MT804 Analysis Homework I

Eudoxus

September 27, 2008

p. 119, Exercise 4.2.1 a,f,g. If necessary you can use parts b,c,d,e without proof.

p. 123 prove Corollary 4.3.1

p. 125 Exercises 4.3.2, 4.3.3

Exercise 4.2.1

(a) Show that $\lambda(\varnothing) = 0$. More generally, show that $\lambda(S) = 0$ for any countable set $S \subset \mathbb{R}$.

Since S is countable, we can write $S = \{s_n : \ n \in \mathbb{N}\}$. Let $\epsilon > 0$ and let $I_n = (s_n - 2^{-n-1}\epsilon, s_n + 2^{-1-n}\epsilon)$. Then

$$\lambda(S) < \sum_{n=1}^{\infty} \lambda(I_n) = \sum_{n=1}^{\infty} 2^{-n}\epsilon = \epsilon.$$

Since ϵ was arbitrary, we have $\lambda(S) = 0$.

(f) Let C_n be as in the construction of the Cantor set. Find $\lambda(C_n)$ for each n and show that $\lambda(C_n) \to 0$.

Since C_n is obtained from C_{n-1} by removing 2^{n-1} intervals of length $1/3^n$, we have

$$\lambda(C_n) = \lambda(C_{n-1}) - (2^{n-1}/3^n),$$

so by induction we have

$$\lambda(C_n) = 1 - \tfrac{1}{2}\textstyle\sum_{k=1}^{n}(2/3)^n = 1 - \tfrac{1}{2}\tfrac{(2/3)-(2/3)^{n+1}}{1-(2/3)} = (2/3)^n,$$

which converges to zero.

(g) Show that $\lambda(C) = 0$.

Let $\epsilon > 0$ and choose n large enough that $(2/3)^n < \epsilon$. Since $C \subset C_n$ we have $\lambda(C) \leq \lambda(C_n) = (2/3)^n < \epsilon$. Since ϵ was arbitrary, we have $\lambda(C) = 0$.

Corollary 4.3.1 Prove that $f : I \to \mathbb{R}$ is continuous if and only if, $f^{-1}(F')$ is closed in I for every closed set $F' \subset \mathbb{R}$.

Proof: Assume that $f \in C(I)$ and let $F' \subset \mathbb{R}$ be a closed set. The complement $U' = \mathbb{R} \setminus F'$ is open. By Theorem 4.3.1 there is an open set $U \subset \mathbb{R}$ such that $f^{-1}(U') = U \cap I$. The complement $F = \mathbb{R} \setminus U$ is closed in $\mathbb{R}$ and $f^{-1}(F') = I \setminus f^{-1}(U') = I \setminus (U \cap I) = I \cap F$, so $f^{-1}(F')$ is closed in I.

Conversely, suppose that $f^{-1}(F')$ is closed in I for every closed set $F' \subset \mathbb{R}$. Let $U' \subset \mathbb{R}$ be an open set. The complement F' of U' is closed, so $f^{-1}(F') = F \cap I$ for some closed set $F \subset \mathbb{R}$. We have $f^{-1}(U') = F^c \cap I$, and F^c is open, so $f^{-1}(U')$ is open in I. Hence f is continuous on I. ■

Exercise 4.3.2 Show that Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$.

Proof: Let $x \in \mathbb{R}$. For $n \in \mathbb{N}$ there exist a rational $x_n \in (x, x + 1/n)$ and an irrational $y_n \in (x, x + 1/n)$, by the density of rational and irrational numbers in $\mathbb{R}$. We have

$$\lim_{n\to\infty} x_n = x = \lim_{n\to\infty} y_n,$$

but

$$\lim_{n\to\infty} f(x_n) = 1 \neq 0 = \lim_{n\to\infty} f(y_n).$$

Hence f is not continuous at x, and since x was arbitrary in $\mathbb{R}$, f is not continuous at any point in $\mathbb{R}$. ■

Exercise 4.3.3 Given $x \in [0, 1]$ with ternary expansion $x = (.x_1x_2\cdots)_3$, define $N_x = +\infty$ if $x_n \neq 1 \forall n \in \mathbb{N}$ and

$$N = \min\{k : \ x_k = 1\}$$

otherwise. Finally, let $y_n = x_n/2$ for $n \leq N$ and $y_N = 1$.

(a) Show that $\sum_{n=1}^{N} y_n/2^n$ is independent of the ternary expansion of x if x has two such expansions.

