f||_\infty - 1 > 0$ and suppose $g \in C(S)$ with $||f - g||_\infty < \delta$. Then

$$||f - g||_\infty < ||f||_\infty - 1,$$

so

$$1 < ||f||_\infty - ||f - g||_\infty \leq ||f - f + g||_\infty = ||g||_\infty.$$

Hence the open ball of radius δ centered at f is contained in B^c, so B^c is open.