Proof: This can only happen if there is some $m \in \mathbb{N}$ such that

$$x = (.x_1 x_2 \cdots x_m 1000 \cdots)_3 = (.x_1 x_2 \cdots x_m 0222 \cdots)_3.$$

If $x_k = 1$ for some $k \leq m$ then the two expansions give exactly the same sum. So suppose that $x_k \neq 1$ for all $k \leq m$. For the first expansion we then have $N = m + 1$, so the sum is

$$\sum_{n=1}^{N} \frac{y_n}{2^n} = \sum_{n=1}^{m} \frac{x_n/2}{2^n} + \frac{1}{2^{m+1}}.$$

For the second expansion we then have $N = \infty$, so the sum is

$$\sum_{n=1}^{N} \frac{y_n}{2^n} = \sum_{n=1}^{m} \frac{x_n/2}{2^n} + \frac{0}{2^{m+1}} + \sum_{n=m+2}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{m} \frac{x_n/2}{2^n} + \frac{2^{-m-2}}{1 - 2^{-1}} = 2^{-m-1},$$

which agrees with the first sum.

(b) Show that the function $\kappa : [0, 1] \to [0, 1]$ defined by

$$\kappa(x) := \sum_{n=1}^{N} \frac{y_n}{2^n}$$

is a continuous, monotone function onto $[0, 1]$.

Proof: Let $\epsilon > 0$. Following the hint in the book, choose $k \in \mathbb{N}$ such that $2^{-k} < \epsilon$ and let $\delta = 3^{-k}$. Then if $x, x' \in [0, 1]$ satisfy $|x - x'| < \delta$, they agree in the first k digits:

$$x = (.x_1 x_2 \cdots x_k x_{k+1} \cdots)_3 \qquad \text{and} \qquad x' = (.x_1 x_2 \cdots x_k x'_{k+1} \cdots)_3.$$

Let N, y_i and N', y'_i be as above for x and x', respectively. Set $y_i = 0$ for $i > N$ and $y'_i = 0$ for $i > N'$. The first possibility is that $x_i \neq 1$ for all $i \leq k$. then

$$\kappa(x) = \sum_{i=1}^{k} \frac{x_i/2}{2^i} + \sum_{i=k+1}^{N} \frac{y_i}{2^i} \qquad \text{and} \qquad \kappa(x) = \sum_{i=1}^{k} \frac{x_i/2}{2^i} + \sum_{i=k+1}^{N'} \frac{y'_i}{2^i}$$

so

$$|\kappa(x) - \kappa(x')| = \left| \sum_{i=k+1}^{\infty} \frac{y_i - y'_i}{2^i} \right| \leq \sum_{i=k+1}^{\infty} \frac{|y_i - y'_i|}{2^i} \leq \sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{2^{-k-1}}{1 - 1/2} = 2^{-k} < \epsilon.$$

The other possibility is that $N = N' \leq k$. In this case, we have

$$\kappa(x) = \sum_{i=1}^{N-1} \frac{x_i/2}{2^i} + \frac{1}{2^N} = \kappa(x'),$$

so

$$|\kappa(x) - \kappa(x')| = 0 < \epsilon.$$

Next, we show that κ is monotonic. Assume $x < x'$. We must show that $\kappa(x) \leq \kappa(x')$. Write ternary expansions so that we never have all zeros to the right of some digit. Again assume $x_i = x'_i$ for all $i \leq k$ and $x_{k+1} \neq x'_{k+1}$. We can also assume that $x_i \neq 1$ for all $i \leq k$, since otherwise $\kappa(x) = \kappa(x')$ and we're done. Let $r = \sum_{i=1}^{k} \frac{x_i/2}{2^i}$

Since $x < x'$, we have $x_{k+1} < x'_{k+1}$. If $x_{k+1} = 1$ then $N = k+1$ and we have $\kappa(x) = r + 2^{-k-1}$. We must then have $x'_{k+1} = 2$. Let x_ℓ be the next nonzero digit. Then $\kappa(x') \geq r + 2^{-k-1} + 2^{-\ell} > \kappa(x)$, as desired.

The other possibility is $x_{k+1} = 0$. In this case we have

$$\kappa(x) \leq r + \sum_{i=k+2}^{\infty} 2^{-i} = r + 2^{-k-1},$$

and $x_{k+1} \in \{1, 2\}$, so

$$\kappa(x') \geq r + 2^{-k-1} \geq \kappa(x),$$

again as desired. So κ is monotonic.

Finally, we show that $\kappa : [0,1] \to [0,1]$ is onto. Let $y = (.y_1 y_2 \cdots)_2$ be the binary expansion of a number $y \in [0,1]$, with $y_i \in \{0,1\}$ for all i. Let $x_i = 2y_i$ for all i. Then no $x_i = 1$, so

$$\kappa(x) = \sum_{i=1}^{\infty} \frac{x_i/2}{2^i} = \sum_{i=1}^{\infty} \frac{y_i}{2^i} = y.$$

Note that x is actually in the Cantor set, so in fact $\kappa : C \to [0,1]$ is already surjective, as well as being injective and continuous.